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ON FERMIONIC GREEN'S FUNCTION AND FERMIONIC DETERMINANT IN MASSLESS QCD_2

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Exact fermionic Green's function in massless QCD_2 is obtained by introducing generalized gauge transformations. An explicit, compact expression for the causal fermionic Green's function is given provided that the gauge potentials satisfy certain conditions. The functional determinant arising from functional integration over the fermionic fields is calculated. The results are expressed in terms of path ordered exponentials.

1. Introduction

Two dimensional QCD is a well-established source of ideas and guesses about color dynamics in the original 4-dimensional QCD. Lack of two spatial dimensions causes great simplifications of the theory. Parallelly, it is believed that results obtained in such a restricted space-time can be nevertheless useful in studies of the 4-dimensional theory. Intensive studies of QCD_2 have been conducted (for a review, see e.g. Ref. [1]), taking full advantage of simplifications present in 2-dimensional space-time.

It is reasonable to expect that, similarly as for many others 2-dimensional models, one can construct an explicit, exact solution of QCD_2, at least for the massless version, which is expected to be the simplest one. By an analogy with thoroughly investigated Abelian QED_2, the first step in this direction could be obtaining the exact, massless causal fermionic Green's function, i.e. exact fermionic propagator. In fact, it was already done to some extent in papers [2, 3] (but not in completely satisfactory way — see Sections 3 and 5 below), of which we became aware after the substantial part of this work was completed. In those papers, apart from an expression for the fermionic propagator, also fermionic current and axial anomaly relation for massless QCD_2 were constructed within the framework of functional approach. The gluonic part of QCD_2 is unsolved as yet. However, some information about exact gluonic propagator in weak and strong coupling limits is already available [2, 3]. In particular, it seems that color is not confined in QCD_2.

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The results contained in our paper and new in comparison with Refs. [2] and [3] are the following. We rederive the fermionic propagator by a method different from that presented in Refs. [2] and [3]. Namely, we introduce a generalized gauge transformation, in a strict analogy with the approach to QED_2 [4]. We find factorisation property of the fermionic propagator. In order to satisfy the causality condition and at the same time to be able to write an explicit, compact expression (not having the form of a formal perturbative expansion) for the fermionic propagator we are forced to assume that the gauge potentials satisfy certain conditions explained in Section 3. We argue that for general gauge potentials, an explicit compact expression for the fermionic propagator is still not known (see more on this point in Section 5). Under those conditions, the fermionic propagator takes very elegant form, in which the gauge potentials are present only in the form of path ordered exponentials (generalized phase factors in terminology of C. N. Yang). Next, we calculate the fermionic functional determinant Det $(\tilde{\partial}-ie\hat{A}\gamma)$ for potentials $\hat{A}_\mu$ satisfying the above mentioned condition. We use the external field technique which guarantees explicit gauge invariance of the results on each step of calculations. Thus obtained, compact expression (45) for the determinant contains path ordered exponentials of the gauge potential. The knowledge of the determinant can be useful, e.g., when one wants to perform explicitly integration over fermion fields in the path integral formula for n-point Green's functions. In a sense, our paper provides also a comparison between massless QED_2 and QCD_2.

Our considerations are carried out on the level parallel to the zero-instanton sector of QED_2 (i.e. Schwinger's model). In particular, we do not consider a possible existence and possible effects of zero-modes connected with topologically nontrivial configurations (in principle, such configurations in QCD_2 are possible — they are given by planar cross sections of nonabelian vortices [5].

Let us mention that while in papers [2, 3], and also Ref. [6], fermionic part of QCD_2 is studied mainly with respect to problem of color confinement, our motivation for this work is quite different. We are interested mainly in getting some insight into structure of the effective Lagrangian for a model of a nonabelian gauge theory. For example — to what extent can this structure be expressed in terms of some path ordered exponentials?

2. *Fermionic Green's function*

The equation for the fermionic Green's function is

$$\gamma_\mu(\partial^\mu - ieT^a A^{a\mu})S(x, y) = \mathbf{1}\cdot\delta(x-y), \tag{1}$$

where $\mathbf{1}$ is the unit matrix in isospace, T^a are generators of the fundamental representation of SU(N). Conventions are the following:

$$\gamma_0 = i\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = \sigma_2, \quad \gamma_1 = i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = i\sigma_1,$$

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}, \quad (g_{\mu\nu}) = (1, -1).$$

In order to solve (1), we generalize to the nonabelian case the procedure [4] used in Schwinger's model. Gauge transformations of the external potential A_μ^a have the form

$$\hat{A}'_\mu = g^{-1}\hat{A}_\mu g + \frac{i}{e} g^{-1}\partial_\mu g, \tag{2}$$

where $\hat{A}_\mu \equiv T^a A_\mu^a$, $g \in SU(N)$. We generalize the gauge transformations (2) by assuming that g has mixed $\gamma_\mu - T^a$ character,

$$g(x) = \begin{pmatrix} I_-^{-1}(x) & 0 \\ 0 & I_+^{-1}(x) \end{pmatrix}, \tag{3}$$

and we look for $I_\pm$ such that

$$\gamma_\mu \hat{A}^\mu = \frac{i}{e} \gamma_\mu g^{-1}\partial^\mu g. \tag{4}$$

We have

$$\gamma_\mu \hat{A}^\mu = i\begin{pmatrix} 0 & -\hat{A}^- \\ \hat{A}^+ & 0 \end{pmatrix},$$

$$\gamma_\mu g^{-1}\partial^\mu g = 2i\begin{pmatrix} 0 & -I_+ \dfrac{\partial}{\partial x^+} I_+^{-1} \\ I_- \dfrac{\partial}{\partial x^-} I_-^{-1} & 0 \end{pmatrix},$$

where $x^\pm = x^0 \pm x^1$, $\hat{A}^\pm = \hat{A}^0 \pm \hat{A}^1$, $\partial_0 \pm \partial_1 = 2\dfrac{\partial}{\partial x^\pm}$.

Thus,

$$\frac{\partial}{\partial x^+} I_+ = \frac{ie}{2}\hat{A}^- I_+, \qquad \frac{\partial}{\partial x^-} I_- = \frac{ie}{2}\hat{A}^+ I_-. \tag{5}$$

These equations can also be written in the form

$$D_+ I_+ = 0, \qquad D_- I_- = 0,$$

where

$$D_\pm = \partial_\pm - \frac{ie}{2}\hat{A}_\pm. \tag{6}$$

Their general solutions are

$$I_+(x) = T_+(x)\tau_+(x^-), \qquad I_-(x) = T_-(x)\tau_-(x^+), \tag{7}$$

where

$$T_+(x) = P\exp\left[\frac{ie}{2}\int_{-\infty}^{x^+} dx'^+ \hat{A}^-(x'^+, x^-)\right],$$

$$T_-(x) = P\exp\left[\frac{ie}{2}\int_{-\infty}^{x^-} dx'^- \hat{A}^+(x^+, x'^-)\right]. \tag{8}$$

Here P denotes path ordering and $\tau_+(x^-)$, $\tau_-(x^+)$ are arbitrary functions of indicated variables. These functions should be appropriately adjusted in order to satisfy chosen boundary conditions for $S(x, y)$. Note that the exponentials are the path ordered exponentials

$$P\exp\left[ie \int_{C_1, C_2} dx_\mu \hat{A}^\mu\right],$$

where C_1, C_2 are straight lines from $-\infty$ to x^- or x^+, parallel to x^--axis or x^+-axis, respectively. These lines are in fact sides of 2-dimensional light-cone of the past for the point $x = (x^+, x^-)$.

Now it is possible to factorize $\hat{A}_\mu$-dependence of $S(x, y)$. Substitute

$$S(x, y) = g^{-1}(x) S_0(x, y) h(y), \tag{9}$$

together with (4) to (1). Here S_0 is the $\hat{A}_\mu = 0$ Green's function of (1), and $h(y)$ is to be determined. Because of the identity

$$\gamma_\mu \begin{pmatrix} I_- & 0 \\ 0 & I_+ \end{pmatrix} = \begin{pmatrix} I_+ & 0 \\ 0 & I_- \end{pmatrix} \gamma_\mu \tag{10}$$

we obtain

$$\begin{pmatrix} I_+(x) & 0 \\ 0 & I_-(x) \end{pmatrix} [\gamma_\mu \partial^\mu S_0(x, y)] h(y) = \mathbf{1}\delta(x-y).$$

Because

$$\gamma_\mu \partial^\mu S_0(x, y) = \delta(x-y),$$

then

$$h(y) = \begin{pmatrix} I_+^{-1}(y) & 0 \\ 0 & I_-^{-1}(y) \end{pmatrix}. \tag{11}$$

Thus, the Green's function has the form (9), with $g(x)$ given by (3), (7) and $h(y)$ given by (11). We observe that all dependences on the external potential $\hat{A}_\mu$ have factorized out. Thus, we have obtained the nonabelian generalization of the factorization property of Schwinger's model. The solution (9) is equivalent to the solution given in Ref. [3].

Let us observe that the above derivation of the Green's function applies without any changes also to the Schwinger model. The only difference is that the path ordering P is superficial in that case. This provides us with useful test of our formulae — at least for gauge potentials $\hat{A}_\mu$ commuting at different space-time points they should reduce to the correscponding formulae for the Schwinger's model. Also this fact makes it possible to compare between QED_2 and QCD_2.

3. *Causality versus explicit solvability*

In this Section we show how the original Schwinger [4] expression for the causal Green's function in the Abelian case can be recovered from expression (9) for the Green's function. This analysis allows us to find a condition for the gauge potentials under which the fermion propagator is given directly by the path ordered exponentials (i.e. $\tau_\pm = \mathbf{1}$). Next, we find a generalization of this condition for the nonabelian case. In that way we have a criterium for picking up a class of gauge potentials distinguished by the fact that the causal Green's function has the elegant form (9) with $\tau_\pm = \mathbf{1}$. Moreover, in general the functions $\tau_\pm(x^\mp)$ are extremely complicated and their explicit form for QCD_2 is still unknown (concerning this problem, we have nothing to add to paper [3], where the rather complicated integral equations for $\tau_\pm$ are given). Therefore, the only case where we can write an explicit expression for the fermionic propagator (apart from a perturbative expansion given in Ref. [6]) is when the above mentioned condition is satisfied. In this sense, the causality requirement is up to now partially at odds with explicit solvability.

Let us recall that in the original approach [4] to QED_2 the fermion propagator has the form

$$S_c(x, y) = \exp\left[ie\Phi(x)\right] \exp\left[-ie\Phi(y)\right] S_0(x-y), \tag{12}$$

where

$$\Phi = \Phi_1 - \gamma_5 \Phi_0; \qquad \gamma_5 = \gamma_0 \gamma_1, \tag{13}$$

and

$$\Phi_0 = \frac{1}{\Box_c} \varepsilon^{\mu\nu} \partial_\mu A_\nu, \qquad \Phi_1 = \frac{1}{\Box_c} \partial^\mu A_\mu, \tag{14}$$

where the free scalar Green's function $\frac{1}{\Box}$ is chosen to be the causal one.

In order to recover (12) from (9), we observe first that it is always possible to write that

$$A^+ = \partial^+ \chi_1, \qquad A^- = \partial^- \chi_2, \tag{15}$$

where χ_1, χ_2 are some functions. These formulae reflect the trivial fact that any locally integrable function (and we assume this property for A_μ) can be written as a derivative of another function. They also follow from Eqs. (5) after the substitution $I_\pm = \exp\left[-\frac{ie}{2}\chi_{1,2}\right]$

which is possible because we assume that $I_\pm \neq 0$ (as follows from the assumed existence of g in Eq. (3)). What is nontrivial in (15) is that specifically in 2-dimensional space--time χ_1, χ_2 are Lorentz scalars. Formulae (15) allow us to perform the x^+, x^- integrations in (8). For instance,

$$T_-(x) = \exp\left[\frac{ie}{2}\chi_1(x)\right]\exp\left[-\frac{ie}{2}\chi_1(x^+, -\infty)\right]. \tag{16}$$

From (16) it is clear that if we can construct the functions χ_1, χ_2 such that they contain only positive frequencies for $x_0 \to +\infty$ and only negative frequencies for $x_0 \to -\infty$ (we shall refer to this condition as to the positive-negative frequency condition), then the Green's function will be the causal one, provided that the last factor in (16) and the $\tau_\pm(x^\mp)$ functions are somehow removed. By choosing $\tau_-(x^+) = \exp\left[\frac{ie}{2}\chi_1(x^+, -\infty)\right]$ (and similarly for $\tau_+(x^-)$) one can make these factors to cancel each other.

The construction of causal χ_1, χ_2 is the following. From (15) we have

$$4\partial^- A^+ = \Box\chi_1, \quad 4\partial^+ A^- = \Box\chi_2. \tag{17}$$

As the boundary conditions for χ_1, χ_2 we choose the positive-negative frequency condition. Then, Eqs. (17) have unique solutions of the form

$$\chi_1 = \frac{4}{\Box_c}\partial^- A^+, \quad \chi_2 = \frac{4}{\Box_c}\partial^+ A^-. \tag{18}$$

However, the fact that Eqs. (17) were obtained from (15) does not immediately mean that (18) also satisfies Eqs. (15). For instance, if (15) have had the form

$$A^\pm = \partial^\pm \chi_{1,2} + \partial^\pm f_{1,2} \tag{15'}$$

with some fixed functions f_1, f_2 satisfying $\Box f_{1,2} = 0$ we would obtain also Eqs. (17) with the same causal solution (18), and, of course, it cannot satisfy both (15) and (15'). Thus we still have to check whether $A^\pm$ satisfy the identities

$$\partial^\pm\left[\frac{4}{\Box_c}\partial^\pm A^\pm\right] = A^\pm. \tag{19}$$

For potentials which do not vanish within an infinite domain in space-time, it is necessary to assume that they satisfy the positive-negative frequency condition in order to have (19) [7]. However, for fields with a restricted space-time domain, (19) is satisfied identically, as it can be easily verified by using the second part of the formula (21) below.

The expression (12) is obtained after introducing Φ_0, Φ_1 by $\chi_1 = 2(\Phi_1 - \Phi_0)$, $\chi_2 = 2(\Phi_1 + \Phi_0)$.

In the original approach [4] the condition (19) is obtained when one inserts (12) as an ansatz in the Dirac equation (1), as the condition for cancellation of the $A^\mu\gamma_\mu$ term present in (1).

For future use in the nonabelian case, let us notice that if we want to put $\tau_\pm = 1$, then for causality it is necessary to have $\chi_1(x^+, -\infty) = \chi_2(-\infty, x^-) = 0$. These conditions for χ_1, χ_2 are satisfied when

$$\int_{-\infty}^{+\infty} A^+ dx^- = \int_{-\infty}^{+\infty} A^- dx^+ = 0, \tag{20}$$

as it can be easily seen from the formula

$$\chi_{1,2} = \frac{4}{\Box_c} \partial^\mp A^\pm = \frac{i}{\pi} \text{P.v.} \int \frac{1}{x^\pm - z^\pm} A^\pm(z) d^2 z + \int \delta(x^\pm - z^\pm) \text{ sign } (x^\mp - z^\mp) A^\pm(z) d^2 z. \tag{21}$$

The second part of this formula is easily derived from the formula [7]

$$\frac{1}{\Box_c} f(x) = \frac{i}{4\pi} \int \ln [\mu^2 (x-z)^2 + i0] f(z) d^2 z,$$

where μ is a dimensional parameter.

The conditions (20) can always be satisfied formally after performing a suitable gauge transformation. However, such a gauge transformation does not vanish at infinity and the gauge transformed potential can be easily shown not to vanish at infinity. But then, on the whole, our consideration does not apply to it (in particular, the condition (19) becomes difficult to satisfy). Thus, the gauge transformations are no remedy for the conditions (20).

Unfortunately, it is difficult to generalize the above recipe (15–19) for obtaining the causal Green's function to the nonabelian case. The difference between the two cases comes from nontriviality of the path ordering operation P in (8) in the nonabelian case. This causes that the integrations in the exponents cannot be performed directly by repeating the QED_2 trick with χ's. In particular, it is difficult to find explicitly the correct $\tau_\pm$ for arbitrary potentials $\hat{A}^\pm$.

Therefore we set $\tau_\pm = \mathbf{1}$. Then we expand the exponentials $T_\pm$,

$$T_\pm(x) = \sum_{k=0}^{\infty} \left(\frac{ie}{2}\right)^k X_k^\pm(x), \tag{22}$$

where, for instance,

$$X_k^-(x^+, x^-) = \int_{-\infty}^{x^-} dx_1^- \hat{A}^+(x^+, x_1^-) \int_{-\infty}^{x_1^-} dx_2^- \hat{A}^+(x^+, x_2^-) \dots \int_{-\infty}^{x_{k-1}^-} dx_k^- \hat{A}^+(x^+, x_k^-), \tag{23}$$

i.e.:

$$X_{k+1}^-(x^+, x^-) = \int_{-\infty}^{x^-} dx_1^- \hat{A}^+(x^+, x_1^-) X_k^-(x^+, x_1^-). \tag{24}$$

Substituting (22) into Eq. (5), which $T_\pm$ satisfy, one gets

$$\frac{\partial}{\partial x^\pm} X^\pm_{k+1} = \hat{A}^\mp X^\pm_k. \tag{25}$$

These formulae form the nonabelian generalization of (15) ($X^\pm_k$ correspond to $\chi_{1\ 2}$). An analogon of (17) is

$$\tfrac{1}{4}\,\Box X^\pm_{k+1} = \frac{\partial}{\partial x^\mp}(\hat{A}^\mp X^\pm_k), \tag{26}$$

which has unique causal solution

$$X^\pm_{k+1} = \frac{4}{\Box_c}\frac{\partial}{\partial x^\mp}(\hat{A}^\mp X^\pm_k). \tag{27}$$

The assumption (19) has now the form

$$\hat{A}^\pm X^\mp_k = \partial^\pm\left(\frac{4}{\Box_c}\partial^\mp(\hat{A}^\pm X^\mp_k)\right). \tag{28}$$

For $k = 0$, (28) takes exactly the form (19). For $\hat{A}^\pm$ with bounded support, (28) is satisfied identically.

However, we still face one problem, namely compatibility of the two recursive relations (24) and (27). Using the formula (21) with $\hat{A}^\pm$ replaced by $\hat{A}^\mp X^\pm_k$ it is easy to see that (24) and (27) are compatible if

$$\int_{-\infty}^{+\infty} dx^\mp \hat{A}^\pm X^\mp_k = 0 \tag{29}$$

for all k. This condition is equivalent to

$$T_-(x^+, x^- = +\infty) = T_+(x^+ = +\infty, x^-) = \mathbf{1}. \tag{30}$$

Conditions (29) are the nonabelian analogon of the condition (20).

When the condition (30) is not satisfied, it is necessary to have $\tau_\pm \neq \mathbf{1}$. In this case one can approach the problem of obtaining the fermion propagator in two ways. One can try to preserve structure (7) of the $I_\pm$ by keeping factors $T_\pm$ and calculating the needed for causality $\tau_\pm$ [3]. Then one obtains a rather complicated integral equations for $\tau_\pm$. Their solution is not known. The other approach is based on a formal perturbative solution of Eqs. (5) for $I_\pm$ without any reference to path ordered exponentials. The result is written as an ∞, formal series in powers of the coupling constant e [6]. Both approaches can turn out to be useful in discussing particular problems, but nevertheless they do not yield an explicit compact expression for $I_\pm$. Therefore we take the point of view that unless (30) is satisfied, an explicit solution for the fermionic, causal Green's function in QCD_2 is still not known (apart from a formal perturbative expansion).

Let us summarize the results of Sections 2 and 3. The fermionic (massless) causal Green's function has the form

$$S_c(x, y) = \begin{pmatrix} T_-(x) & 0 \\ 0 & T_+(x) \end{pmatrix} S_0(x-y) \begin{pmatrix} T_+^{-1}(y) & 0 \\ 0 & T_-^{-1}(y) \end{pmatrix}, \tag{31}$$

where $T_\pm(x)$ are given by (8), provided that the gauge potentials $\hat{A}^\pm$ satisfy the conditions (30). For $\hat{A}^\pm$ which do not satisfy (30), an explicit expression for the Green's function is still not known (apart from a formal perturbative expansion).

4. *Calculation of the fermionic determinant*

It is well-known that in order to obtain a gauge-invariant effective Lagrangian for gluon fields one can use the background field method [8]. We shall find that a slight modification of this method is also useful in obtaining the gauge invariant effective Lagrangian for fermions, provided that one knows the fermionic Green's function.

The method consists of splitting off the gauge potential on two parts

$$e\hat{A}_\mu = e\hat{A}_{\mu\mathrm{cl}} + e'\hat{A}_{\mu\mathrm{qu}}, \tag{32}$$

called, correspondingly, classical and quantum. $\hat{A}_{\mu\mathrm{cl}}$ can be regarded as normalized vacuum expectation value of the operator gluon field in the presence of external sources $\hat{j}_{\mu\mathrm{ext}}$ for gauge fields [9]. The external sources are introduced into the Lagrangian in order to generate gluonic n-point Green's functions. $\hat{A}_{\mu\mathrm{cl}}$ vanishes in absence of the external sources. $\hat{A}_{\mu\mathrm{qu}}$ describes quantum fluctuating part of the gauge field with the vanishing vacuum expectation value. For future use we have slightly generalized the standard approach by introducing intermediate (auxiliary) independent coupling constants e, e' for $\hat{A}_{\mu\mathrm{cl}}$ and $\hat{A}_{\mu\mathrm{qu}}$, respectively. The case of interest is, of course, $e = e'$. The gauge transformations of these fields have the following form

$$\hat{A}'_{\mu\mathrm{cl}} = g^{-1}\hat{A}_{\mu\mathrm{cl}}g + \frac{i}{e}\, g^{-1}\partial_\mu g, \tag{33a}$$

$$\hat{A}'_{\mu\mathrm{qu}} = g^{-1}\hat{A}_{\mu\mathrm{qu}}g. \tag{33b}$$

Then, the full gauge potential $\hat{A}_\mu$, given by (32), has the correct gauge transformation (2).

We want to calculate $\mathrm{Det}\,(\partial\!\!\!/ - ieA\gamma)$ $(= \exp(-iS_{\mathrm{eff}})$, where S_{eff} is effective fermionic action). To this end we shall use the following generalization of the well-known [10] Pauli's formula (derived in Appendix A)

$$\log \mathrm{Det}\,(\partial\!\!\!/ - ie\hat{A}\gamma) = \log \mathrm{Det}\,(\partial\!\!\!/ - ie\hat{A}_{\mathrm{cl}}\gamma) - i\int_0^e de'\, \mathrm{Tr}\,[S_c(;\, e\hat{A}_{\mathrm{cl}} + e'\hat{A}_{\mathrm{qu}})\hat{A}_{\mathrm{qu}}\gamma]. \tag{34}$$

The first term on the right-hand side does not need to be calculated because it is constant with respect to the quantum gluonic field $\hat{A}_{\mu\mathrm{qu}}$. The second term contains the causal fermionic Green's function calculated in Sections 2 and 3. The trace refers to the trace over Lorentz and isospin matrix indices and also to the trace over "space-time indices" x, y

in $S_c(x, y; e\hat{A})$. This latter trace means $\int d^2x S_c(x, x; e\hat{A})$. This requires some special treatment because the integrand is not well-defined (the limit $x \to y$ is singular). The most convenient approach consists of use of the standard point-splitting technique.

From Eq. (1) it follows that under gauge transformation (2)

$$S(x, y; e\hat{A}') = g^{-1}(x)S(x, y; e\hat{A})g(y). \tag{35}$$

This, together with (33b), ensures formal gauge invariance of the effective action (34).

Thus, we want to calculate

$$\mathrm{Tr}\,[S_c(; e\hat{A}_{cl}+e'\hat{A}_{qu})\hat{A}_{qu}\gamma] = \int d^2x \lim_{\substack{x''\to x\\ x'\to x}} \mathrm{Tr}\,[S_c(x'', x'; e\hat{A})\hat{A}_{qu}\gamma]. \tag{36}$$

The trace on the right-hand side is taken with respect to Lorentz and color indices, and for S_c we take (31), where

$$S_0(x''-x') = \frac{1}{2\pi}\frac{\gamma_\mu(x''-x')^\mu}{(x''-x')^2+i0}. \tag{37}$$

We set $x' = x-\frac{\varepsilon}{2}$, $x'' = x+\frac{\varepsilon}{2}$ and we consider the limit $\varepsilon \to 0$. It is known [11] that when taking this limit, one should approach the singularity in S_0 from a space-like direction. Thus, we first take $\varepsilon_0 \to 0$ and then $\varepsilon_1 \to 0$. In the light-cone coordinates it means that $\varepsilon^+ = -\varepsilon^- \to 0$. In order to ensure that the limiting procedure will yield a gauge and Lorentz invariant result, it is necessary to perform the parallel transport of $S_c(x'', x')$ from points x'', x' to point x [11]. This amounts to multiplying $S(x'', x')$ by

$$P\exp\left[ie\int_x^{x'} dx^\mu \hat{A}_\mu\right] \cong \mathbf{1}-\frac{ie}{4}(\varepsilon^-\hat{A}^+ +\varepsilon^+\hat{A}^-) \tag{38}$$

from the right, and by

$$P\exp\left[ie\int_{x''}^{x} dx^\mu \hat{A}_\mu\right] \cong \mathbf{1}-\frac{ie}{4}(\varepsilon^-\hat{A}^+ +\varepsilon^+ A^-) \tag{39}$$

from the left. The identity (10) applied to γ's present in S_0 allows us to write $S_c(x'', x'; e\hat{A})$ in the form

$$S(x'', x'; e\hat{A}) = \begin{pmatrix} T_-(x'')T_-^{-1}(x') & 0 \\ 0 & T_+(x'')T_+^{-1}(x') \end{pmatrix} S_0(x'', x'), \tag{40}$$

where

$$T_\pm(x'')T_\pm^{-1}(x') \cong \mathbf{1}+\varepsilon^-\left(\frac{\partial}{\partial x^-}T_\pm(x)\right)T_\pm^{-1}(x)+\varepsilon^+\left(\frac{\partial}{\partial x^+}T_\pm(x)\right)T_\pm^{-1}(x). \tag{41}$$

Inserting (40), (41) in (36), multiplying the integrand in (36) by factors (38), (39) and calculating the trace with respect to spinor indices one obtains in the $\varepsilon^+ = -\varepsilon^- \to 0$ limit

$$\mathrm{Tr}\,[S\hat{A}_{\mathrm{qu}}\gamma] = \frac{1}{2\pi}\int d^2x\,\mathrm{Tr}\,[(D_-T_+)T_+^{-1}\hat{A}^-_{\mathrm{qu}}+(D_+T_-)T_-^{-1}\hat{A}^+_{\mathrm{qu}}], \tag{42}$$

where D_+, D_- are given by (6) and $e\hat{A}_\mu$ is splitted according to (32). The trace is only with respect to color indices. In obtaining (42) we have used Eqs. (5) which hold also for $T_\pm$ and the fact that $Tr\hat{A}^\pm_{\mu\mathrm{qu}} = 0$ (because $\hat{A}_\mu = A^a_\mu T^a$ and the SU(N) generators T^a are traceless). The quantities $T_\pm$ are Lorentz scalars despite their indices "$\pm$", as it can be easily seen from (8). Under gauge transformation (2) they transform like

$$T_\pm(A') = g^{-1}(x)T_\pm(A)g(x^\pm = -\infty). \tag{43}$$

In the nonabelian gauge theories one usually assumes that the gauge transformations satisfy condition $g(\infty) = \mathbf{1}$. Therefore the last factor in (43) is equal to $\mathbf{1}$ and

$$T_\pm(A') = g^{-1}(x)T_\pm(A). \tag{44}$$

Observe that the condition (30) as well as the basic requirement of vanishing of $\hat{A}^\pm$ at ∞, are gauge invariant under the above class of gauge transformations. The Lorentz and gauge invariance of (42) is now obvious. The covariant derivatives $D_\pm T_\mp$ can be calculated explicitly, i.e. expressed by the gauge potential and some path ordered exponentials. We present this calculation in the Appendix B.

Thus, the effective action generated by fermions has the form

$$S_{\mathrm{eff}} = i\log\mathrm{Det}\,(\tilde{\partial}-ie\hat{A}_{\mathrm{cl}}\gamma)+\frac{1}{2\pi}\int_0^e de'\int d^2x\,\mathrm{Tr}\,[(D_-T_+)T_+^{-1}\hat{A}^-_{\mathrm{qu}}+(D_+T_-)T_-^{-1}\hat{A}^+_{\mathrm{qu}}], \tag{45}$$

where the trace is taken with respect to color indices. The potential $\hat{A}_\mu$ present in $T_\pm$ and $D_\pm$ is decomposed according to (32). The formula (45) contains only c-numbers and matrices T^a. Therefore we hope that it will be possible to analyse in detail the dependence of S_{eff} on $\hat{A}_{\mu\mathrm{qu}}$ in order to gain some information about dynamical properties of gluons in QCD_2. Certainly this effective Lagrangian is not simply quadratic in $\hat{A}_{\mu\mathrm{qu}}$, as it is the case in QED_2.

The above derivation of the effective Lagrangian holds also in the general case when the assumption (30) is not satisfied. The result still has the form (45), with $T_\pm$ replaced by $I_\pm$, respectively.

5. *Final remarks*

Here we want to stress once more that in our opinion the condition (30) is up to now necessary in order to have an explicit and causal expression for the Green's function, not having the form of a formal perturbative expansion. In particular, Ref. [3], where such assumption is not made, does not give an explicit expression. The procedure chosen there ensures causality, but the constructed Green's function still contains unknown functions (likely with a very complicated dependence on the potential A^a_μ), for which only integral equations are given. On the other hand, expression given in Ref. [2] is explicit, equivalent to our (31), but it should be completed with an assumption of type (30) in order to ensure causality (the fact that the solution given in Ref. [2] in general is not causal was noticed already in Refs. [3] and [6]). In our opinion, the expression which is causal and does not assume anything of type (30) about $\hat{A}_\mu$ is still not known (apart from a formal perturbative expansion). When the condition (30) is satisfied, the fermionic propagator has the very simple and elegant form (31).

We also want to point out that path ordered exponentials, present in the effective Lagrangian (45), in the nonabelian case in general cannot be reduced to a simpler form by introducing some analogons of the χ's from QED_2. The presence of such path ordered exponentials in the effective Lagrangian (45) is very interesting because it is a confirmation of the point of view that dynamics of the nonabelian gauge theory should be formulated in terms of objects containing explicitly such generalized phase factors. Just such path ordered exponentials recently were used to recover string model from QCD in four dimensional space-time [12]. It would be very desirable to carry out a mathematical study of the path ordered exponentials regarded as a kind of primary mathematical objects in nonabelian gauge theories.

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APPENDIX A

Here we present the derivation of the formula (34). We have

$$S(;e\hat{A}_\mu) = \frac{1}{\gamma\partial - ie\hat{A}_{cl}\gamma - ie'\hat{A}_{qu}\gamma} = (1 - ie'S_{cl}\hat{A}_{qu}\gamma)^{-1}S_{cl},$$

where

$$S_{cl} = \frac{1}{\gamma\partial - ie\hat{A}_{cl}\gamma},$$

and

$$\text{Det}\, S^{-1} = \text{Det}\, S_{cl}^{-1} \cdot \text{Det}\,(1 - e'X), \tag{A1}$$

where $X = iS_{cl}\hat{A}_{qu}\gamma$. On the other hand,

$$\frac{d}{de'}\log \text{Det}\,(1-e'X) = \lim_{de'\to 0} \frac{\text{Det}\left(1 - \dfrac{de'X}{1-e'X}\right) - 1}{de'} = -\text{Tr}\,\frac{X}{1-e'X}$$

$$= -i\,\text{Tr}\,[S(;e\hat{A}_{cl} + e'\hat{A}_{qu})\hat{A}_{qu}\gamma],$$

from which

$$\log \mathrm{Det}\,(1-e'X) = -i\int_0^{e'} de''\,\mathrm{Tr}\,[S(;\,e\hat{A}_{\mathrm{cl}}+e''\hat{A}_{\mathrm{qu}})\hat{A}_{\mathrm{qu}}\gamma]. \tag{A2}$$

Thus, setting $e' = e$ in (A2) and substituting it to logarithm of (A1) one obtains the formula (34).

APPENDIX B

In this Appendix we present the calculation of $D_\pm T_\mp$. Let us concentrate on $D_- T_+$. D_- is defined by (6). We have

$$\left(\frac{\partial}{\partial x^-}T_+\right)T_+^{-1} = \lim_{dx^-\to 0}\frac{T_+(x^+,x^-+dx^-)T_+^{-1}(x)-\mathbf{1}}{dx^-}. \tag{B1}$$

Further,

$$T_+(x^+,x^-+dx^-) = P\exp\left[\frac{ie}{2}\int_{-\infty}^{x^+}dx'^+\hat{A}^-(x'^+,x^-+dx^-)\right]$$

$$= \prod_{x'^+=x^+}^{-\infty}\exp\left[\frac{ie}{2}dx'^+\hat{A}^+(x'^+,x^-+dx^-)\right]$$

$$= \prod_{x'^+=x^+}^{-\infty}\left(1+\frac{ie}{2}dx'^+\hat{A}^-(x'^+,x^-)+\frac{ie}{2}dx'^+dx^-\frac{\partial}{\partial x^-}\hat{A}^-(x'^+,x^-)\right)$$

$$= \int_{-\infty}^{x^+}d\sigma^+P\exp\left[\frac{ie}{2}\int_{\sigma^+}^{x^+}dx'^+\hat{A}^-(x'^+,x^-)\right]\left(\frac{ie}{2}dx^-\frac{\partial}{\partial x^-}\hat{A}^-(\sigma^+,x^-)\right)$$

$$\times P\exp\left[\frac{ie}{2}\int_{-\infty}^{\sigma^+}dx'^+\hat{A}^-(x'^+,x^-)\right]+T_+(x^+,x^-).$$

Substituting this to (B1) we obtain

$$\left(\frac{\partial}{\partial x^-}T_+\right)T_+^{-1}(x) = \frac{ie}{2}\int_{-\infty}^{x^+}d\sigma^+T_+(x;\sigma)\frac{\partial}{\partial x^-}\hat{A}^-(\sigma^+,x^-)T_+^{-1}(x;\sigma), \tag{B2}$$

where we have introduced the notation

$$T_+(x;\sigma) = T_+(x^+,x^-)T_+^{-1}(\sigma^+,x^-) = P\exp\left[\frac{ie}{2}\int_{\sigma^+}^{x^+}dx'^+\hat{A}^-(x'^+,x^-)\right].$$

Similarly, introducing

$$T_-(x;\sigma) = T_-(x^+, x^-)T_-^{-1}(x^+, \sigma^-) = P\exp\left[\frac{ie}{2}\int_{\sigma^-}^{x^-} dx'^- \hat{A}^+(x^+, x^{-\prime})\right],$$

one obtains

$$\left(\frac{\partial}{\partial x^+} T_-\right) T_-^{-1}(x) = \frac{ie}{2}\int_{-\infty}^{x^-} d\sigma^- T_-(x;\sigma)\frac{\partial}{\partial x^+}\hat{A}^-(x^+, \sigma^-)T_-^{-1}(x;\sigma).$$

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