ELECTRIC, CYLINDRICALLY SYMMETRIC SOLUTIONS IN EINSTEIN'S UNIFIED FIELD THEORY

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(Received May 29, 1979)

Some particular solutions of Einstein's unified field equations are derived for the electric, cylindrically symmetric case.

1. Introduction

Exact solutions of Einstein's unified field equations are expected to play an important role in the interpretation of the theory but the complexity of the equations has made solutions elusive.

The static, cylindrically symmetric case, first examined by Klotz and Russell [1, 2, 3], was expected to represent a charged carrying wire and possibly lead to locally testable predictions. Although Klotz and Russell found the general solution for the magnetic and electromagnetic cases, in one article [1] it was claimed that solutions corresponding to the pure electric case did not exist.

The purpose of this paper is to re-examine the electric case and derive some particular solutions.

2. The field equations

Einstein's weak field equations are

$$g_{\underset{+}{\mu}\underset{-}{\nu};\lambda} \equiv g_{\mu\nu,\lambda}-\Gamma^{\sigma}_{\mu\lambda}g_{\sigma\nu}-\Gamma^{\sigma}_{\lambda\nu}g_{\mu\sigma} = 0, \tag{1}$$

$$\Gamma_{\mu} \equiv \tfrac{1}{2}(\Gamma^{\sigma}_{\mu\sigma}-\Gamma^{\sigma}_{\sigma\mu}) = 0, \tag{2}$$

$$R_{(\mu\nu)} = 0, \tag{3}$$

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and

$$R_{[\mu\nu,\lambda]} = 0, \tag{4}$$

where

$$R_{\mu\nu} = -\Gamma^{\sigma}_{\mu\nu,\sigma} + \Gamma^{\sigma}_{\nu\sigma,\mu} + \Gamma^{\varrho}_{\mu\sigma}\Gamma^{\sigma}_{\varrho\nu} - \Gamma^{\varrho}_{\mu\nu}\Gamma^{\sigma}_{\varrho\sigma}$$

is the generalised Ricci tensor, $\Gamma^{\lambda}_{\mu\nu}$ the non-symmetric affine connection and $g_{\mu\nu}$ the non--symmetric fundamental tensor. The round and square brackets about the indices denote symmetrisation and skew-symmetrisation respectively.

Using cylindrical polar coordinates $x^{\mu} = (r, z, \phi, t)$ the fundamental tensor takes the form [1]

$$g_{\mu\nu} = \begin{bmatrix} -\alpha & 0 & 0 & 0 \\ 0 & -\alpha & e & 0 \\ 0 & -e & -\beta & 0 \\ 0 & 0 & 0 & \gamma \end{bmatrix},$$

where α, β, γ and e are functions of r only.

This is expected to represent the purely electric case.

The following notation is used throughout this paper:

$$p = \frac{e^2}{\alpha\beta}, \quad k = \beta\gamma(1+p), \quad l = \alpha\beta(1+p), \quad \alpha = e^{2A}, \quad \beta = e^{2B},$$

$$\gamma = e^{2C}, \quad e = e^{2E}, \quad k = e^{2K} \quad \text{and} \quad l = e^{2L}.$$

The algebraic equations (1) can be solved uniquely for the affine connection (in terms of the fundamental tensor and its derivatives) if $p \neq \pm 1$, and the solution is

$$\Gamma^1_{11} = A', \quad \Gamma^1_{22} = A' - 2L' + 2N, \quad \Gamma^1_{[23]} = e/\alpha(L' - E'), \quad \Gamma^1_{33} = \beta/\alpha(B' - 2N),$$

$$\Gamma^1_{44} = \gamma/\alpha C', \quad \Gamma^2_{(12)} = L' - N, \quad \Gamma^2_{(13)} = e/\alpha(L' - E' - N), \quad \Gamma^3_{(12)} = e/\beta(E' - N),$$

$$\Gamma^3_{(13)} = N \quad \text{and} \quad \Gamma^4_{(14)} = C', \tag{5}$$

where $N = \Gamma^3_{(13)}$ is defined by

$$(p-1)(L' - 2N) = B' - A' \tag{6}$$

and all the remaining components of $\Gamma^{\lambda}_{\mu\nu}$ vanish. Dashes denote derivatives with respect to r.

From the definition of K, it can easily be shown that

$$K' = N + C' + p(E' - N) = L' + C' - A', \tag{7}$$

$$B' - N = p(L' - E' - N), \tag{8}$$

$$(p+1)N = p(L' - E') - (B' - 2N),$$

$$(p+1)(E' - N) = (2L' - 2N - A') + (E' - L'),$$

and

$$(p+1)(L'-2N) = (2L'-2N-A')+(B'-2N).$$

With the help of these identities, the five non-trivial field equations

$$R_{11} = R_{22} = R_{33} = R_{44} = 0, \quad R_{[34]} = \text{constant} = d$$

are equivalent to the system

$$C''+C'K' = 0, \tag{9}$$

$$(L'-2N)'+(L'-2N)K' = 0, \tag{10}$$

$$(p+1)\left[(E'-N)'+(E-N)(K'-2L'+2E'-2B'+4N)\right] = -\alpha e^{-1}d, \tag{11}$$

$$(p+1)\left[N'+NK'\right] = p\alpha e^{-1}d, \tag{12}$$

$$(p+1)\left[L'(C'+N)-N^2+(E'-N)(B'-N)\right] = p\alpha e^{-1}d, \tag{13}$$

plus the redundant equation

$$(p+1)\left[(B'-N)'+(B'-N)(K'+2L'-2E'+2B'-4N)\right] = p\alpha e^{-1}d. \tag{14}$$

Klotz and Russell worked with what appeared to be an overdetermined system (five equations in four unknowns) but it will now be proved that only four of the equations are independent.

Equations (9) and (10) can be integrated and if new variables are defined by

$$y = e^K = k^{\frac{1}{2}}, \quad u = y(E'-N), \quad v = y(B'-N);$$

equations (9) to (14) change to

$$C' = cy^{-1}, \quad c \text{ a constant}, \quad L'-2N = my^{-1}, \quad m \text{ a constant},$$

$$yu'+2u(u-v-m) = -e\gamma p^{-1}d, \tag{15}$$

$$y(yN)' = e\gamma d, \tag{16}$$

$$(yN)(m+2c+yN)+mc+uv = e\gamma d, \tag{17}$$

$$yv'-2v(u-v-m) = e\gamma d. \tag{18}$$

Adding equations (15) and (18) gives the new equation

$$y(uv)' = e\gamma(2u-m)d. \tag{19}$$

However, from equation (17)

$$(uv)' = d(e\gamma)'-(yN)'(m+2c+2yN)$$

$$= y(yN)'\,\frac{(e\gamma)'}{e\gamma} -(yN)'(m+2c+2yN) \text{ using (16)}$$

$$= (yN)'[y(2E'+2C')-m-2c-2yN] = (yN)'(2u-m) = e\gamma(2u-m)dy^{-1}.$$

Hence, in the new notation the field equations are

$$C' = cy^{-1}, \quad (L'-2N) = my^{-1}, \quad y(yN)' = e\gamma d,$$

$$uv+yN(m+2c+yN)+mc = e\gamma d,$$

where from equations (7) and (8) $v = p(m-u)$ and $y' = yN+c+pu$.

Two more useful relations follow from the definition of p and equations (7) and (8). They are:

$$p' = 2(p+1)(L'-A'-B') = 2(p+1)(pu-v)y^{-1}$$

and

$$yy'' = 2p(p+1)u^2.$$

If $m = 0$, it immediately follows that α is proportional to β. Since this is a very restrictive condition, it will be assumed that $m \neq 0$.

3. Einstein's strong field equations

Considerable simplification occurs when Einstein's strong field equations are considered. Equations (3) and (4) are replaced by the more stringent conditions $R_{\mu\nu} = 0$.

For the weak system this forces d to be zero and the equations reduce to

$$C' = cy^{-1}, \tag{20}$$

$$L'-2N = my^{-1}, \quad m \neq 0, \tag{21}$$

$$N = ny^{-1}, \quad n \text{ a constant}, \tag{22}$$

$$uv+n(m+2c+n)+mc = 0, \tag{23}$$

where

$$y' = c+n+pu, \tag{24}$$

$$v = p(m-u), \tag{25}$$

$$yy'' = 2p(p+1)u^2, \tag{26}$$

and

$$p' = 2(p+1)(pu-v)y^{-1}, \tag{27}$$

The general solution of the strong field equations is not known but solutions have been found for the special cases $u = 0$, $v \neq 0$ and $v = 0$, $u \neq 0$. The case $u = v = 0$ violates the condition $m = 0$ and therefore can be eliminated.

4. *Integration for the case* $u = 0$, $v \neq 0$

If $u = 0$, equation (24) integrates to give

$$y = (c+n)r+k_2 = k_1 r+k_2, \quad k_1, k_2 \text{ constants}$$

and equation (27) implies that p cannot be a constant. Hence, the unknown functions α, β, γ and e can be found from the equations

$$A' = L'-N = (m+n)y^{-1}, \quad C' = cy^{-1}, \quad E' = N = ny^{-1},$$

$$\beta = \frac{y^2}{\gamma} - \frac{l^2}{\alpha},$$

subject to the conditions $n(n+m+2c)+mc = 0$ and $p \neq$ constant.

The solution for $mnc \neq 0$ is

$$\alpha = a_1(k_1 r+k_2)^{\frac{1}{2}x^2-x}, \quad \beta = [c_1^{-1}-e_1^2 a_1^{-1}(k_1 r+k_2)^{2x-\frac{1}{2}x^2}]\,(k_1 r+k_2)^x,$$

$$\gamma = c_1(k_1 r+k_2)^{2-x}, \quad e = e_1(k_1 r+k_2)^x;$$

where k_1, k_2, a_1, c_1, e_1 and x are constants and $x \neq 0$, 2 or 4.

Klotz and Russell [1] worked with the strong field equations subject to the condition $C' = L'-2N$, and conjectured that no solutions existed in this case. However, it is easy to show that the solution for $x = 1+\sqrt{5}$ or $x = 1-\sqrt{5}$ provides a counterexample to the conjecture.

5. *Integration for the case* $v = 0$, $u \neq 0$

Under the condition $v = 0$, equations (24), (25) and (26) reduce to

$$y' = n+c+pm, \quad u = m = \text{constant},$$

and

$$yy'' = 2p(p+1)m^2.$$

Hence y is given by the differential equation

$$yy''-2(y'-n-c)(y'-n-c+m) = 0 \tag{28}$$

which has as a first integral

$$y^2 = D(y'-n-c)^{m(n+c)^{-1}}(y-n-c+m)^{1-m(n+c)^{-1}}$$

$$= Dmp^{m(n+c)^{-1}}(p+1)^{1-m(n+c)^{-1}}, \quad D \text{ a constant},$$

for $n+c \neq 0$, or if $n+c = 0$

$$y^2 = D(y'+m) = Dm(p+1)$$

The expression for y in terms of p can be used to derive an equation for p', namely,

$$p' = 2(mD^{-1})^{\frac{1}{2}} p^{1-\frac{1}{2}m(n+c)^{-1}} (p+1)^{\frac{1}{2}[1+m(n+c)^{-1}]}, \qquad n+c \neq 0.$$
$$= 2(mD^{-1})^{\frac{1}{2}} (p+1)^{\frac{1}{2}}, \qquad n = c = 0 \tag{29}$$

Note that equation (23) and $n+c = 0$ force $n = c = 0$.

Once either y or p is known as a function of r, the variables α, β, γ and e can be found from the equations

$$B' = ny^{-1}, \quad C' = cy^{-1}, \quad E' = (m+n)y^{-1}, \quad \alpha = \frac{e^2}{\beta p},$$

$$n(m+n+2c)+mc = 0.$$

As an example, consider the restrictions $v = 0$ and $m+n = 0$. Equations (23) and (29) state that $C' = 0$

and $p' = -2k_1 p^{\frac{3}{2}}$, $k_1 = -(mD^{-1})^{\frac{1}{2}}$

giving, as a solution to the strong field equations

$$\alpha = \frac{e^2}{b^2} \frac{(k_1 r+k_2)^2}{[1+(k_1 r+k_2)^2]}, \quad \beta = b^2[1+(k_1 r+k_2)^2], \quad \gamma = \gamma, \quad e = e,$$

where γ, e and b are arbitrary constants.

This special solution was first derived by Klotz and Russell [2] and was used to examine the equations of motion in a cylindrical field.

The assumption that either u or v is a constant leads to the condition: $u = v+m$. Since $p \neq -1$, equation (25) is only satisfied if $v = 0$. Hence u and v can only be constant if $v = 0$ and $u = m$.

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