Vol. B10 (1979) ACTA PHYSICA POLONICA No 12

RELATIVISTIC ANALOGA OF THE PHASE SPACE AND THE LIOUVILLE THEOREM OF CLASSICAL STATISTICS

By H. Strobel

Sektion Physik, Friedrich-Schiller-Universität, Jena*

(*Received May 11, 1979*)

Different anholonomic subspaces of the general relativistic μ-space (space of states of a free particle in the gravitational field; i.e. the tangent bundle $V_8 = T(V_4)$ on the space time V_4) are introduced which, respectively, represent relativistic analoga of the classical phase space or the space on which the classical theory of radiation defines the distribution function of spectral intensity. These subspaces of V_8 prove to be holonomic, if the 4-velocity field in V_4 the extensions (prolongations) of which are involved in their defining multivector fields is submitted to certain conditions. In this case, from local equations holding for the mapping of sets of points of the subspaces of V_8 and describing the development in time of states of particles, laws of conservation in integral form can be derived.

1. *Introduction*

According to [8] and adapting the notation applied in [1], the general relativistic μ-space, or, as can be said, the "state space" of a test particle freely moving in the space-time V_4 is a tangent bundle $\bigcup_{x\in V_4} E_4(x)$ on V_4. In [1] this tangent bundle has been investigated as a special Riemannian space V_8 with the methods of Ricci-calculus[1].

Due to the principles outlined in (I.1) each statistical theory should apply to its proper state space. The state space of statistical thermodynamics on the basis of classical mechanics or quantum mechanics is called phase space. One should only call a space "relativistic phase space" if it is correspondingly connected with the phase space of these theories (cp. [2, 8])[2]. In the following, after having generally investigated the question of the existence of holonomic subspaces of V_8, defined by the fields introduced in [1], it will be regarded

* Address: Sektion Physik, Friedrich-Schiller-Universität, DDR-69 Jena, Max-Wien-Platz 1.

[1] In the following we quote paragraphs, formulae and theorems of [1] as (I. ...).

[2] In [9, 10] the term "phase space" has been connected with the existence of a canonical formalism and applied to V_8, since locally the equations (I. 1.1) may be written in canonical form. Notwithstanding the application of a canonical formalism the same notation as here has been chosen in [11].

as different (holonomic or anholonomic) relativistic analoga of the classical phase space and analyzed laws of conservation for a projection of sets of points within them, which is induced by the development in the time of possible states of particles.

2. *Holonomic subspaces of the relativistic state space*

An m-vector field $v^{\kappa_1 \dots \kappa_m} = v^{[\kappa_1}_{(1)} \cdots v^{\kappa_m]}_{(m)}$ $(\kappa = 1, \dots, n)$ in a Riemannian space V_n defines m-dimensional subspaces E_m in the local pseudo-Euclidian tangent spaces E_n of V_n, i.e., it defines an E_m-field in V_n, or as is said, an anholonomic Riemannian space V_n^m [7, 13–15]. Accordingly, the $(n-m)$-pseudovector field

$$\overset{*}{v}_{\kappa_{m+1} \cdots \kappa_n} = \frac{(-1)^{\alpha_{n-m}}}{m!} \overset{*}{\varepsilon}_{\kappa_1 \cdots \kappa_m \kappa_{m+1} \cdots \kappa_n} v^{\kappa_1 \cdots \kappa_m} \tag{2.1}$$

dual to $v^{\kappa_1 \dots \kappa_m}$ $(\overset{*}{\varepsilon}_{\kappa_1 \dots \kappa_n} = \overset{(n)}{(g)}{}^{1/2} n! \, \delta^1_{[\kappa_1} \cdots \delta^n_{\kappa_n]}$, $\overset{(n)}{g} = \sigma_n \cdot \mathrm{Det}\,(g_{\mu\nu}) = |\mathrm{Det}\,(g_{\mu\nu})|$, $\sigma_n = \pm 1$ according to the signature of V_n; $\alpha_{2p} = 0$, $\alpha_{2p-1} = n-1$ for $p = 1, 2, \dots$) defines an E_{n-m} field in V_n or a V_n^{n-m}. The necessary and sufficient condition for the E_m-field in V_n to be tangent to a family of ∞^{n-m} V_m (or: to have enveloping V_m, to be V_m-forming; cp. [7], p. 81) can be expressed, equivalently, by

$$v^{\mu\kappa_2 \cdots [\kappa_m}{}_{;\mu} v^{\lambda_1 \cdots \lambda_m]} = 0, \qquad \overset{*}{v}_{[\kappa_{m+1} \cdots \kappa_n;\mu} \overset{*}{v}_{\lambda_{m+1}]\lambda_{m+2} \cdots \lambda_n} = 0. \tag{2.2}$$

In this form written with the covariant derivative these conditions hold also with respect to anholonomic coordinates. With the second of the criteria (2.2), the theorems (I.4.A., B., E.) and (I.4.10) it follows for the fields $^{(H)}u^\kappa$, $^{(V)}u^\kappa$, $^{(H)}p^\kappa$, $^{(V)}p^\kappa$ (cp. I.3.3, 4.1):

A. In V_8 the field $^{(H)}u^\kappa$ is V_7-normal if and only if the 4-velocity field u^k in V_4 is V_3-normal, i.e. $u_{[n;m}u_{k]} = 0$ or $u_{[n;m]} = -\dot{u}_{[n}u_{m]}$; $^{(V)}u^\kappa$ is V_7-normal if and only if u^k in V_4 is covariantly constant, i.e. $u_{k;l} = 0$; $^{(V)}p^\kappa$ is always V_7-normal and $^{(H)}p^\kappa$ never.

To investigate the properties of the fields (I.4.27, 28) it is advantageous to apply the following criteria, which may be proved easily,[3] instead of calculation with the conditions (2.2) themselves and (I.4.9, 10, 29–32):

Corollary I. An m-vector field $\underset{h}{a}_{\kappa_1 \dots \kappa_m} = \overset{1}{a}_{[\kappa_1} \cdots \overset{m}{a}_{\kappa_m]}$ in a V_n is V_m-normal if and only if the covariant derivative of the $\overset{h}{a}_\kappa$ $(h = 1, \dots, m)$ may be represented as

$$\overset{h}{a}_{\kappa;\lambda} = \sum_{i=1}^{m} (\overset{hi}{\varphi}_\kappa \overset{i}{a}_\lambda + \overset{i}{a}_\kappa \overset{hi}{\psi}_\lambda) + \sum_{s=m+1}^{n} \overset{hs}{\alpha} \overset{s}{b}_\kappa \overset{s}{b}_\lambda, \tag{2.3}$$

where $\overset{hi}{\varphi}_\kappa$, $\overset{hi}{\psi}_\kappa$ $(h, i = 1, \dots, m)$ are arbitrary vectors, $\overset{s}{b}_\kappa (s = m+1, \dots, n)$ arbitrary vectors with $\overset{s}{b}_\kappa \overset{i}{a}^\kappa = 0$ and completing the $\overset{i}{a}_\kappa$ to an n-leg, and the $\overset{hs}{\alpha}$ $(h = 1, \dots, m; s = m+1, \dots, n)$ are arbitrary scalar functions.

---

[3] Apply (2.2) and an expansion of $\overset{h}{a}_{\kappa;\lambda}$ in terms of $\overset{i}{a}_\kappa \overset{j}{a}_\lambda$, $\overset{i}{a}_\kappa \overset{s}{b}_\lambda$, $\overset{s}{b}_\kappa \overset{t}{b}_\lambda$, or, for the proof of (2.4), the definition of the Lie-derivative (cp. footnote 12 to I.4.F.).

Corollary II. An m-vector field $a^{\kappa_1 \ldots \kappa_m} = \underset{1}{a}^{[\kappa_1} \ldots \underset{m}{a}^{\kappa_m]}$ in a V_n is V_m-tangent if and only if the Lie-derivatives of the $\underset{i}{a}^{\kappa}$ $(i = 1, \ldots, m)$ after each other may be represented as

$$\underset{\underset{i}{a}\,j}{\mathscr{L}} a^{\kappa} = \sum_{h=1}^{m} \overset{h}{\underset{ij}{\alpha}} \underset{h}{a}^{\kappa}, \quad (i, j = 1, \ldots, m), \tag{2.4}$$

where $\overset{h}{\underset{ij}{\alpha}}$ $(h = 1, \ldots, m)$ are arbitrary scalar functions.

The results of the application of (2.3, 4) on (I.4.27, 28) with regard to (I.4.9, 4.10, 4.F. — i.e. Table I) are the following theorems:

B. In V_8 the E_2-fields defined by $p^{\kappa\lambda}$, $r^{\kappa\lambda}$, $v^{\kappa\lambda}$ are always tangent to a family of V_2; $u^{\kappa\lambda}$ is V_2-forming if and only if $\dot{u}^k = 0$ holds for the field u^k in V_4; $w^{\kappa\lambda}$ is V_2-forming if and only if $u_{k;l} = 0$; $q^{\kappa\lambda}$ cannot be V_2-forming.

C. In V_8 the E_6-field defined by $v^{\kappa\lambda}$ is tangent to a family of V_6 if and only if the field u^k in V_4 is V_3-normal; $u^{\kappa\lambda}$ is V_6-normal if and only if $u_{k;l} = -\dot{u}_k u_l$ holds in V_4; $r^{\kappa\lambda}$ is V_6-normal if and only if $u_{k;l} = 0$; the fields $p^{\kappa\lambda}$, $q^{\kappa\lambda}$, $w^{\kappa\lambda}$ cannot be V_6-normal.

D. In V_8 the E_3-field defined by $u^{\kappa\lambda\mu}$ is tangent to a family of V_3 if and only if $\dot{u}^k = 0$ holds for the field u^k in V_4; $q^{\kappa\lambda\mu}$ and $v^{\kappa\lambda\mu}$ are V_3-forming if and only if $u_{k;l} = 0$; $p^{\kappa\lambda\mu}$ cannot be V_3-forming.

E. In V_8 the E_5-field defined by $u^{\kappa\lambda\mu}$ is tangent to a family of V_5 if and only if $u_{k;l} = -\dot{u}_k u_l$ holds for the field u^k in V_4; $p^{\kappa\lambda\mu}$, $q^{\kappa\lambda\mu}$ and $v^{\kappa\lambda\mu}$ cannot be V_5-normal.

3. *Relativistic analoga of the classical phase space*

We apply the notation

$$V_8 = \bigcup_x \{V_4\}_x \oplus E_4(x) \tag{3.1}$$

for the characterization of the relativistic state space as a tangent bundle $V_8 = \bigcup_{x^k \in V_4} E_4(x^k)$ on the space-time [8]. At (3.1) it is seen that, locally, the V_8 may be regarded as a direct product of a coordinate neighbourhood $\{V_4\}_x$ of points x^k of V_4 and the local tangent spaces (physically, the 4-momentum spaces) $E_4(x)$ at these points.

The anholonomic subspaces defined by the fields (I.3.3, 4.1, 4.27, 4.28) may be characterized with the same notation as in (3.1). In the following, we are going to regard the spaces

$$^{(V)}p_{\kappa} \leftrightarrow \overset{V}{V_7}(\sigma) = \bigcup_x \{V_4\}_x \oplus V_3(x; m), \tag{3.2}$$

$$^{(H)}u_{\kappa} \leftrightarrow \overset{H}{V_8^7}(\tau) = \bigcup_x \{V_4^3(\tau)\}_x \oplus E_4(x), \tag{3.3}$$

$$v_{\kappa\lambda} \leftrightarrow V_8^6(\overset{H}{\tau}, \overset{V}{\sigma}) = \bigcup_x \{V_4^3(\tau)\}_x \oplus V_3(x; m), \tag{3.4}$$

$$u_{\kappa\lambda} \leftrightarrow V_8^6(\overset{H}{\tau}, \overset{V}{\tau}) = \bigcup_x \{V_4^3(\tau)\}_x \oplus E_3(x; \varepsilon), \tag{3.5}$$

$$u_{\kappa\lambda\mu} \leftrightarrow V_8^5(\overset{H}{\tau}, \overset{V}{\tau}, \overset{V}{\sigma}) = \bigcup_x \{V_4^3(\tau)\}_x \oplus V_2(x; \varepsilon, m). \tag{3.6}$$

In (3.2–6) the parameters $\overset{H}{\tau}(\tau)$, $\overset{V}{\tau}(\varepsilon)$, $\overset{V}{\sigma}(m)$ run through all their values and are indicated to symbolize the fields (and their basic congruences) with which the spaces are connected. The $V_4^3(\tau)$ is the anholonomic 3-space orthogonal to the 4-velocity field u^k in V_4. The $V_4^3(\tau)$ represents the relativistic analogue of the configuration 3-space of classical mechanics. In (3.3–6) for constant values of the parameters one gets sets of patches of the anholonomic spaces, which we call "layers" of these spaces. The above theorems (2.A.–E.) give conditions for the anholonomic spaces to be families of Riemannian subspaces of V_8.

In (3.2) $V_3(x; m)$ symbolizes the family of hypersurfaces with constant rest mass m in the local tangent E_4 at the point x^k of V_4. The $V_7(\overset{V}{\sigma})$ represents a family of Riemannian spaces the members $\overset{V}{\sigma}(m) = \text{const}$ of which have the structure of tangent sphere bundles on V_4 (cp. [3, 4]) and are equivalent to isotropic Finslerian spaces (cp. [5, 6]). The development in time of the states of a particle with constant rest mass m can be described in a $V_7(\overset{V}{\sigma})$ with $\overset{V}{\sigma}(m) = \text{const}$.

The $V_8^7(\overset{H}{\tau})$ in (3.3) turns out to be a parametrized by $\overset{H}{\tau}$ family $V_7(\overset{H}{\tau})$ of "state hypersurfaces" in V_8 if the field u^k in V_4 is tangent to a normal to hypersurfaces $\tau = \text{const}$ congruence (cp. 2.A.).

The $V_8^6(\overset{H}{\tau}, \overset{V}{\sigma})$ in (3.4) is a first relativistic analogue of the classical phase space which may be called "general relativistic phase space of particles with constant rest mass". Providing the field u^k is normal to hypersurfaces $\tau = \text{const}$ in V_4, the $V_8^6(\overset{H}{\tau}, \overset{V}{\sigma})$ goes over into a family $V_6(\overset{H}{\tau}, \overset{V}{\sigma}) = V_3(\tau) \oplus V_3(m)$ of ∞^2 holonomic subspaces in V_8 (cp.2.C.). For flat V_4 and covariantly constant fields u^k the $V_8^6(\overset{H}{\tau}, \overset{V}{\sigma})$ obtains the form of a 2-parameter family $V_6(\overset{H}{\tau}, \overset{V}{\sigma}) = E_3(\tau) \oplus V_3(m)$.

The $V_8^6(\overset{H}{\tau}, \overset{V}{\tau})$ in (3.5) represents a second relativistic analogue of the classical phase space. Since only those changes of ${}^{(H)}p^\kappa$, ${}^{(V)}p^\kappa$ (caused by interactions of a particle with an initially given 4-momentum p^k with other ones) lie within $V_8^6(\overset{H}{\tau}, \overset{V}{\tau})$ for which $\varepsilon = -p^k u_k$ and $\overset{V}{\tau}(\varepsilon)$ remain constant, the space $V_8^6(\overset{H}{\tau}, \overset{V}{\tau})$ may be called "general relativistic phase space of particles with constant relative energy". The $V_8^6(\overset{H}{\tau}, \overset{V}{\tau})$ goes over into a family $V_6(\overset{H}{\tau}, \overset{V}{\tau}) = \bigcup_x \{V_3(\tau)\}_x \oplus E_3(x; \varepsilon)$, if u^k is tangent to a rigid and non-twisting congruence of worldlines in V_4 (cp.2.C.). In particular, for cases with flat V_4 and covariantly constant fields u^k the $V_8^6(\overset{H}{\tau}, \overset{V}{\tau})$ proves to be a family $E_6(\overset{H}{\tau}, \overset{V}{\tau}) = E_3(\tau) \oplus E_3(\varepsilon)$ of flat spaces. The members of this family with different constant values $\overset{H}{\tau}, \overset{V}{\tau}$ are identical with respect to their geometrical structure. By the last example it may be seen that, in comparison with the classical phase space itself, its relativistic analogue $V_8^6(\overset{H}{\tau}, \overset{V}{\tau})$ is split in two ways into infinite sets of layers. Since the Euclidian subspaces $E_3(x; \varepsilon)$ in the local $E_4(x)$ for different $\varepsilon = \text{const}$ are of identical structure, a superposition of the distinct layers with $\overset{V}{\tau}(\varepsilon) = \text{const}$ effected by summation $\bigcup_{\varepsilon} V_8^6(\overset{H}{\tau}, \overset{V}{\tau}(\varepsilon))$ over ε between 0 and ∞ would be possible and would improve the analogy of the relativistic phase space $V_8^6(\overset{H}{\tau}, \overset{V}{\tau})$ with the classical one. The splitting of $V_8^6(\overset{H}{\tau}, \overset{V}{\tau})$ into layers with different $\overset{H}{\tau} = \tau = \text{const}$ is unavoidable (except for the case of V_4 with $u_{k;l} = -\dot{u}_k u_l$). This is the reason why in relativistic statistics the development in time of the states of particles cannot be described within the set of their possible initial values, even if the particles have the same relative energy $\varepsilon = -p^k u_k$ with respect to the system of reference defined by u^k.

We remark that the 4-momentum p^k may be split up by

$$p^k = \pi^k + \varepsilon u^k \qquad (\pi^k = h_l^k p^l,\ \varepsilon = -p^k u_k,\ h_{kl} = g_{kl} + u_k u_l). \tag{3.7}$$

From that, it follows $m^2 = \varepsilon^2 - \pi^2$. The vector π^k with $\pi = \varepsilon$ ($m = 0$) is adequate to the classical wave 3-vector. The space on which the classical theory of radiation defines the intensity distribution function is given by the direct product of the configuration 3-space and the wave-vector 3-space. A relativistic analogue of this classical space is the special layer with $m = 0$ of the space (3.4).

In the classical theory of radiation the distribution function of spectral intensity of radiation is defined on the 5-dimensional space given by the direct product of the configuration 3-space and a spherical 2-surface around the origin of the 3-dimensional wave-vector space. A relativistic analogue of this space is represented by the layer with $m = 0$ ($\pi = \varepsilon$) of the space $V_8^5(\overset{H}{\tau}, \overset{V}{\tau}, \overset{V}{\sigma})$. In (3.6) the $V_2\ (x; \varepsilon, m)$ have the structure of hyperboloids with one non-vanishing, depending on m main-curvature.

We observe that a general mathematical framework for the investigation of the curvature and embedding of anholonomic spaces as (3.3–6) has already been developed in [7, 12–14]. Similar methods have been rederived in the special case of $V_4^3(\tau)$ for the purpose of interpreting relativistic equations in terms of classical physics (cp., e.g. [15–17]).

4. *Laws of conservation*

The horizontal flow (cp.I.3.) intersects all the layers $\overset{H}{\tau} = \text{const}$ of the anholonomic space $V_8^7(\overset{H}{\tau})$ (cp.3.3). In this way, it induces a homeomorphic map of point sets $B_{8\overset{H}{\tau}}^7$ on these layers onto each other. This map characterizes the development in time of the set of possible states of a particle (cp. I.1.). It is submitted to the constraint given by the incompressibility of the horizontal flow (cp.I.4.19), which can be expressed with respect to anholonomic or holonomic coordinates in V_8, respectively, by

$$^{(H)}p^{\kappa}{}_{;\kappa} = \frac{1}{\overset{(4)}{g}} (\overset{(4)}{g}\, p^k)_{,k} = 0, \tag{4.1}$$

$$^{(H)}p^{\hat{\kappa}}{}_{;\hat{\kappa}} = \frac{1}{\overset{(4)}{g}} \left[(\overset{(4)}{g}\, p^k)_{,\hat{k}} - \left(\overset{(4)}{g} \begin{Bmatrix} k \\ lm \end{Bmatrix} p^l p^m \right)_{,\widehat{4+k}} \right] = 0. \tag{4.2}$$

Here, $\overset{(8)}{g} = \left[\overset{(4)}{g}\right]^2$ with $\overset{(8)}{g} = \mathrm{Det}\,(g_{\mu\nu})$, $\overset{(4)}{g} = -\mathrm{Det}\,(g_{ik})$ and (I.2.7, 3.3) have been applied. With (I.1.1) equation (4.2) may be rewritten as

$$\frac{\partial \left(\overset{(4)}{g} \dfrac{dx^k}{d\sigma} \right)}{\partial x^k} + \frac{\partial \left(\overset{(4)}{g} \dfrac{dp^k}{d\sigma} \right)}{\partial p^k} = 0. \tag{4.3}$$

Chernikov [8] has given a coordinate-dependent equation equivalent to (4.1–3).

In particular, if the $V_8^7(\overset{H}{\tau})$ is supposed to be a family $V_7(\overset{H}{\tau})$, a domain $B_{7\overset{H}{\tau}_1}$ on the hypersurface $\overset{H}{\tau}_1$ = const obtains images $B_{7\overset{H}{\tau}_1}$ on all other hypersurfaces $\overset{H}{\tau}$ = const ($\overset{H}{\tau} \neq \overset{H}{\tau}_1$) of this $V_7(\overset{H}{\tau})$ in V_8. Then, from (4.1) after application of Stokes theorem[4] on an 8-dimensional cylinder along the horizontal flow, one gets the law of conservation of the integral

$$\int_{B_{7\overset{H}{\tau}}} {}^{(H)}p^{\kappa} df_{\kappa} = \text{const}; \qquad df_{\kappa} = df_{(7)}{}^{(H)}u_{\kappa}. \tag{4.4}$$

Locally, the volume element df_{κ} in (4.4) may be split into the volume elements $df_k = d^3x \cdot u_k$ of the configuration 3-space and $dg_{(4)}$ of the 4-momentum space $E_4(x)$, so that ${}^{(H)}p^{\kappa}df_{\kappa} = u_k p^k df_{(7)} = -\varepsilon d^3x \cdot dg_{(4)}$ results.

---

[4] The theorem (cp. [7], p. 94) for the transformation of integrals on simply connected regions $B_{m+1} \subset V_{m+1} \subset V_n$ into integrals on the boundaries B_m of B_{m+1} may be rewritten as

$$(n-m) \int_{B_{m+1}} \overset{*}{a}^{\kappa_{m+2}\ldots\kappa_n\varrho}{}_{;\varrho} d\overset{*}{f}_{\kappa_{m+2}\ldots\kappa_n} = \int_{B_m} \overset{*}{a}^{\kappa_{m+1}\ldots\kappa_n} d\overset{*}{f}_{\kappa_{m+1}\ldots\kappa_n}$$

where, e.g., $d\overset{*}{f}_{\kappa_{m+1}\ldots\kappa_n}$ is the dual (according to (2.1)) of the volume element $df^{\mu_1\ldots\mu_m} = m!\, dx^{[\mu_1}_{(1)} \ldots dx^{\mu_m]}_{(m)}$. In this form of the theorem (written with the covariant derivative) the tensors may be taken with respect to holonomic or anholonomic coordinates as well. We put $df^{\mu_1\ldots\mu_m} = df_{(m)} e^{\mu_1\ldots\mu_m}$, where $e^{\mu_1\ldots\mu_m} = m! \underset{1}{e}^{[\mu_1} \ldots \underset{m}{e}^{\mu_m]}$ is the unity m-vector tangent to V_m with $e^{\mu_1\ldots\mu_m} e_{\mu_1\ldots\mu_m} = \sigma_m \cdot m!$ ($\underset{k}{e}^{\mu} \underset{k}{e}_{\mu} = \pm 1$ for each $k = 1, \ldots, m$; $\sigma_m = +1$ if an even, $\sigma_m = -1$ if an odd number of the $\underset{k}{e}^{\mu}$ have negative norm). From this it follows $d\overset{*}{f}^{\mu_{m+1}\ldots\mu_m} = df_{(m)} \overset{*}{e}^{\mu_{m+1}\ldots\mu_n}$ and

$$df_{(m)} = [df^{\mu_1\ldots\mu_m} df_{\mu_1\ldots\mu_m} / \sigma_m \cdot m!]^{\frac{1}{2}} = [d\overset{*}{f}^{\mu_{m+1}\ldots\mu_n} d\overset{*}{f}_{\mu_{m+1}\ldots\mu_n} / \sigma_{n-m} \cdot (n-m)!]^{\frac{1}{2}}$$

($\sigma_n = \sigma_m \cdot \sigma_{n-m}$). In the following, elements with exterior orientation shall be applied only, and all the asterisks will be omitted.

In (4.4) ${}^{(H)}p^\kappa$ may be substituted according to

$$p^{\kappa\lambda}{}_{;\lambda} = \tfrac{5}{2}\,{}^{(H)}p^\kappa \tag{4.5}$$

(cp. I.4.29). The boundary of the region $B_{7\underset{\tau}{\mathrm{H}}}$ can be chosen as a 7-cylinder along the vertical flow (cp.I.3). The basis area of the 7-cylinder shall be taken within the hypersurface $\overset{V}{\sigma}_1 = \overset{V}{\sigma}(m = 0) =$ const. Its upper cross-sectional area can be chosen as the projected along ${}^{(V)}p^\kappa$ domain $B_{6\overset{H}{\tau}\overset{V}{\sigma}}$ on a 6-surface $\overset{H}{\tau} =$ const, $\overset{V}{\sigma} =$ const $\neq \overset{V}{\sigma}_1$ of the family $V_6(\overset{H}{\tau}, \overset{V}{\sigma})$ which (3.4) is supposed to be[5]. Then, from (4.4, 5) one gets

$$\int_{B_{6\overset{H}{\tau}\overset{V}{\sigma}}} p^{\kappa\lambda} df_{\kappa\lambda} = \text{const}\,(\overset{V}{\sigma}), \qquad df_{\kappa\lambda} = \frac{2}{m}\, df_{(6)} v_{\kappa\lambda} \tag{4.6}$$

after repeated application of the Stokes theorem and with $p^{\kappa\lambda} v_{\kappa\lambda} = m^2\varepsilon/2 = 0$ on $\overset{V}{\sigma}_1 =$ const. i.e. $m = 0$. In (4.6) the notation const $(\overset{V}{\sigma})$ shall indicate that the integral is independent of $\overset{H}{\tau}$ but depends on the chosen value $\overset{V}{\sigma}$. Until now it has been supposed that the field u^k in V_4 is hypersurface normal. If, additionally, u^k is assumed to be rigid, it holds that $u_{n;m} = -\dot{u}_n u_m$. Then, a u^k-normal family $V_6(\overset{H}{\tau}, \overset{V}{\tau})$ exists (cp. 2.C.), and, the basis area of the 7-cylinder along the field ${}^{(V)}p^\kappa$ can be taken within the 6-surface $\overset{H}{\tau} =$ const, $\overset{V}{\tau}_1 = \overset{V}{\tau}(\varepsilon = 0) =$ const; its upper cross-sectional area as the projected domain $B_{6\overset{H}{\tau}\overset{V}{\tau}}$ on a 6-surface $\overset{H}{\tau} =$ const, $\overset{V}{\tau} =$ const $(\overset{V}{\tau} \neq \overset{V}{\tau}_1)$. Providing this, one gets

$$\int_{B_{6\overset{H}{\tau}\overset{V}{\tau}}} p^{\kappa\lambda} df_{\kappa\lambda} = \text{const}\,(\overset{V}{\tau}), \qquad df_{\kappa\lambda} = 2 df_{(6)} u_{\kappa\lambda} \tag{4.7}$$

instead of (4.6). Here $p^{\kappa\lambda} u_{\kappa\lambda} = \varepsilon^2/2 = 0$ on $\overset{V}{\tau}_1 \doteq$ const, i.e. $\varepsilon = 0$, has been taken into account.

The equations (4.6, 7) may be said of to exhibit the "relativistic Liouville theorem" in integral form. Accordingly, one has to look at (4.5) as representing a differential form of this theorem. Equations (4.5, 6) are equivalent to those already given by Lindquist [4].

In (4.6) $df_{\kappa\lambda}$ can be defined with special elements, horizontal and vertical extensions, respectively, of $df_k = \varepsilon_{km_1m_2m_3} dx^{[m_1}_{(1)} \dots dx^{m_3]}_{(3)} = df_{(3)} u_k = d^3x \cdot u_k$ and $dg_k = \varepsilon_{km_1m_2m_3} Dp^{[m_1}_{(1)} \dots Dp^{m_3]}_{(3)} = \dfrac{1}{m} dg_{(3)}\, p_k = dP \cdot p_k$ so, that $p^{\kappa\lambda} df_{\kappa\lambda}$ splits into $p^{\kappa\lambda} df_{\kappa\lambda} = \varepsilon m df_{(6)} = \varepsilon m^2 d^3x \cdot dP$.

The above $dg_{(4)}$ may be built as $dg_{(4)} = m dm dP$. With $dP = dg_{(3)}/m$ instead of the 3-element $dg_{(3)}$ of the mass-shell a quantity has been introduced which is defined in the limiting case $m \to 0$ also, and, according to [18] is called the absolute 2-content of the mass-shell. Applying $dP = -d^3p/p_k u^k = d^3p/\varepsilon$ (cp. [18]) in (4.6) we get $p^{\kappa\lambda} df_{\kappa\lambda}$

[5] This boundary of $B_{7\underset{\tau}{\mathrm{H}}}$ corresponds to a set of cones in the local $E_4(x^k)$ having their tops at the points of contact with the points x^k of a region on a 3-surface $\tau =$ const in V_4, and, as their basis an arbitrary mass-shell $m \neq 0$.

$= m^2 \cdot d^3x \cdot d^3p$. Similarly, in (4.7) $df_{\kappa\lambda}$ can be defined with $df_k = d^3x \cdot u_k$ and $dg_k = d^3p \cdot u_k$ so, that in this relation $p^{\kappa\lambda} df_{\kappa\lambda} = \varepsilon^2 df_{(6)} = \varepsilon^2 d^3x \cdot d^3p$ holds.

Because $\overset{V}{\sigma}(m) = \text{const}$ for 6-surfaces on which (4.6) is calculated and $\overset{V}{\tau}(\varepsilon) = \text{const}$ for 6-surfaces on which (4.7) is calculated, comparison of these two results for $p^{\kappa\lambda} df_{\kappa\lambda}$ shows, that both integrals (4.6) and (4.7) are generalizations of the same measure $\int d^3x \cdot d^3p$ on the classical phase space. However, since $p^{\kappa\lambda} u_{\kappa\lambda} = \varepsilon^2/2$ remains constant for members with $\overset{H}{\tau}(\tau)$ varying and $\overset{V}{\tau}(\varepsilon)$ constant of the family $V_6(\overset{H}{\tau}, \overset{V}{\tau})$, but $p^{\kappa\lambda} v_{\kappa\lambda} = \varepsilon \cdot m^2/2$ varies with ε for members with $\overset{H}{\tau}(\tau)$ varying and $\overset{V}{\sigma}(m)$ constant of $V_6(\overset{H}{\tau}, \overset{V}{\tau})$, only (4.7) allows us to conclude that the measure $\int df_{(6)}$ itself of the underlying domain of integration is conserved at its homeomorphic projection along the trajectories of the horizontal flow onto members of these families with other $\overset{H}{\tau} = \text{const}$. For this reason and similarly to the manner of Gibbs it might be spoken of a "principle of conservation of extension in relativistic phase" with respect to (3.5, 4.7) only, and not (3.4, 4.6).

Assuming the field u^k in V_4 to be tangent to a rigid and non-twisting congruence of world-lines, u^k is incompressible in V_4, of course, and its horizontal extension ${}^{(H)}u^\kappa$ is incompressible in V_8 (cp. I.4.12). Further, ${}^{(V)}u^\kappa$, $u^{\kappa\lambda}$, $u^{\kappa\lambda\mu}$ have properties dealt with in (2.A., C., E.). Then from ${}^{(H)}u^\kappa{}_{;\kappa} = 0$ and $v^{\kappa\lambda}{}_{;\lambda} = 2\,{}^{(H)}u^\kappa$ or $q^{\kappa\lambda}{}_{;\lambda} = ({}^{(H)}u^\kappa - {}^{(V)}l^\kappa)/2$ (cp. I.4.29, 30), respectively, one gets

$$\int_{B_{6\overset{H}{\tau}\overset{V}{\tau}}} v^{\kappa\lambda} df_{\kappa\lambda} = \text{const}\,(\overset{V}{\tau}), \qquad \int_{B_{6\overset{H}{\tau}\overset{V}{\tau}}} q^{\kappa\lambda} df_{\kappa\lambda} = \text{const}\,(\overset{V}{\tau}) \tag{4.8}$$

after repeated application of Stokes theorem, similarly as (4.7), if here the domains $B_{6\overset{H}{\tau}\overset{V}{\tau}}$ with different $\overset{H}{\tau}$ are projected along the field ${}^{(H)}u^\kappa$.

Assuming that domains of integration in (4.8) are identical and applying

$$p^{\kappa\lambda\mu}{}_{;\mu} = \tfrac{1}{3} v^{\kappa\lambda} - \tfrac{1}{3}\, {}^{(V)}l^{[\kappa}\, {}^{(V)}p^{\lambda]} - \tfrac{4}{3} q^{\kappa\lambda} \tag{4.9}$$

(cp. I.4.31) we get

$$\int_{B_{6\overset{H}{\tau}\overset{V}{\tau}}} p^{\kappa\lambda\mu}{}_{;\mu} df_{\kappa\lambda} = \text{const}\,(\overset{V}{\tau}). \tag{4.10}$$

Now, if the domain $B_{6\overset{H}{\tau}\overset{V}{\tau}} \subset V_6(\overset{H}{\tau} = \text{const}, \overset{V}{\tau} = \text{const})$ is chosen as a 6-cylinder along the vertical flow with its basis area $B_{5\overset{H}{\tau}\overset{V}{\tau}\overset{V}{\sigma}_1}$ on the 5-surface $\overset{H}{\tau} = \text{const}$, $\overset{V}{\tau} = \text{const}$, $\overset{V}{\sigma}_1(m = \varepsilon) = \text{const}$ and its top area $B_{5\overset{H}{\tau}\overset{V}{\tau}\overset{V}{\sigma}}$ on a 5-surface $\overset{H}{\tau} = \text{const}$, $\overset{V}{\tau} = \text{const}$, $\overset{V}{\sigma} = \text{const}$ $(\overset{V}{\sigma} \neq \overset{V}{\sigma}_1)$ from (4.10) it follows

$$\int_{B_{6\overset{H}{\tau}\overset{V}{\tau}\overset{V}{\sigma}}} p^{\kappa\lambda\mu} df_{\kappa\lambda\mu} = \text{const}\,(\overset{V}{\tau}, \overset{V}{\sigma}), \qquad df_{\kappa\lambda\mu} = \frac{6}{\pi} df_{(5)} u_{\kappa\lambda\mu}. \tag{4.11}$$

Here $p^{\kappa\lambda\mu} u_{\kappa\lambda\mu} = \varepsilon(m^2 - \varepsilon^2)/6 = 0$ (cp. 3.7) on $\overset{V}{\sigma}_1 = \text{const}$, i.e. $m = \varepsilon$, has been observed.

Equation (4.11) means, that the integral on the left hand side on arbitrary domains within a 5-surface $\overset{H}{\tau} = \text{const}$, $\overset{V}{\tau} = \text{const}$, $\overset{V}{\sigma} = \text{const}$ is independent of $\overset{H}{\tau}$, if the domain is projected onto 5-surfaces of a family $V_5(\overset{H}{\tau}, \overset{V}{\tau}, \overset{V}{\sigma})$ with other values $\overset{H}{\tau}$ along a congruence with the tangent field ${}^{(H)}u^{\kappa}$, being a horizontal extension of a field u^k in V_4 with $u_{k;l} = -\dot{u}_k u_l$. If, additionally, the field u^k is assumed to be covariantly constant, then its tangent world--lines coincide with geodesics in V_4, and, the projection of the boundary of $B_{5\overset{H}{\tau}\overset{V}{\tau}\overset{V}{\sigma}}$ along ${}^{(H)}u^{\kappa}$ in V_8 is a projection along the horizontal flow at once. In this case equation (4.11) represents a "principle of conservation of extension of a hypersurface in relativistic phase". The equation (4.9), being most important for obtaining (4.12), may be regarded as a local expression of this principle.

The laws of conservation given by (4.4, 6, 7, 11) can be rederived applying a theorem communicated in [7] which in our context (cp.[4]) reads as follows.

Theorem of E. J. Post.[6] In a V_m a $(n-m)$-vector field $v^{\kappa_{m+1}\ldots\kappa_n}$ and a set of ∞^1 V_m's depending on a parameter t are given. Let a set of curves intersect each V_m at one point, such that t is also a parameter on each of these curves. If B_m is an m-dimensional part of one V_m there is a one to one correspondence between the points of this B_m and ∞^1 B_m's on the ∞^1 V_m's. Then a function of t and its derivative are given by

$$\Psi(t) = \int_{B_m} v^{\kappa_{m+1}\cdots\kappa_n} df_{\kappa_{m+1}\cdots\kappa_n}, \tag{4.12}$$

$$\frac{d\Psi}{dt} = \int_{B_m} (\underset{s}{\mathcal{L}} v^{\kappa_{m+1}\cdots\kappa_n} + s^{\mu}{}_{;\mu} v^{\kappa_{m+1}\cdots\kappa_n}) df_{\kappa_{m+1}\cdots\kappa_n}; \qquad s^{\kappa} = \frac{dx^{\kappa}}{dt}.$$

By the help of the equation

$$\frac{d\Psi}{dt} = (n-m) \int_{B_m} v^{\kappa_{m+2}\cdots\kappa_n\varrho}{}_{;\varrho} \cdot s^{\mu} df_{\mu\kappa_{m+2}\cdots\kappa_n} \tag{4.13}$$

appearing as an intermediate result in the proof of (4.12) a corollary to the theorem of Post may be formulated, which also permits a simplified rederivation of (4.4, 6, 7, 11). Applying (4.12) and the equations

$${}^{(H)}p^{\kappa}{}_{;\kappa} = 0, \qquad \underset{(H)p}{\mathcal{L}} p^{\kappa\lambda} = 0, \qquad \underset{(H)p}{\mathcal{L}} p^{\kappa\lambda\mu} = p^{[\kappa\lambda(V)}l^{\mu]} - v^{\kappa\lambda\mu} \tag{4.14}$$

(cp. theorems I.4. G., H.) one immediately gets the integral laws of conservation (4.4, 6, 7, 11) with the same suppositions as above: 1. families of subspaces exist, on which the

---

[6] This "dual" form of the theorem can be obtained from that one given in ([7], p. 111) if the above dualization conventions (cp. 2.1) are applied. It can be proved directly with the help of the theorem of Stokes in its "dual" form (cp.[4]) and an identity connecting the divergence of $v^{\kappa_{m+1}\ldots\kappa_n} = s^{[\kappa_{m+1}}v^{\kappa_{m+2}\ldots\kappa_n]}$ and $\underset{s}{\mathcal{L}} v^{\kappa_{m+2}\ldots\kappa_n}$ (cp.[14] in [1]).

B_m's may be chosen (cp. theorems 2.A., C., E.); 2. for the proof of (4.11) instead of $u_{k;l} = -\dot{u}_k u_l$ even $u_{k;l} = 0$ must be assumed, since $p^{[\kappa\lambda}\,{}^{(V)}l^{\mu]}u_{\kappa\lambda\mu} = \varepsilon^2 u_{m;s}p^m p^s/6$ vanishes for arbitrary p^k only in this case.

It is reasonable to regard the last two equations in (4.14) or their transvections $v_{\kappa\lambda} \underset{[H]p}{\mathscr{L}} p^{\kappa\lambda} = 0$, $u_{\kappa\lambda} \underset{[H]p}{\mathscr{L}} p^{\kappa\lambda} = 0$, $u_{\kappa\lambda\mu} \underset{[H]p}{\mathscr{L}} p^{\kappa\lambda\mu} = \frac{\varepsilon^2}{6} u_{m;s}p^m p^s = 0$ if $u_{(m;s)} = 0$ as laws of conservation in local form. These equations express the same facts as (4.5, 9) or their transvections ${}^{(V)}p_\kappa p^{\kappa\varrho}{}_{;\varrho} = 0$, ${}^{(V)}u_\kappa p^{\kappa\varrho}{}_{;\varrho} = 0$, $r_{\kappa\lambda}p^{\kappa\lambda\varrho}{}_{;\varrho} = u_{k;s}p^k p^s/6 \underset{H}{=} 0$ if $u_{(k;s)} = 0$[7] for the mapping of sets of points (states) between the different layers $\tau = \text{const}$, respectively, of the anholonomic spaces (3.4, 5, 6).

With respect to the application of the theorem of Post in its original (not dualized) form [7] the equations

$$\underset{(H)p}{\mathscr{L}} p_{\kappa_1 \cdots \kappa_7} = 0, \qquad \underset{(H)p}{\mathscr{L}} p_{\kappa_1 \cdots \kappa_6} = 0,$$

$$\underset{(H)p}{\mathscr{L}} p_{\kappa_1 \cdots \kappa_5} = -\tfrac{1}{3}\, p_{\kappa_1 \cdots \kappa_5 \varrho}\,{}^{(V)}l^\varrho - v_{\kappa_1 \cdots \kappa_5} \tag{4.15}$$

(multivectors defined as duals of ${}^{(H)}p^\kappa$, $p^{\kappa\lambda}$, $p^{\kappa\lambda\mu}$, $v^{\kappa\lambda\mu}$) have to be regarded as further local expressions of the laws of conservation given by (4.14).

REFERENCES

[1] H. Strobel, *Acta Phys. Pol.* **B10**, 863 (1979),
[2] R. Hakim, *J. Math. Phys.* **8**, 1315 (1967).
[3] S. Sasaki, *Tôhoku Math. J.* **10**, 338 (1958); **14**, 146 (1962).
[4] R. W. Lindquist, *Ann. Phys.* (USA) **37**, 487 (1966).
[5] J. I. Horvath, *Wiss. Z. Schiller-Univ. Jena* **15**, 149 (1966); *Acta Phys. Acad. Sci. Hung.* **24**, 205 (1968); **24**, 347 (1968); **25**, 1 (1968).
[6] H. Rund, *The Differential Geometry of Finsler Spaces*, Springer 1959.
[7] J. A. Schouten, *Ricci-Calculus*, Springer 1954.
[8] N. A. Chernikov, *Acta Phys. Pol.* **23**, 629 (1963).
[9] G. E. Tauber, J. W. Weinberg, *Phys. Rev.* **122**, 1342 (1961).
[10] P. Bergmann, *Phys. Rev.* **84**, 1026 (1951).
[11] C. Møller, *K. Dan. Vidensk. Selsk. Mat.-Fys. Medd.* **36**, No. 16 (1968).
[12] G. Vranceanu, *Les espaces non holonomes*, Mém. d. Sc. Math., fasc. 56, Gauthier-Villars, Paris 1936.
[13] G. Vranceanu, *Vorlesungen über Differentialgeometrie*, Bd. II, Akademie-Verlag, Berlin 1961 — transl. of: Leçons ..., Edit. l'Académie Republ. Roumaine, Bucarest 1957.
[14] J. A. Schouten, E. R. van Kampen, *Math. Ann.* **103**, 752 (1930).
[15] A. L. Zelmanov, *Dokl. Akad. Nauk SSSR* **107**, 815 (1956).
[16] N. Salié, *Wiss. Z. Friedrich-Schiller-Univ. Jena, Mat.-nat.* **15** No. 1, 45–62; 63–70 (1966).
[17] E. Schmutzer, *Relativistische Physik*, Teubner-Verlag, Leipzig 1968.
[18] J. L. Synge, *Relativity: The Special Theory*, North-Holland Publ. Comp., Amsterdam 1967.

---

[7] Cp. the proof of (4.6, 7, 11) with application of (4.13) where ${}^{(H)}p^\mu v_{\kappa\mu} = -\varepsilon^{(V)}p_\kappa/2$, ${}^{(H)}p^\mu u_{\kappa\mu} = -\varepsilon^{(V)}u_\kappa/2$, ${}^{(H)}p^\mu u_{\kappa\lambda\mu} = \varepsilon r_{\kappa\lambda}/3$ can be put (cp. I. 4.27, 28).