DIFFRACTION OF A PLANE ELECTROMAGNETIC WAVE BY A SCHWARZSCHILD BLACK HOLE: THE POYNTING VECTOR IN THE VICINITY OF THE HORIZON

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The diffraction of a plane electromagnetic wave by a Schwarzschild black hole is considered. Measurable quantities of the diffraction field (Poynting vector, frequencies) are calculated in the high-frequency approximation for (freely falling) observers very near the horizon. Within two focal regions on opposite sides of the black hole the intensity is strongly amplified but finite, in contrast to the results of geometrical optics. Outside these regions no interference takes place, i.e., geometrical optics holds. According to the different light rays passing through each point of the horizon, there are several images (redshifted or blueshifted) of the radiation source (distant star). The positions and relative intensities of these images are given in terms of the observer's position at the horizon.

1. Introduction

In a series of papers, Herlt and Stephani ([1], [2]) have investigated the field of a plane electromagnetic wave diffracted by the spherically symmetric gravitational field of a star or a black hole. Though a high-frequency approximation was used, a waveoptical treatment of this problem (including interference effects) has been given. Furthermore, the observer was assumed to be far away from the deflecting mass. In the present paper the optical appearance of the radiation source of the plane wave to observers near the event horizon is studied.

Because it is of little importance to astrophysics, the existing literature offers only few works dealing with optical phenomena measured by observers in the vicinity of the horizon. Breuer and Ryan [3] and Cunningham [4] have considered the appearance of the external universe as seen by observers near the black hole in the framework of geometrical optics. They find an infinite number of images of all distant objects which can be seen within a circular region of the observer's sky. Pineault and Roeder [5] used an observer

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near a rotating black hole but not at the horizon. Though Nugaev [6] goes beyond the scope of geometrical optics he is unable to sum up the partial waves. Trümper and Wayland [7] have calculated frequency shifts of photons which would be measured by observers inside a cloud of freely falling radiating particles. A similar investigation has been performed by Debney [8].

In the present paper the above-mentioned images are calculated in terms of the observer's position on the black hole's surface for the special case of an incident plane wave. However, the starting point is not the null geodesic equation but the exact expression for the field of the plane wave in Schwarzschild's space-time. This waveoptical approach enables us to determine the Poynting vector associated with each image and the relative intensities; it becomes essential in the focal regions. The approximation procedure used here is similar to that of Herlt and Stephani. It is described briefly in the following section.

2. The Debye potentials of a plane wave near the horizon

In the background of the Schwarzschild metric (with the horizon at $r = 1$)

$$ds^2 = \frac{r}{r-1} \cdot dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2) - \frac{r-1}{r} \cdot dt^2 \tag{1}$$

the electromagnetic field of a generalized plane wave incident from the direction $\vartheta = \pi$ can be derived from a single function $P(r, \vartheta)$. As shown by Herlt and Stephani P is closely connected with the Debye potentials and has the structure

$$P(r, \vartheta) = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot (2n+1)}{2\omega^2 \cdot n(n+1)} \cdot e^{i\omega(\frac{1}{2} - \ln 2)} \cdot R_n(r) \cdot P_n^1(\cos\vartheta), \tag{2}$$

where ω is the frequency of the wave, i.e., 2π times the number of flat space wavelengths per Schwarzschild radius. According to Maxwell's equations the radial functions $R_n(r)$ have to fulfil

$$\frac{d^2R_n}{dr^2} + \frac{1}{r(r-1)} \cdot \frac{dR_n}{dr} + \omega^2 \cdot \left[1 - a^2 \cdot \frac{r-1}{r^3}\right] \cdot \frac{r^2}{(r-1)^2} \cdot R_n = 0, \quad a^2 = n(n+1)/\omega^2. \tag{3}$$

In the high-frequency limit ($\omega \gg 1$), which is assumed throughout the paper, an expression of the form (2) can be further elucidated using a four-step procedure proposed by Ford and Wheeler [9]. Firstly, the radial equation (3) is solved by means of the WKB approximation. Near the horizon ($1 \leqslant r \leqslant 1.5$), where only ingoing waves contribute, the WKB solution is given by

$$R_n(r) = \frac{1}{\sqrt[4]{1 - a^2 \cdot \frac{r-1}{r^3}}} \cdot e^{-i\omega(r + \ln(r-1) + T)}. \tag{4}$$

T denotes the phase shifts of the partial waves coming from infinity:

$$T(r, n) = \int_r^\infty \left(1 - \sqrt{1 - a^2 \cdot \frac{r-1}{r^3}}\right) \cdot \frac{r}{r-1}\, dr. \tag{5}$$

In the next step the asymptotic representations (for $n \sin \vartheta \gg 1$) of the Legendre functions $P_n^1(\cos \vartheta)$ are inserted into the sum (2). Then this sum has to be replaced by an integral which, finally, can be evaluated by application of the method of stationary phase. The points of stationary phase $n_0(m, \pm)$ or $a_0(m, \pm)$ are solutions of

$$(2m+1)\pi \pm \vartheta - \omega \cdot \frac{\partial T}{\partial n} = 0 \tag{6}$$

(m integer). It can easily be seen that the lines $n_0(m, \pm) = \text{const}$ coincide with the light rays of geometrical optics, $\pm$ indicating the direction of revolution and m the number

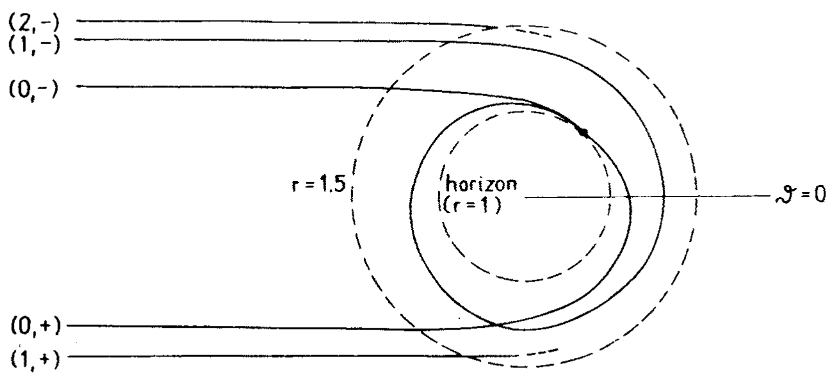

Fig. 1. Light rays near the horizon ($r = 1$)

of revolutions the ray has performed. Only rays with impact parameters $a_0(m, \pm)$ in the range $0 \leqslant a_0^2 < 27/4$ can enter the region between $r = 1$ and $r = 1.5$. (Fig. 1).
At the horizon approximate solutions of (6) are

$$a_0(0, -) = \pi - \vartheta \quad \text{for } 0 \leqslant a_0 \leqslant 1$$

$$a_0^2(m, \pm) = \tfrac{27}{4} - \frac{54^2}{14 \cdot (2+\sqrt{3})} \cdot e^{-[(2m+1)\pi \pm \vartheta]} \quad \text{for } m \lesssim 1. \tag{7}$$

There is an infinite number of points of stationary phase, which accumulate near $a_0^2 = 27/4$. Consequently, the sum (2) is transformed into the infinite sum

$$P = -\sum_{(m,\pm)} \frac{1}{\omega^2} \cdot \frac{1}{\sqrt{\sin \vartheta}} \cdot \frac{1}{\sqrt{S_0^{\pm\prime\prime}(m, \pm)}} \cdot \frac{1}{\sqrt{n_0(m, \pm)}} \cdot \frac{e^{iS_0^{\pm}(m,\pm)}}{\sqrt[4]{1 - a_0^2(m, \pm) \cdot \dfrac{r-1}{r^3}}}, \tag{8}$$

where

$$S_0^{\pm}(m, \pm) = \omega \cdot (\tfrac{1}{2} - \ln 2) + n_0(m, \pm) \cdot (2m+1) \cdot \pi - \omega \cdot \{r + \ln(r-1) + T[r, n_0(m, \pm)]\}$$
$$\pm \left[(n_0(m, \pm) + \tfrac{1}{2}) \cdot \vartheta - \frac{3\pi}{4} \right] + \frac{\pi}{4} . \tag{9}$$

$S_0^{\pm''}$ is the second derivative of the phase with respect to n. At the horizon it is given by the expressions

$$\omega S_0^{-''}(0, -) = -[1 + a_0^2(0, -)/8] \quad \text{for } 0 \leqslant a_0 \lesssim 1,$$
$$\omega S_0^{\pm''}(m, \pm) = -2a_0(m, \pm) \cdot [\tfrac{27}{4} - a_0^2(m, \pm)]^{-1} \quad \text{for } m \geqslant 1. \tag{10}$$

Fortunately, because $S_0^{\pm''}(m, \pm)$ increases rapidly for large m, only few terms in the sum (8) are essential, i.e., in the vicinity of the horizon the contributions of rays with large values of m are negligible.

3. *Poynting vector and frequency shifts for a freely falling observer*

To simplify the calculations we choose an observer who is initially at rest at infinity and then radially falls into the black hole. His proper reference frame is characterized by the orthonormal tetrad vectors (in Schwarzschild coordinates)

$$h_i^{(1)} = \left(\frac{r}{r-1}, 0, 0, \frac{1}{\sqrt{r}} \right), \qquad h_i^{(2)} = (0, r, 0, 0),$$
$$h_i^{(3)} = (0, 0, r \sin \vartheta, 0), \qquad h_i^{(4)} = \left(\frac{\sqrt{r}}{r-1}, 0, 0, 1 \right). \tag{11}$$

The last formulas can be obtained by using the line element (1) in Lemaître coordinates (see e.g. [10]). In the field of the diffracted wave the observer measures the frequencies

$$\Omega = -(S_0^{\pm} - \omega t)_{,i} \cdot h^i_{(4)} \tag{12}$$

(tetrad indices are raised and lowered with the flat space metric $\eta_{(i)(j)}$). Inserting (11) and the phase $S_0^{\pm}$ from (9) we obtain at the horizon

$$\Omega(m, \pm) = [1 + a_0^2(m, \pm)] \cdot \frac{\omega}{2} . \tag{13}$$

Consequently, for a freely falling observer the various terms in the sum (8) have different frequencies. Therefore, the interference terms of light rays cancel out and incoherent superposition takes place as in geometrical optics. These interferences must be taken into account only in the vicinity of the points $\vartheta = 0$ and $\vartheta = \pi$, where frequency differences between certain rays become small (this case is treated in Section 4). Due to gravitational frequency shift and Doppler effect, both redshifts ($a_0 < 1$) and blueshifts ($a_0 > 1$) occur (Fig. 2).

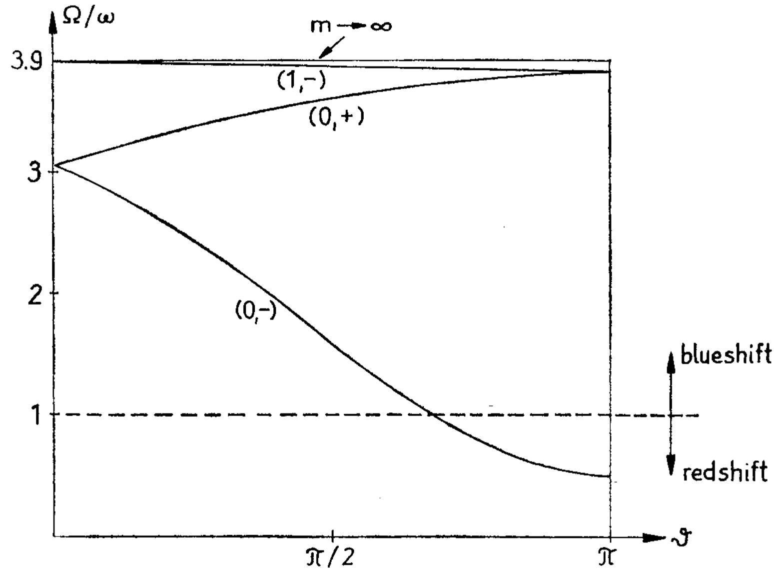

Fig. 2. Frequencies $\Omega(m, \pm)$ as measured by a freely falling observer in terms of his position at the horizon

Generally, the Poynting vector in stationary space-times is expressed in terms of the metric tensor g_{ij}, the energy-momentum tensor T_{ij} and the timelike Killing vector ξ_i by

$$S^i = \sqrt{-g_{44}} \cdot T^{ij}\xi_j. \tag{14}$$

Since $g_{44} = -1$ in Lemaître coordinates the components of the energy current as measured by the observer with the tetrad (11) are

$$S^{(k)} = T^{ij}\xi_j h_i^{(k)}, \tag{15}$$

where $\xi^i = (0, 0, 0, -1)$ is timelike outside the horizon, even for very small $r-1 > 0$. In the plane wave case the Poynting vector has the structure

$$S^r = -\frac{1}{r^2} \cdot \frac{1}{\sin^2 \vartheta} \cdot \mathrm{Re}\,(\beta e^{-i\omega t}) \cdot \mathrm{Re}\,(\delta e^{-i\omega t}),$$

$$S^\vartheta = \frac{1}{r^4} \cdot \frac{1}{\sin^2 \vartheta} \cdot \mathrm{Re}\,(\alpha e^{-i\omega t}) \cdot \mathrm{Re}\,(\delta e^{-i\omega t}),$$

$$S^\varphi = 0,$$

$$S^t = \tfrac{1}{2} \cdot \frac{1}{r^4} \cdot \frac{1}{\sin^2 \vartheta} \cdot \mathrm{Re}^2\,(\alpha e^{-i\omega t}) + \tfrac{1}{2} \cdot \frac{1}{r(r-1)} \cdot \frac{1}{\sin^2 \vartheta} \cdot [\mathrm{Re}^2\,(\beta e^{-i\omega t}) + \mathrm{Re}^2\,(\delta e^{-i\omega t})] \tag{16}$$

with the abbreviations

$$\alpha = -\sin\vartheta \frac{\partial}{\partial\vartheta} \frac{1}{\sin\vartheta} \frac{\partial}{\partial\vartheta} P \sin\vartheta,$$

$$\beta = \sin\vartheta \frac{\partial}{\partial\vartheta} \frac{r-1}{r} \frac{\partial P}{\partial r} + i\omega P,$$

$$\delta = -i\omega \cdot \sin\vartheta \frac{\partial P}{\partial\vartheta} - \frac{r-1}{r} \cdot \frac{\partial P}{\partial r}. \tag{17}$$

Because no interference of terms of the sum (8) takes place the time-averaged Poynting vector $\bar{S}^{(k)}$ (averaged with respect to the observer's proper time) is simply the sum of the quantities $\bar{S}^{(k)}(m, \pm)$ connected with the various rays. In the high-frequency approximation the radial components $\bar{S}^{(1)}(m, \pm)$ and the tangential components $\bar{S}^{(2)}(m, \pm)$ at the horizon are

$$\bar{S}^{(1)}(m, \pm) = \frac{a_0^3(m, \pm) - a_0(m, \pm)}{4 \sin\vartheta \cdot \omega |S_0^{\pm\prime\prime}(m, \pm)|}$$

$$\bar{S}^{(2)}(m, \pm) = \frac{\pm a_0^2(m, \pm)}{2 \sin\vartheta \cdot \omega |S_0^{\pm\prime\prime}(m, \pm)|}. \tag{18}$$

From (7) and (18) it follows that the intensity of the images $(m, \pm)$ of the star decreases by a factor $e^{-2\pi} \approx 0.002$ if m increases by one in accordance with the predictions of geometrical optics for rays going round the black hole outside the circle $r = 1.5$ [11].

The angle Δ between the (outward) radial direction and the direction in which the image of the star is seen by the observer at the horizon is given by

$$\cot \Delta(m, \pm) = -\frac{\bar{S}^{(1)}(m, \pm)}{|\bar{S}^{(2)}(m, \pm)|} = \frac{1 - a_0^2(m, \pm)}{2a_0(m, \pm)}. \tag{19}$$

Intensity (i.e. magnitude of the Poynting vector $\bar{S}^{(k)}(m, \pm)$) and direction of the images are plotted against the observer's position in Fig. 3.

The freely falling observer receives all images within a circular region of his sky with half an opening angle $\Delta_{max} = 138°$. This phenomenon has already been discussed by Breuer and Ryan [3] and Cunningham [4] and was called the "porthole effect". Redshifted rays have $\Delta < 90°$ (outward half of the observer's sky) whereas for blueshifted rays $\Delta > 90°$ holds (inward half of the observer's sky).

To summarize, the results of this section agree with the predictions of geometrical optics as a consequence of the approximation procedure used here: interference terms rapidly oscillate and cannot be observed. Nevertheless, the details of the images as shown in the figures (especially the relative intensities) have not yet been given in the literature for the case of an incident plane wave.

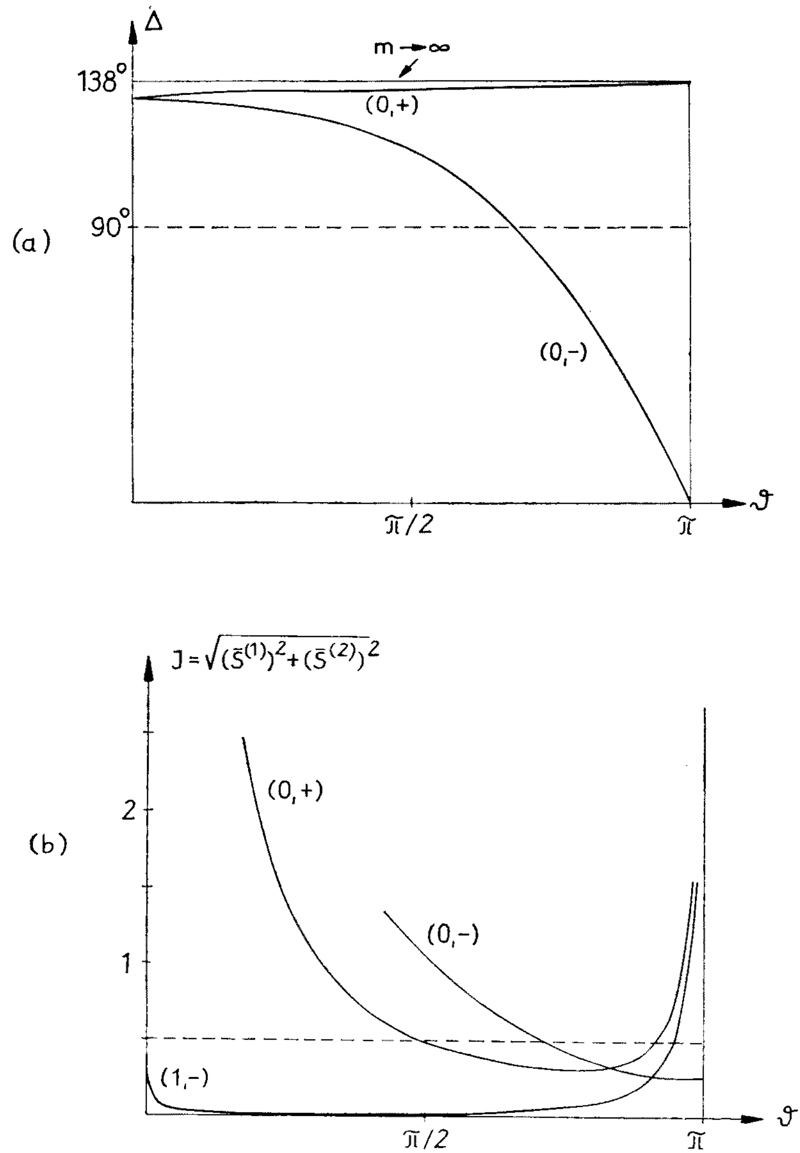

Fig. 3. (a) Direction of the Poynting vector and (b) intensity of the images (m, $\pm$) as measured by a freely falling observer at the horizon. The intensity of the incident plane wave far away from the black hole is normalized to 1/2

4. The focal regions near $\vartheta = 0$ and $\vartheta = \pi$

In the vicinity of the points $\vartheta = 0, \pi$ interference between certain rays takes place and, therefore, the predictions of wave optics differ from those of geometrical optics. More precisely, near $\vartheta = 0$ the interference between the rays (m, $+$) and (m, $-$) and near $\vartheta = \pi$

the interference between the rays $(m, +)$ and $(m+1, -)$ are essential, because the corresponding frequency differences become zero. At these points geometrical optics yields an infinite intensity. Contrary to this, wave optics is expected to yield an extreme but finite amplification of the intensity.

To succeed in this question the expression (8) has to be replaced by a similar formula involving the asymptotic representations of the Legendre functions holding true for $\vartheta \ll 1$, $n \gg 1$ and $\pi - \vartheta \ll 1$, $n \gg 1$, respectively. Using this function we find that the interference of the rays $(m, +)$ and $(m, -)$ at $\vartheta \approx 0$ gives the averaged components $\bar{S}^{(k)}(m)$ of the Poynting vector as measured from a freely falling observer at the horizon as

$$\bar{S}^{(1)}(m) = \frac{\pi}{2} \cdot \frac{1}{|S_0''(m)|} \cdot \left[a_0^4(m) \cdot J_1^2(n_0(m) \cdot \vartheta) - a_0^2(m) \cdot J_2^2(n_0(m) \cdot \vartheta)\right]$$

$$\bar{S}^{(2)}(m) = 0, \tag{20}$$

where $a_0(m, +) \approx a_0(m, -)$ and $S_0''(m) = S_0^{+''}(m, +) \approx S_0^{-''}(m, -)$. J_1, J_2 are Bessel functions. Fig. 4 shows the radial component $\bar{S}^{(1)}(m = 0)$ and the corresponding rays connected with this (nearly time-independent) interference pattern. The intensity behaves

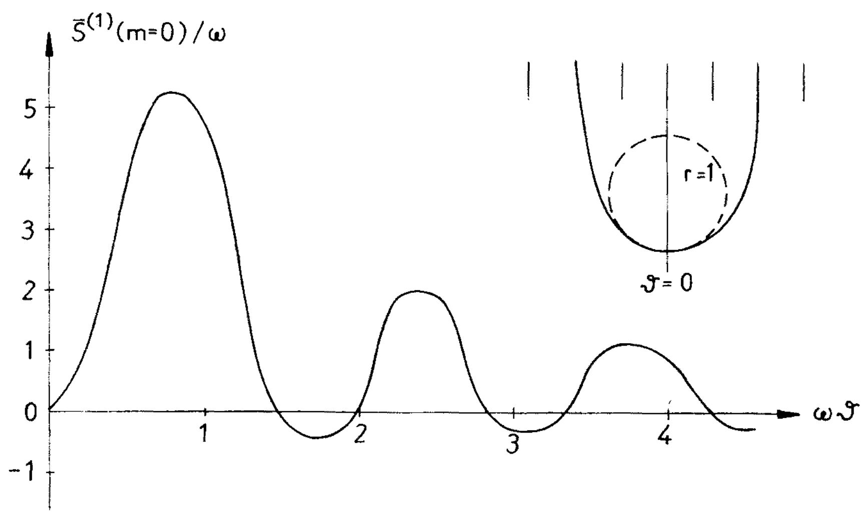

Fig. 4. The radial component of the Poynting vector close to $\vartheta = 0$.

in an oscillatory fashion and is amplified by a factor of order ω. Thus at the horizon the magnitude of the Poynting vector is enlarged up to the same remarkable factor as in the focal beam far away from the black hole ([1], [2]). Again, the intensity of the interference pattern arising from rays having revolved the black hole several times is negligible because it decreases by a factor $e^{-2\pi} \approx 0.002$ if m increases by one. At $\vartheta = \pi$ the situation is similar to that at $\vartheta = 0$. Therefore, the expression "focal regions" is justified here.

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