CANONICAL QUANTIZATION OF GRAVITY AND QUANTUM FIELD THEORY IN CURVED SPACE-TIME

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It is shown that canonical quantization of gravity and matter leads to quantum field theory in curved space-time in the case when the gravitational field can be regarded as a classical background field. The Tomonaga-Schwinger equation for matter in a background field is derived. The physical meaning of momentum constraints of quantum gravity in the presence of matter is established. As an example, the Bianchi type I cosmology is studied and the direction of alteration of evolution caused by created particles is found.

1. Introduction

It is well known that canonical formulation of general relativity leads to constraints imposed on canonical variables and the full Hamiltonian of the theory vanishes up to surface terms [1]. Following the Dirac-Wheeler–De Witt method of quantization of gravity [2, 3] we assume that canonical variables of the theory obey usual commutational relations and the constraints of general relativity are the only equations imposed on the state vector. Thus, in the case of empty space-time the main equations are:

$$\mathscr{H}^{\mu}(\hat{\pi}^{ij}, \hat{g}_{ij})\,|\Psi\rangle = 0, \tag{1.1}$$

where

$$\mathscr{H}^0 \equiv \tfrac{1}{2}\,\hat{g}^{-1/2}(\hat{g}_{ik}\hat{g}_{jl}+\hat{g}_{il}\hat{g}_{jk}-\hat{g}_{ij}\hat{g}_{kl})\hat{\pi}^{ij}\hat{\pi}^{kl}-\hat{g}^{1/2}\hat{R},$$

$$\mathscr{H}^i \equiv -2(\hat{\pi}^{ij})_{|j}.$$

Here $\hat{g}_{ij}$ denotes an operator of spatial metric, $\hat{\pi}^{ij}$ is its canonically conjugate momentum, $g \equiv \det ||g_{ij}||$, R denotes a spatial curvature. A generalization for the case of gravitational field interacting with the fields of matter is straightforward [1]. Let $\hat{\varphi}_a(x)$, $\hat{\pi}^a(x)$ be canonical variables of the fields of matter, obeying standard commutational relations.

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Then the presence of matter leads to additional terms $\mathscr{E}^\mu(\hat{\varphi}_a, \hat{\pi}^a, \hat{g}_{ij})$ on the left-hand side of Eq. (1.1):

$$R^\mu|\Psi\rangle \equiv (\mathscr{H}^\mu + \mathscr{E}^\mu)\,|\Psi\rangle = 0. \tag{1.2}$$

Note, that the Hamiltonian of matter in a background gravitational field is exactly

$$H_\mu = \int \mathscr{N}_\mu \mathscr{E}^\mu d^3x, \tag{1.3}$$

where $\mathscr{N}_0$ and $\mathscr{N}_i$ are well known lapse and shift functions [1, 4].

This method of quantization is widely used in the investigations of quantum cosmological models [5]. But the validity of this method and its general consequences have not been studied sufficiently (see also [6]).

However it was proved, that the quasiclassical limit of Eq. (1.1) leads to Einstein equations of general relativity [7]. For the readers convenience we reproduce here some results obtained by Gerlach [7].

Wheeler's representation of canonical commutational relations is chosen. The state vector $|\Psi\rangle$ becomes a functional of spatial metric:

$$|\Psi\rangle \to \Psi[g_{ij}].$$

The operator $\hat{g}_{ij}$ acts as an operator of multiplication, $\hat{\pi}^{ij}$ acts as an operator of the functional derivative,

$$\hat{g}_{ij}(x) \to g_{ij}(x); \qquad \hat{\pi}^{ij}(x) \to -i\delta/\delta g_{ij}(x). \tag{1.4}$$

The quasiclassical limit is given by the state functional

$$\Psi = \exp(iS), \tag{1.5}$$

where the action functional $S[g_{ij}]$ obeys the Einstein–Hamilton–Jacobi equation:

$$\mathscr{H}^\mu(\delta S/\delta g_{ij};\, g_{ij}) = 0. \tag{1.6}$$

Eq. (1.6) is equivalent to Eqs (1.1) and (1.5) if the second functional derivatives in (1.1) are negligible. In fact, they are of the order $\hbar$, while Eq. (1.6) is of zeroth order in $\hbar$.

Every solution of Eq. (1.6) corresponds to classical four geometry obeying Einstein equations and vice versa. The four geometry is represented by a certain subset of a set of all spatial metrics. It is possible to introduce a Tomonaga bubble-time parameter, $\sigma(x)$ enumerating elements of the subset and to write down the equations of motion in a covariant way:

$$\delta g_{ij}/\delta\sigma(y) = \int \frac{\delta \mathscr{H}^\mu(\delta S/\delta g_{kl}(x);\, g_{kl}(x))}{\delta(\delta S/\delta g_{ij}(z))} \cdot \frac{\delta M_\mu(z)}{\delta\sigma(y)} \cdot d^3z$$

$$= 2G_{ijkl} \cdot \frac{\delta S}{\delta g_{kl}} \cdot \frac{\delta M_0}{\delta\sigma(y)} + \left(\frac{\delta M_{i/j}(x)}{\delta\sigma(y)} + \frac{\delta M_{j/i}(x)}{\delta\sigma(y)}\right). \tag{1.7}$$

Here

$$G_{ijkl} \equiv \tfrac{1}{2}\, g^{-1/2}(g_{ik}g_{jl} + g_{il}g_{jk} - g_{ij}g_{kl}),$$

S obeys Eq. (1.6), functionals $\delta M_\mu(x)/\delta\sigma(y)$ are connected with the lapse and shift functions

$$\mathcal{N}_\mu(x) = \int \frac{\delta M_\mu(x)}{\delta\sigma(y)} d^3y. \tag{1.8}$$

Choosing $\sigma(y) = \text{const} = t$ and integrating Eq. (1.7) over d^3y, one derives the canonical equation of motion

$$\frac{\partial g_{ij}(x)}{\partial t} = \frac{\delta \int \mathcal{N}_\mu \mathcal{H}^\mu d^3z}{\delta\pi^{ij}(x)}\bigg|_{\pi^{ij}=\delta S/\delta g_{ij}}.$$

Equations $\mathcal{H}^i|\Psi\rangle = 0$ and $\mathcal{H}^i(\delta S/\delta g_{ij}; g_{ij}) = \sigma$ mean that functionals $\Psi[g_{ij}]$ and $S[g_{ij}]$ are invariant under the transformation of spatial coordinates [2, 7]:

$$x^i \to x'^i(x^i). \tag{1.9}$$

The rest of this paper is organized in the following way. In Section 2 the last statement is generalized for the presence of any kind of matter. In Section 3 the Tomonaga–Schwinger equation for the fields of matter in background space-time is derived from Eq. (1.2). Some applications of the results of Sections 2 and 3 to a simple cosmological model are considered and the direction of alteration of evolution of the universe caused by created particles is found in Section 4. Section 5 contains some concluding remarks.

2. *The role of momentum constraints*

In this section unified notations $(g_{ij}, \varphi_a) \equiv \phi_A$ and $(\pi^{ij}, \pi^a) \equiv \pi^A$ are used. Consider the infinitesimal transformation of spatial coordinates, Eq. (1.9):

$$x'^i = x^i + \xi^i(x). \tag{2.1}$$

The dynamic coordinates and momenta transform according to:

$$\delta\phi_A(x) = \int \eta^i_A(x, y)\xi_i(y)dy, \tag{2.2}$$

$$\delta\pi^A(x) = \int \theta^{A,i}(x, y)\xi_i(y)dy. \tag{2.3}$$

It was shown in [1], that in the classical case this transformation is generated by the quantities R^i (see Eq. (1.2):

$$\{\phi_A(x), R^i(y)\} \equiv \delta R^i(y)/\delta\pi^A(x) = \eta^i_A(x, y), \tag{2.4a}$$

$$\{\pi^A(x), R^i(y)\} \equiv -\delta R^i(y)/\delta\phi_A(x) = \theta^{A,i}(x, y). \tag{2.4b}$$

From Eqs (2.2) and (2.3) it follows that functions $\eta_A{}^i$ and $\theta^{A,i}$ depend only on ϕ_A and π^A, respectively. Thus the functions R^i are linear and homogeneous functions of π^A and ϕ_A (see Eq. (2.4)). Consequently,

$$R^i[\phi_A; \pi^A; y] = \int \frac{\delta R^i(y)}{\delta\pi^A(z)} \pi^A(z)d^3z \tag{2.5}$$

and $\delta R^i/\delta\pi^A$ is independent of momenta.

Now, consider the change of some functional $U[\phi_A(x)]$ under the transformation of (2.1) and (2.2)

$$\delta U = \int \frac{\delta U}{\delta \phi_A(x)} \delta \varphi_A(x) d^3x$$
$$= \int \frac{\delta U}{\delta \phi_A(x)} \eta^i_A(x, y) \xi_i(y) d^3x d^3y$$
$$= \int \frac{\delta U}{\delta \phi_A(x)} \frac{\delta R_i(y)}{\delta \pi^A(x)} \xi_i(y) d^3x d^3y$$
$$= \int R^i[\varphi_A; \delta U/\delta \phi_A; y] \xi_i(y) d^3y.$$

Here Eqs (2.2), (2.4a) and (2.5) and the independence of $\delta R^i/\delta \pi^A$ of momenta were used.

Thus, the necessary and sufficient condition for the invariance of the functional U under the transformation is

$$R^i[\phi_A; \pi^A; y]/_{\pi^A = \delta U/\delta \phi_A} = 0. \tag{2.6}$$

In generalized Wheeler representation, which is

$$\hat{\phi}_A \to \phi_A; \quad \hat{\pi}^A \to -i\delta/\delta\phi_A,$$

the momentum constraints are exactly in the form of Eq. (2.6) because of the linearity and homogeneity of R^i as functions of the canonical variables. Thus, the meaning of momentum constraints in the presence of matter is precisely the same as that of empty space-time (see Section 1).

3. *Derivation of Tomonaga–Schwinger equation*

Consider Eq. (1.2) in a particular case. Let the gravitational field be classical and the matter field be weak so that the gravitational field may be regarded as background. In this case the main dependence of the state vector on the gravitational variables is quasiclassical. We construct the space of state vectors of the system as a set of mappings from the set of all spatial metrics into the Hilbert space of state vectors of matter, i.e., into Fock space. In other words, the space of state vectors is a set of all functions of spatial metric with values in the Hilbert space of matter. Each state vector can be written in the form $|\Psi[g_{ij}]\rangle$ and for a particular metric it is an element of Hilbert space, $\mathscr{H}_M$, of the state vectors of matter. Operators $\hat{g}_{ij}$ and $\hat{\pi}^{ij}$ are realized as in the Wheeler representation, Eq. (1.4), operators $\hat{\varphi}_a$ and $\hat{\pi}^a$ act in $\mathscr{H}_M$ in a common way, i.e., operators $\hat{\varphi}_a$ and $\hat{\pi}^a$ act on $|\Psi[g_{ij}]\rangle$ as they do on an element of $\mathscr{H}_M$.

We search for the solution of Eq. (1.2) in the form:

$$|\Psi[g_{ij}]\rangle = \exp(iS) \cdot |\Psi_M[g_{ij}]\rangle. \tag{3.1}$$

Here, S is the classical action of empty space, obeying Eq. (1.6). The terms $\delta^2 S/\delta g_{ij}\delta g_{kl}$ in Eq. (1.2) are negligible because of the classical character of geometry. The terms $\delta^2|\Psi_M[g_{ij}]\rangle/\delta g_{ij}\delta g_{kl}$ are also negligible because of the weakness of matter (see below).

Then Eq. (1.2) has the form:

$$2G_{ijkl}\frac{\delta S}{\delta g_{ij}}\cdot i\frac{\delta|\Psi_{\rm M}\rangle}{\delta g_{kl}} = \mathscr{E}^0/\Psi_{\rm M}\rangle,$$

$$-2i\left(\frac{\delta|\Psi_{\rm M}\rangle}{\delta g_{ij}}\right)_{|j} = \mathscr{E}^i|\Psi_{\rm M}\rangle. \qquad (3.2)$$

Using Eq. (1.7) we derive from Eqs (3.2) and (3.3):

$$\left[\int\frac{\delta M_\mu(x)}{\delta\sigma(y)}\mathscr{E}^\mu(x)d^3x\right]|\Psi_{\rm M}(\sigma)\rangle = i\frac{\delta|\Psi_{\rm M}(\sigma)\rangle}{\delta\sigma(y)}. \qquad (3.4)$$

The dependence of $|\Psi_{\rm M}\rangle$ on $\sigma(y)$ appears here instead of the dependence of $|\Psi_{\rm M}\rangle$ on $g_{ij}(x)$ because the spatial metric describing classical four geometry is in its turn a functional of σ (see Eq. (1.7)). In the derivation of Eq. (3.4) we used the identity

$$\frac{\delta|\Psi_{\rm M}(\sigma)\rangle}{\delta\sigma(y)} \equiv \int\frac{\delta|\Psi_{\rm M}[g_{ij}]\rangle}{\delta g_{ij}(x)}\cdot\frac{\delta g_{ij}(x)}{\delta\sigma(y)}d^3x.$$

From (3.4) we observe that $\delta^2|\Psi_{\rm M}\rangle/\delta g_{ij}\delta g_{kl}$ is of the order of $\mathscr{E}^2|\Psi_{\rm M}\rangle$ and is thus a small quantity.

Eq. (3.4) is in fact the Tomonaga–Schwinger equation for the matter in background space-time. In the case where σ = const = t integration over y in Eq. (3.4) leads to the Schrödinger equation:

$$i\frac{\partial|\Psi_{\rm M}\rangle}{\partial t} = H_{\rm M}|\Psi_{\rm M}\rangle.$$

Here, $H_{\rm M}$ is determined by Eq. (1.3) and $\mathscr{N}_\mu$ being given by Eq. (1.8).

4. *Application to Bianchi type I cosmology*

4.1. Empty space-time

We choose the metric tensor for the model in the form [8]:

$$g_{\mu\nu} = \mathrm{diag}\,[-\mathscr{N}^2;\exp(2\Omega)\exp 2(\beta_+ + \sqrt{3}\,\beta_-);$$

$$\exp(2\Omega)\exp 2(\beta_+ - \sqrt{3}\,\beta_-);\exp(2\Omega)\exp(-4\beta_+)].$$

In the Hamiltonian approach the variables Ω, $\beta_\pm$ are canonical ones. Following the standard procedure of quantum cosmology [5], we neglect other degrees of freedom of the gravitational field and obtain the only constraint [8]:

$$\mathscr{H}^0 = \frac{\exp(3\Omega)}{24}(p_+^2+p_-^2-p_\Omega^2) \equiv \frac{\exp(3\Omega)}{24}G_{AB}p^Ap^B,$$

$$G_{AB} = \mathrm{diag}(-1,+1,+1), \quad A,B = \Omega,+,-.$$

The classical equations of motion are [8]:

$$\dot{\beta}_+ = (1/12)\mathcal{N}\exp(3\Omega)p_+,$$
$$\dot{\beta}_- = (1/12)\mathcal{N}\exp(3\Omega)p_-,$$
$$\dot{\Omega} = (1/12)\mathcal{N}\exp(3\Omega)p_\Omega,$$
$$\dot{p}_+ = \dot{p}_- = \dot{p}_\Omega = 0.$$

The collapsing universe corresponds to $p_\Omega > 0$, and singularity is reached at $\Omega = -\infty$. In the space with axes β_+, β_- and Ω (minisuperspace [8]) the evolution of the universe is represented by a straight line, being null in the metric G_{AB}, Eq. (4.1). In the quantum case the analog of Eq. (1.1) for this model is:

$$(-i)^2(1/24)\exp(3\Omega)(\partial_+^2+\partial_-^2-\partial_\Omega^2)\Psi = 0,$$
$$\partial_\pm \equiv \partial/\partial\beta_\pm; \qquad \partial_\Omega \equiv \partial/\partial\Omega. \tag{4.2}$$

The solution of Eq. (4.2) for the collapsing universe is given by:

$$\Psi = \exp[i(p_+\beta_+ + p_-\beta_- + p_\Omega\cdot\Omega)], \qquad p_\Omega = (p_+^2+p_-^2)^{1/2}. \tag{4.3}$$

This solution is analogous to the wave function of a zero mass particle in three dimensional space-time with space coordinates $\beta_\pm$, the time coordinate $\tau = -\Omega$ and with metric G_{AB}. Just as in the classical case the universe collapses up to singularity, $\Omega = -\infty$.

4.2. Space-time with matter

The addition of matter can break the symmetry of the model. However, we consider the case of the weak matter (see Eq. (4.5), where this difficulty does not appear. So we choose the action in the form

$$S = \int d^3x dt[\dot{\beta}_+p_+ + \dot{\beta}_-p_- + \dot{\Omega}p_\Omega + \varphi_a(x)\pi^a(x) - \mathcal{N}(\mathcal{H}^0+\mathcal{E})].$$

We do not write the normalization volume for simplicity of notation, so that Eq. (1.2) for this model is:

$$[(-i)^2(1/24)\exp(3\Omega)(\partial_+^2+\partial_-^2-\partial_\Omega^2)+\int \mathcal{E}d^3x]\,|\Psi\rangle = 0. \tag{4.4}$$

According to Section 3 we search for the solution of Eq. (4.2) in the form of

$$|\Psi\rangle = \exp[i(p_+\beta_+ + p_-\beta_- + p_\Omega\Omega)]\,|\Psi_M(\beta_\pm,\Omega)\rangle,$$
$$p_\Omega = (p_+^2+p_-^2)^{1/2}.$$

In the case of weak matter, $|\Psi_M\rangle$ obeys the equation

$$i\frac{d|\Psi_M\rangle}{dt} = (\mathcal{N}\int \mathcal{E}_M d^3x)\,|\Psi_M\rangle.$$

It may be proved that this approximation is valid if:

$$\exp(3\Omega)p_\Omega^2 \gg \langle \Psi_M, \int \mathcal{E} d^3x \Psi_M \rangle, \tag{4.5}$$

i.e., when matter does not change the evolution of the universe considerably.

Now, we introduce the "conserved current" $\mathcal{J}_A$ [2, 9]:

$$\mathcal{J}_\pm = \frac{1}{2i}(\langle \Psi, \partial_\pm \Psi \rangle - \langle \partial_\pm \Psi, \Psi \rangle),$$

$$\mathcal{J}_\Omega = \frac{i}{2}(\langle \Psi, \partial_\Omega \Psi \rangle - \langle \partial_\Omega \Psi, \Psi \rangle). \tag{4.6}$$

In Eqs (4.5) and (4.6) the brackets $\langle , \rangle$ denote the inner product in the Hilbert space of matter. The current $\mathcal{J}_A$ is a function of $\beta_\pm$ and Ω. In the case of an empty universe with the wave function (4.2) it is

$$\mathcal{J}_\pm = p_\pm, \mathcal{J}_\Omega = -p_\Omega.$$

Thus, the current reflects the evolution of the system [10]. For an empty universe the current is null in the metric G_{AB}, but in the presence of matter we get

$$G_{AB}\mathcal{J}^A\mathcal{J}^B = -24\exp(-3\Omega)\langle \Psi_M, \int \mathcal{E} d^3x \Psi_M \rangle$$

up to terms negligible when Eq. (4.5) holds. Due to the positiveness of $\langle \int \mathcal{E} d^3x \rangle$, the current is directed inside the "light cone". This corresponds to the isotropisation of collapse. Note that if no particles exist at the beginning of collapse, $\langle \int \mathcal{E} d^3x \rangle > 0$ at $t > 0$ due to the particle creation [11], and the particle creation leads to the isotropisation.

One can show [11] that Eq. (4.5) holds up to diameters of the universe on the order of 10^{-33} cm and only then the investigation of a complete equation, Eq. (1.2), is necessary[1].

5. *Concluding remarks*

The results presented in this paper show that the canonically quantized theory of gravitation interacting with matter leads in proper limits to the theory of quantized matter fields in curved background space-time. Authors hope that these results will help to find a solution of such important problems as the back reaction of created particles on geometry. The example in Section 4 shows that the relevant object for this problem is the "conserved current" and it is likely that the full solution of this problem, including the question about the energy source of creation will take into account the quantum properties of space--time.

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[1] Note that the value of $\langle \int \mathcal{E} d^3x \rangle$ can be calculated both in Schrödinger and Heisenberg representations because of the usual connection between them.

Editorial note. This article was proofread by the editors only, not by the authors.

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