A NEW UNIFIED FIELD THEORY BASED ON DE SITTER GAUGE INVARIANCE

BY P. K. SMRZ

Department of Mathematics, University of Newcastle*

(*Received June 25, 1979*)

All observable transformations (rotations, Lorentz transformations, and translations) of space-time are assumed to be contained in a de Sitter gauge group. A de Sitter structured connection on a five-dimensional base manifold is considered. The fifth component of the Lorentz gauge potential A_5^{ik} is interpreted as the electromagnetic field tensor. Requirement of zero torsion gives an interpretation to the fifth coordinate in terms of a length scale factor, as well as the first pair of Maxwell equations in a flat space. A de Sitter gauge invariant Lagrangian reduces to the Langrangian of the Maxwell–Einstein theory on the physical four-dimensional space-time, providing the radius of de Sitter "translations" is small.

Introduction

The unified theory presented in this article is a result of combining two well explored ideas: (1) Enlarging the gauge group of the gravitational field to include translations (see [1], [2], [3]). (2) Replacing Poincaré group by a de Sitter group.

References relating to (2) are numerous, but they have only a marginal relation to what is considered in the present work. In the conventional approach the "radius" of the de Sitter "translations" is considered to be large (radius of the Universe), while here it is expected to be much smaller than a typical radius of curved space-time. In fact, a related work [4] suggests that it could have the meaning of an elementary subatomic length.

The basic idea of the theory is the concept of a five-dimensional base manifold not accessible to direct coordinate measurements. A connection with de Sitter structure based on such a manifold is described by ten five-dimensional gauge potentials. Four of them are interpreted as the tetrads in space-time, while remaining six describe the 24 gravitational gauge potentials together with 6 components of the electromagnetic field tensor. A quadratic de Sitter invariant Lagrangian leads to approximate Einstein–Maxwell equations.

---

* Address: Department of Mathematics, University of Newcastle, Newcastle, N.S.W. 2308, Australia.

1. A de Sitter connection on a five-dimensional manifold

Consider a five-dimensional manifold N as a base manifold for a de Sitter structured connection. Let $A^{ab}_\alpha = -A^{ba}_\alpha$, $\alpha = 1, \ldots, 5$, a, $b = 1, \ldots, 5$, be the components of the connection (gauge potentials). The curvature of the connection is then given by the usual gauge covariant expression.

$$R^{ab}_{\alpha\beta} = \partial_\alpha A^{ab}_\beta - \partial_\beta A^{ab}_\alpha + g_{cd}(A^{ac}_\alpha A^{db}_\beta - A^{ac}_\beta A^{db}_\alpha), \tag{1}$$

where $g_{cd} = 0$ if $c \neq d$, and $g_{11} = g_{22} = g_{33} = -g_{44} = \pm 1$, $g_{55} = +1$.

The two different signs correspond to the two types of de Sitter group: (3,2) and (4,1).

We shall work with a system of units in which both the speed of light and the gravitational constant are equal to one. All physical dimensions are then expressible as powers of length.

We interpret A^{ab}_α in a close analogy with the usual interpretation of A^{ik}_μ, $\mu = 1, \ldots, 4$, $i, k = 1, \ldots, 4$, the components of the Lorentz structured connection in space-time. Recall that A^{ik}_μ characterize the observable Lorentz rotation of the local frames when one proceeds in the μ-direction. If $A^{ik}_\mu = 0$, the local frames happen to be chosen in such a way that they follow the parallel transport in the μ-direction. Let us consider A^{i5}_α, $\alpha = 1, \ldots, 5$, $i = 1, \ldots, 4$. These are the components corresponding to the four de Sitter "translations". Thus A^{i5}_α characterize the observable translations when one proceeds in the α-direction. This interpretation provides an immediate explanation for the four dimensions of space--time despite the five-dimensionality of the base manifold: Components A^{i5}_α define four vectors in the five-dimensional tangent vector space. Hence there exists such a direction in the tangent vector space that the projections of all four vectors on that direction are zero. Moreover, if the four vectors are linearly independent, the direction is unique. We choose it as the direction of the fifth coordinate, i.e. $A^{i5}_5 = 0$. Components A^{i5}_μ, $\mu = 1, \ldots, 4$, then form a non-singular 4×4 matrix. They characterize a local change of the observable (but in general non-holonomic) Lorentz coordinates when a small shift of position is made in the μ-direction. Hence they should be interpreted as the tetrads of space-time. We write

$$A^{i5}_\mu = l^{-1} h^i_\mu \tag{2}$$

on a four-dimensional submanifold M of N defined by $x^5 = \text{const.}$ l is a constant with the dimension of length. Let L_{i5} be the generators of the de Sitter rotations and T_i the corresponding generators of translations, $T_i = \frac{1}{R} L_{i5}$, where R is the "radius" of the de Sitter group. Then writing

$$A^{i5}_\mu L_{i5} = h^i_\mu T_i \tag{3}$$

we see that l may be interpreted as the "radius". In this way a de Sitter structured connection on a five-dimensional manifold N satisfying the conditions that A^{i5}_μ are independent define a four-dimensional submanifold of N with a metric

$$g_{\mu\nu} = h^i_\mu h^j_\nu g_{ij}. \tag{4}$$

If A^{i5}_μ are linearly dependent, the dimension of the submanifold may be lower than four, but it can never be greater than four.

Components A^{ij}_μ are to be interpreted as the usual components of a Lorentz structured connection in M. Finally, there are components A^{ij}_5 which still need an interpretation. We may say that the theory predicts existence of a field described by A^{ij}_5 alongside the gravitational field A^{ij}_μ, $\mu = 1, \ldots, 4$. We shall see that an interpretation as the electromagnetic field tensor is quite feasible.

The conventional Christoffel symbols $\Gamma^\sigma_{\mu\nu}$ can be expressed in terms of A^{ij}_μ and the tetrads by a gauge transformation (see e.g. [5])

$$\Gamma^\sigma_{\mu\nu} = h^\sigma_k(\partial_\mu h^k_\nu) + h^\sigma_k A^{kl}_\mu h^j_\nu g_{lj}. \tag{5}$$

We should observe, that an equation

$$\Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\nu\mu} = l h^\sigma_k R^{k5}{}_{\mu\nu}, \qquad \mu, \nu = 1, \ldots, 4, \tag{6}$$

bounds the torsion with the translational components of the curvature.

So far the connection over N was defined abstractly and did not have anything to do with the linear frames of N, i.e. it was not a linear (or affine) connection in N. Interpretation of the components A^{i5}_μ as tetrads by Eq. (2), however, provides at least the four-dimensional submanifold of N with a linear frame structure. We shall complement this structure to that of N by

$$h^i_5 = 0, \qquad h^5_\mu = 0, \qquad h^5_5 = 1. \tag{7}$$

This, together with the requirement of zero torsion, leads to an interesting interpretation of the fifth coordinate.

2. *Structure of the five-dimensional base manifold*

Let us assume now that the connection described in Section 1 is in fact a torsion free linear connection in N, with a linear frame structure defined by Eq. (2) on $x^5 = \text{const.}$, and Eq. (7) everywhere. Using Eq. (5) generalized for five dimensions, i.e.

$$\Gamma^\gamma_{\alpha\beta} = h^\gamma_a(\partial_\alpha h^a_\beta) + h^\gamma_a A^{ab}_\alpha h^c_\beta g_{bc}, \tag{8}$$

we obtain as the conditions of zero torsion

$$\Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\nu\mu} = l k^\sigma_k R^{k5}{}_{\mu\nu} = 0, \tag{9a}$$

$$\Gamma^5_{\mu\nu} - \Gamma^5_{\nu\mu} = (A^{5l}_\mu h^j_\nu - A^{5l}_\nu h^j_\mu) g_{lj} = 0, \tag{9b}$$

$$\Gamma^5_{\mu 5} - \Gamma^5_{5\mu} = 0, \tag{9c}$$

$$\Gamma^\nu_{\mu 5} - \Gamma^\nu_{5\mu} = -h^\nu_k(\partial_5 h^k_\mu) - h^\nu_i A^{ik}_5 h^l_\mu g_{kl} + \delta^\nu_\mu l^{-1} = 0. \tag{9d}$$

Eq. (9b) is satisfied because of Eq. (2), while (9d) yields

$$A^{ik}_5 = l^{-1} g^{ik} - (\partial_5 h^i_\nu) h^\nu_l g^{lk}. \tag{10}$$

Using the skew-symmetry of A_5^{ik} one derives

$$\partial_5 g_{\mu\nu} = 2l^{-1} g_{\mu\nu}. \tag{11}$$

In this way the five-dimensional manifold N may be considered as consisting of infinite number of space-time manifolds with metrics depending on the fifth coordinate by

$$g_{\mu\nu}(x^\mu, x^5) = e^{2x^5/l} g_{\mu\nu}(x^\mu, 0). \tag{12}$$

Thus the fifth coordinate is connected with the length scale in space-time. If $A_5^{ik} = 0$, then Eq. (10) directly yields

$$\partial_5 h_\nu^i = l^{-1} h_\nu^i, \tag{13}$$

taking the notion of the length scale one step down from the metric to linear frames. When A_5^{ik} is interpreted as the electromagnetic field tensor, its relationship to the length scale may remind the reader of the Weyl's unified field theory [6]. We should, perhaps, point out the difference: In Weyl's theory, change of the scale is a part of the gauge group, while in the present theory it is a part of the base manifold.

3. Maxwell–Einstein equations

We shall now construct a de Sitter gauge invariant Lagrangian that yields the Maxwell–Einstein equations by means of a variational principle. Since the relationship between the tangent structure of the base manifold (i.e. its linear frames) and the de Sitter group is brought in only with the macroscopic space-time interpretation of the connection components, we may expect that the Lagrangian will contain two simplest gauge invariant quantities: a constant term, and $R^A_{\alpha\beta} R_A^{\alpha\beta}$, where A is the group index (in our case replaced by the pair (a, b)). Lowering of the group index is by the non-degenerate group metric (the Killing form), while α and β are raised by a metric in the base manifold. It turns out that the appropriate Lagrangian density is of the form

$$L = R^{ab}{}_{\alpha\beta} R_{ab}{}^{\alpha\beta} - 24l^{-2}. \tag{14}$$

We shall assume that the macroscopical space-time interpretation of the connection requires Eqs. (2) and (7). Further, the invariance of the theory with respect to the change of scale will be expressed as $\partial_5 A_\alpha^{ab} = 0$. Under these conditions the Lagrangian (14) reduces to

$$L = \tilde{R}^{ik}{}_{\mu\nu} \tilde{R}_{ik}{}^{\mu\nu} + R^{ik}{}_{\mu 5} R_{ik}{}^{\mu 5} + l^{-2}(-4\tilde{R}^{ik}{}_{\mu\nu} h_i^\mu h_k^\nu + F_{\mu\nu} F^{\mu\nu}), \tag{15}$$

where the range of all indices is only 1 to 4, and

$$\tilde{R}^{ik}{}_{\mu\nu} = \partial_\mu A_\nu^{ik} - \partial_\nu A_\mu^{ik} + g_{lm}(A_\mu^{il} A_\nu^{mk} - A_\nu^{il} A_\mu^{mk}), \qquad F_{\mu\nu} = g_{il} h_\mu^l A_5^{ik} h_\nu^m g_{km}.$$

If l is small compared to an average radius of curved space-time as given by $\tilde{R}^{ik}{}_{\mu\nu}$, only the second part of the Lagrangian with the coefficient l^{-2} needs to be considered. This is the Maxwell–Einstein Lagrangian, providing $F_{\mu\nu}$ is interpreted as the electromagnetic field tensor multiplied by a constant.

It is well known that the Einstein equations with an electromagnetic source term contain the second pair of the Maxwell equations, once the first pair is satisfied. In the present approach the electromagnetic field potential is not introduced as the fundamental quantity, and the first pair of the Maxwell equations is not identically satisfied. Nevertheless, using Eq. (10) one obtains

$$\partial_\mu F_{\sigma\varrho}+\partial_\sigma F_{\varrho\mu}+\partial_\varrho F_{\mu\sigma} = \tfrac{1}{2}\, g_{il}(H^l_{\mu\varrho}\partial_5 h^i_\sigma + H^l_{\sigma\mu}\partial_5 h^i_\varrho + H^l_{\varrho\sigma}\partial_5 h^i_\mu - h^i_\sigma\partial_5 H^l_{\mu\varrho} - h^i_\varrho\partial_5 H^l_{\sigma\mu} - h^i_\mu\partial_5 H^l_{\varrho\sigma}), \tag{16}$$

where $H^l_{\mu\varrho} = \partial_\mu h^l_\varrho - \partial_\varrho h^l_\mu$.

In the case of the Minkowski space with the electromagnetic field only (which, of course, exists only approximately), the Minkowski coordinates may be used as holonomic coordinates, and expression (16) is equal to zero identically. Thus the Maxwell theory in the presence of a weak gravitational field is unchanged, but the general Maxwell–Einstein theory would need a considerable revision in the approach presented here.

4. *Conclusions*

The unified field theory presented in this article is a byproduct of a definite stream in the translational gauge philosophy, and it is fair to recapitulate the main features from that point of view. The basic idea is including translations, rotations, and Lorentz transformations together in a gauge group. Choice of the Poincarè group as the gauge group leads to describing a flat Minkowski space by a flat connection (see [3]). That is not a pleasing situation, since the physical space-time is described by a cross-section[1] which is not horizontal, though a horizontal cross-section exists. With a de Sitter group as the gauge group the flat Minkowski space is described by a nonflat connection and a cross-section which is "as horizontal as possible" in the sense of ${R^{i5}}_{\mu\nu} = 0$. De Sitter groups act naturally in a five-dimensional space, hence the five dimensions of the base manifold. Interpretation of the ten de Sitter transformations as the only observable changes guarantees the four-dimensionality of physical space-time. Yet, six components of A^{ik}_5 remain as an evidence of the existence of the fifth dimension. The base manifold is given its linear frame structure and metric by the connection, and zero torsion is assumed. In fact, interpreting A^{ik}_5 as the electromagnetic field tensor, the condition of zero torsion implies that the first pair of Maxwell equations is satisfied in a flat space. It also gives a physical interpretation to the fifth coordinate in terms of a length scale change factor. Finally, a simple de Sitter gauge invariant Lagrangian density yields approximately the Maxwell–Einstein equations, when the length scale invariance is assumed.

The unification of the electromagnetic and gravitational fields presented here is "non-trivial", i.e. changing both the Einstein theory, as well as the Maxwell theory. The changes are concerned with strong gravitational fields, and are not observable by the present day experiments.

---

[1] Choice of a cross-section is a choice of a gauge; e.g. selection of a linear frame at each point of a manifold.

REFERENCES

[1] K. Hayashi, *Prog. Theor. Phys.* **39**, 494 (1968).
[2] F. W. Hehl, P.v.d. Heyde, G. D. Kerlick, J. M. Nester, *Rev. Mod. Phys.* **48**, 393 (1976).
[3] P. K. Smrz, *J. Austral. Math. Soc.* **20B**, 38 (1977).
[4] P. K. Smrz, *Prog. Theor. Phys.* **57**, 1771 (1977).
[5] P. K. Smrz, *J. Austral. Math. Soc.* **15**, 482 (1973).
[6] H. Weyl, *Ann. Phys.* (Germany) **59**, 101 (1919).