DETERMINATION OF THE LIMITING VALUES FOR ELECTROMAGNETIC FORM FACTORS AT INFINITE FOUR-MOMENTUM TRANSFER*

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We show how the value of the electromagnetic form factor at infinite four-momentum transfer can be estimated using experimental data only over a finite interval of four-momentum transfer and a theorem from the theory of functions.

Studies of elastic electron-nucleon scattering have given a wealth of information on the electromagnetic structure of the nucleon [1, 2]. Likewise, single-pion electroproduction has been used to determine the pion charge form factor [3]. It is generally assumed that these form factors satisfy a once-subtracted dispersion relation [4],

$$F(t) = F(0)+ \frac{t}{\pi}\int_{t_0}^{\infty} \frac{\operatorname{Im} F(w)dw}{w(w-t)}, \tag{1}$$

where $t = 0$ is chosen as the subtraction point. A once-subtracted dispersion relation allows for the possibility that at infinite momentum transfer the form factor has a finite non-zero value [4] i.e.,

$$\lim_{|t|\to\infty} F(t) = F(\infty) < \infty. \tag{2}$$

In the nonrelativistic limit $F(\infty)$ has the following interpretation: an amount of charge having value $F(\infty)$ is concentrated at the center of the particle in a hard core represented by a delta function $\delta(\vec{r})$; the rest of the charge is distributed in a superposition of Yukawa type distributions [5].

It is to be noted that only the electromagnetic form factors of the proton have been measured over a wide range of four-momentum transfers [2]. Since experimental data for

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any electromagnetic form factor will only be over a finite interval of t, i.e., $t_1 \leqslant t \leqslant 0$, if we desire a knowledge of the value of $F(\infty)$, then it will have to be calculated within the context of a given theoretical framework where one of the inputs is the measured values of the form factor over a finite t-interval.

The purpose of this paper is to present a procedure whereby $F(\infty)$, the value of the form factor at infinite four-momentum transfer, can be calculated[1].

Our major assumption is that the form factor, $F(t)$, satisfies a once-subtracted dispersion relation as given by equation (1). Such a relation implies that $F(t)$ and its derivative are both continuous for $t \leqslant 0$. We also assume that the limit as $t \to -\infty$ exists and denote its value by $F(\infty)$.

The following theorem is useful [7]: Assume that $f(x)$ is continuous when $x \geqslant 0$; that the derivative of $f(x)$ is continuous when $x > 0$; that the limit as $x \to \infty$ of $f(x)$ exists and has the value $f(\infty)$; then, for positive a and b, the following relationship holds,

$$\int_0^\infty \frac{f(bx)-f(ax)}{x}\,dx = [f(\infty)-f(0)]\log\left(\frac{b}{a}\right). \tag{3}$$

Let $F(t)$ be an electromagnetic form factor. Given the previous assumptions on $F(t)$, it is easily seen that in terms of the variable $x = -t$, the function $f(x) = F(t)$ satisfies the conditions of the above theorem. Solving for $F(\infty)$, gives,

$$F(\infty) = F(0)+\frac{1}{\log\left(\frac{b}{a}\right)}\int_0^\infty \frac{F(-bx)-F(-ax)}{x}\,dx. \tag{4}$$

As stated earlier, $F(t)$ is, in general, known experimentally only over a finite interval of t. If we let $a = 1$, then equation (4) can be written in the following form,

$$F(\infty) = F(0)+\frac{1}{\log b}\int_0^{\frac{x_1}{b}} \frac{[F(-bx)-F(-x)]}{x}\,dx+\frac{1}{\log b}\int_{\frac{x_1}{b}}^\infty \frac{[F(-bx)-F(-x)]}{x}\,dx, \tag{5}$$

where, $t_1 \leqslant t \leqslant 0$, is the interval of four-momentum transfer over which the form factor is experimentally known and $x_1 = -t_1$. Note that the bracketed expression in the first integral can be determined from the data and thus the first integral can be evaluated.

We must now consider what is to be done with the second integral. We will now show that this term is, for fixed b, of order $(1/x_1)$ and, consequently, may be neglected for sufficiently large x_1.

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[1] As indicated by equation (2), $F(t)$ approaches the same value $F(\infty)$ everywhere as $|t| \to \infty$. See Ref. [6].

Now for large values of $x, F(-x)$ has the asymptotic expansion [8]

$$F(-x) \sim F(\infty) + \frac{A}{x} + O\left(\frac{1}{x^2}\right), \tag{6}$$

where A is a constant. Thus,

$$F(-bx) - F(-x) \sim \left(\frac{1-b}{b}\right)\frac{A}{x} + O\left(\frac{1}{x^2}\right), \tag{7}$$

and,

$$\int_{\frac{x_1}{b}}^{\infty} \frac{[F(-bx)-F(-x)]}{x}\,dx \sim \frac{(1-b)A}{x_1} + O\left(\frac{1}{x_1^2}\right). \tag{8}$$

Under these conditions equation (5) becomes

$$F(\infty) = F(0) + \int_{0}^{\frac{x_1}{b}} \frac{[F(-bx)-F(-x)]}{x}\,dx + O\left(\frac{1}{x_1}\right). \tag{9}$$

As stated above, the third term in equation (5) is of order $(1/x_1)$ and, consequently, for large enough x_1 will make a negligible contribution.

It is of interest to note that the upper limit for the integral in equation (9) is x_1/b, which for $b > 1$ is less than the experimental range of x values, $0 \leqslant x \leqslant x_1$. However, the argument of $F(-bx)$ will range over the full interval of x values.

As an illustration of the use of equation (9), we consider the electric and magnetic form factors of the proton. The electric form factor, $F_e(t)$, is normalized to the value $F_e(0) = 1$. Likewise, we define the "normalized" magnetic form factor $F_M(t) = G_M(t)/\mu$, such that $F_M(0) = 1$. $G_M(t)$ is the usual magnetic form factor and μ is the magnetic moment of the proton. A good simple analytic representation of the experimental data over the measured interval, $0 \leqslant -t \leqslant 25\ (\mathrm{GeV}/c)^2$, is given by the "dipole" formula [2],

$$F_e(t) \simeq F_M(t) \simeq \frac{1}{\left[1 - \dfrac{t}{0.71}\right]^2}, \tag{10}$$

where t is measured in units of $(\mathrm{GeV}/c)^2$. Substitution of equation (10) into equation (9), with $x_1 = 25$ and $b = 1.5$, gives[2],

$$F(\infty) = 0. \tag{11}$$

[2] This result is accurate to about one percent.

The "correction term", i.e., the second integral on the right-side of equation (5), can be estimated from equation (8); it is,

$$\int_{\frac{x_1}{b}}^{\infty} \frac{[F(-bx)-F(-x)]}{x}\, dx \sim 0.05A, \tag{12}$$

which is small, since we expect A to be certainly no larger than order of magnitude one.

The above calculations strongly suggest that at infinite four-momentum transfer the electric and magnetic form factors of the proton are zero. Experimental data over a larger interval of t would permit us to obtain a better estimate of $F(\infty)$ and thus strengthen our belief that $F(\infty) = 0$.

In summary, we have shown that the difference in value of a form factor at infinite and zero four-momentum transfers is related to a certain integral which involves the form factor itself. Further, we have shown that to obtain an accurate estimate of $F(\infty)$, values of $F(t)$ need to be known only over a finite, but sufficiently large, interval of t.

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