# DIFFERENCE EQUATIONS FOR LEPTON AND QUARK SPECTRA

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A new approach to lepton and quark spectra, proposed recently on the base of difference equations for mass and charge in the discrete space of elementary-fermion generations, is here supplemented by a conjecture about a crucial coefficient. This leads to the predictions $1788.035 \pm 0.004$ MeV and $28.70604 \pm 0.00007$ GeV for the masses of $\tau$ lepton and the next charged lepton, respectively. The predicted mass of toponium $t\bar{t}$ is about 36–40 GeV.

## 1. Introduction

In a recent work [1] we raised the question whether the mass and charge spectra of leptons and quarks might be described by a coupled system of two *difference equations* defined in a discrete space of nonnegative integers $n = 0, 1, 2, \ldots$ related to the quantum number $N = 1, 2, 3, \ldots$ numerating the elementary-fermion generations [2]. In fact, we conjectured some difference equations of the second order for mass and charge of leptons and quarks which implied the relationship

$$n = \begin{cases} 2N-2 & \text{for neutrions } \nu_N \text{ and up-quarks } u_N \\ 2N-1 & \text{for charged leptons } l_N \text{ and down-quarks } d_N, \end{cases} \tag{1}$$

where

$$\begin{aligned} \nu_N &= \nu_e, \nu_\mu, \nu_\tau, \ldots, \\ l_N &= e^-, \mu^-, \tau^-, \ldots, \\ u_N &= u, c, (t), \ldots, \\ d_N &= d, s, b, \ldots \end{aligned} \tag{2}$$

corresponding to $N = 1, 2, 3, \ldots$, respectively (notice that in Ref. [1] the fermion generations were numerated by $N = 0, 1, 2, \ldots$ giving, therefore, $n = 2N$ or $2N+1$). The conjec-

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tured difference equations contained four free parameters, two for leptons and two for quarks, which in order to be determined required the use of known lepton and quark masses. This diminished considerably the predictive power of the new equations, especially in their application to the already known leptons and quarks. In the present paper we conjecture a particular value for one of these two parameters, increasing thereby much the ability of our equations to be decidedly confronted with the already known spectra of leptons and quarks. In particular, we predict the mass of $\tau$ lepton surprisingly close to its experimental value. We predict also the masses of two next charged leptons.

## *2. Difference equations for mass and charge*

First, we reproduce the difference equations from Ref. [1]. The difference operator $\Delta$ defined by the equation

$$\Delta f(n) = f(n+1) - f(n) \tag{3}$$

describes the rise of an one-particle physical quantity $f(n)$ when passing from the $n$ lepton or quark to the $n+1$ lepton or quark. Notice that $\Delta+1$ plays the role of a rising operator,

$$(\Delta+1)f(n) = f(n+1). \tag{4}$$

In Ref. [1] we assumed that *the rise of the mass* $m(n)$ *during the displacement* $n \to n+2$ *depends linearly on* $m(n)$ *itself and on the electric charge squared* $Q^2(n)$. Then

$$[(\Delta+1)^2 - \lambda^2]m(n) = \varepsilon Q^2(n), \tag{5}$$

where $\lambda$ and $\varepsilon$ are universal constants, at least for leptons and quarks separately. *About the electric charge* $Q(n)$ we assumed that *it does not rise during the displacement* $n \to n+2$. Then

$$[(\Delta+1)^2 - 1]Q(n) = 0. \tag{6}$$

Eqs. (5) and (6) give us a coupled system of two second-order difference equations for the mass $m(n)$ and electric charge $Q(n)$ of leptons or quarks. This system can be integrated under physical boundary conditions imposed on $m(0)$, $m(1)$ and $Q(0)$, $Q(1)$. These boundary conditions, playing here the role of an input, enable us to determine four integration constants which appear during the integration procedure. In order to determine two free parameters $\lambda$ and $\varepsilon$ appearing from the very beginning in Eq. (5) we need in principle two further masses.

Let us notice, however, that Eq. (5) rewritten in the form

$$m(n+2) = \lambda^2 m(n) + \varepsilon Q^2(n) \tag{7}$$

shows that $\lambda$ measures in a way the ability of the elementary particle to submit to the mass excitation $m(n) \to m(n+1)$ generated by the mass itself. It seems rather plausible that this ability should grow if *the number of internal degrees of freedom in Minkowski space* might increase for the considered particle. Here, we will make the simplest (though, a priori, quite arbitrary) assumption that $\lambda$ *is just equal to this number*. Then for *bare* leptons and quarks we get

$$\lambda = 4 \tag{8}$$

since a Dirac particle, being described by a bispinor, has four internal degrees of freedom in Minkowski space. Some deviation of $\lambda$ from the value 4, if it appeared, might be perhaps related to *a renormalization (or structural) effect*. If so, this deviation should be larger for quarks, due to their strong interactions (or their compositness), than for leptons which display only electroweak interactions. Our argument about the value of $\lambda$, if true and general enough, implies that for other possible elementary particles different from leptons and quarks the constant $\lambda$ in their Eq. (5) should be in general different. For instance, one should have $\lambda = 1$ for bare Higgs particles and $\lambda = 3$ for bare electroweak intermediate bosons and bare gluons. One may wonder if it is possible to construct a reliable theory of electroweak interactions based on a sequence of intermediate bosons (and perhaps also Higgs particles) with growing masses.

At this point we ought to make perhaps the following methodological remark about our standpoint assumed in this paper. Since at the present early stage of the particle theory we know nothing certain about the mechanism of mass generation, we are not bound in our conjectures (5) and (8) concerning this mechanism by any "theoretical prejudices", but only by the experimental check. In this situation, however, it is not immediately clear whether our conjectures have only a purely phenomenological character or rather provide a deeper theoretical description of this mechanism (although, to find even a purely phenomenological description for the lepton and quark spectra would not be so bad!).

### *3. Lepton and quark spectra*

Now, we go over to a discussion of consequences of our conjectures. A simple integration of the coupled system of Eqs. (5) and (6) with the lepton or quark input, $m(0) = 0$ or $m_u$, $m(1) = m_e$ or $m_d$ and $Q(0) = 0$ or $2/3$, $Q(1) = -1$ or $-1/3$, gives us uniquely the following mass and charge spectra of leptons or quarks:

$$\left\{\begin{array}{ll} m_{\nu_N} = 0 , & Q_{\nu_N} = 0 \\ m_{l_N} = \left(m_e + \dfrac{\varepsilon}{\lambda^2-1}\right)\lambda^{2N-2} - \dfrac{\varepsilon}{\lambda^2-1}, & Q_{l_N} = -1 \end{array}\right. \quad (N = 1, 2, 3, \ldots) \tag{9}$$

or

$$\left\{\begin{array}{ll} m_{u_N} = \left(m_u + \frac{4}{9}\dfrac{\varepsilon}{\lambda^2-1}\right)\lambda^{2N-2} - \frac{4}{9}\dfrac{\varepsilon}{\lambda^2-1}, & Q_{u_N} = \frac{2}{3} \\ m_{d_N} = \left(m_d + \frac{1}{9}\dfrac{\varepsilon}{\lambda^2-1}\right)\lambda^{2N-2} - \frac{4}{9}\dfrac{\varepsilon}{\lambda^2-1}, & Q_{d_N} = -\frac{1}{3} \end{array}\right. \quad (N = 1, 2, 3, \ldots), \tag{10}$$

where the free parameter $\varepsilon$ may be different for leptons and quarks. Here, $m(2N-2) = m_{\nu_N}$, or $m_{u_N}$, $m(2N-1) = m_{l_N}$ or $m_{d_N}$ and similarly for $Q(n)$ (cf. Eqs. (1) and (2)).

We can see from the spectral formulae (9) and (10) that this theory predicts *infinite sequences of lepton and quark pairs*, $(\nu_N, l_N)$ and $(u_N, d_N)$. For leptons, we get all $m_{\nu_N} = 0$ whilst $m_{l_N}$ rise exponentially with $N$ like $\exp(2N \ln \lambda) = \exp(2.77N)$ if $\lambda = 4$. For quarks, both $m_{u_N}$ and $m_{d_N}$ increase exponentially with $N$ like $\exp(2N \ln \lambda)$.

Notice that the spectral formulae (9) and (10) imply (irrespectively of the value of $\lambda$) the following mass relations:

$$\frac{m_{l_{N+2}^-} - m_{l_{N+1}^-}}{m_{l_{N+1}^-} - m_{l_N^-}} = \lambda^2, \qquad m_{l_{N+1}^-} - m_{l_N^-}\lambda^2 = \varepsilon \tag{11}$$

and

$$\frac{m_{u_{N+2}} - m_{u_{N+1}}}{m_{u_{N+1}} - m_{u_N}} = \frac{m_{d_{N+2}} - m_{d_{N+1}}}{m_{d_{N+1}} - m_{d_N}} = \lambda^2,$$

$$m_{u_{N+1}} - m_{u_N}\lambda^2 = 4(m_{d_{N+1}} - m_{d_N}\lambda^2) = \tfrac{4}{9}\,\varepsilon. \tag{12}$$

In the case of leptons, the free parameter $\varepsilon$ can be determined through the mass $m_{l_2} = m_\mu$ (and $m_{l_1} = m_e$). Then, taking $\lambda = 4$ we obtain from Eq. (11)

$$\varepsilon = 97.48341 \pm 0.00024 \text{ MeV}. \tag{13}$$

The predicted masses for the last known lepton pair $(\nu_3, l_3) = (\nu_\tau, \tau^-)$ and the next to-be--discovered lepton pairs $(\nu_4, l_4)$ and $(\nu_5, l_5)$ are

$$m_{\nu_\tau} = 0, \qquad m_\tau = 1788.035 \pm 0.004 \text{ MeV} \tag{14}$$

and

$$m_{\nu_4} = 0, \qquad m_{l_4^-} = 28.70604 \pm 0.00007 \text{ GeV},$$

$$m_{\nu_5} = 0, \qquad m_{l_5^-} = 459.3941 \pm 0.0012 \text{ GeV}. \tag{15}$$

The prediction for $m_\tau$ is to be compared with the DELCO [3] and DASP [4] experimental values

$$m_\tau = 1782^{+2}_{-7} \text{ MeV} \qquad \text{and} \qquad m_\tau = 1807 \pm 20 \text{ MeV},$$

respectively [5]. Notice that one gets for $m_\tau$ exactly the DELCO value if $\lambda = 3.99^{+0.004}_{-0.009}$. Then $\varepsilon = 97.51^{+0.01}_{-0.04}$ MeV and $m_{l_4} = 28.5^{+0.2}_{-0.5}$ GeV [1]. So the possible deviation of $\lambda$ from the value 4 is certainly small for leptons. This might be consistent with its interpretation as a renormalization effect originating for leptons in their electroweak interactions.

In the case of quarks, if $m_c \gg m_u$ and $m_b \gg m_d$, we obtain from Eq. (12)

$$\frac{m_t}{m_c} \simeq \frac{m_b}{m_s - m_d} \simeq 1 + \lambda^2, \qquad m_c \simeq 4(m_s - m_d\lambda^2) = \tfrac{4}{9}\,\varepsilon. \tag{16}$$

Putting here $m_c \simeq 1.5$ GeV, we get

$$\varepsilon \simeq 3.4 \text{ GeV} \tag{17}$$

and

$$m_s \gtrsim 0.38 \text{ GeV}. \tag{18}$$

Taking in addition $\lambda = 4$ we obtain

$$m_b \gtrsim 6.4 \text{ GeV}, \quad m_t \simeq 26 \text{ GeV}. \tag{19}$$

The high value of $m_b$, connected here with large $\lambda = 4$, would suggest a considerable negative binding energy for heavy quarks in their quarkonia, since the mass of $\Upsilon = b\bar{b}$ is 9.4 GeV. With a smaller $\lambda$ we can get, however, lower values for $m_b$ and $m_t$. For instance, taking in Eq. (16) $\lambda = 3.5$ we obtain

$$m_b \gtrsim 5 \text{ GeV}, \quad m_t \simeq 20 \text{ GeV}. \tag{20}$$

This possible diminishing of $\lambda$, large in comparison with the rather well fitted $\lambda = 4$ for leptons, might be perhaps interpreted as a renormalization effect originating for quarks in their strong interactions (or as a quark structural effect if quarks could be considered as composite states). The big difference in the mass scale $\varepsilon$ between leptons and quarks (cf. Eqs. (13) and (17)) obviously must also be related to the quark strong interactions (or to the quark structure). It may be interesting to remark that in the case of the good-working value $\lambda = 3.5$ one can write $\lambda = (4+3)/2$, whilst $\lambda = 4$ and $\lambda = 3$ are just those values of $\lambda$ which one should expect for a Dirac and a bare vector particle, respectively. We do not know yet if this numerology suggests that, in some approximation, constituent quarks are composite states of one current quark and one current gluon each [6]. At any rate, this way of explaining the smaller value of $\lambda$ for quarks is certainly appealing.

Notice that if $m_u(\lambda^2-1) \ll (4/9)\varepsilon$ and $m_d(\lambda^2-1) \ll (1/9)\varepsilon$ the spectral formulae (10) imply (irrespectively of the value of $\lambda$) the following approximate mass relation:

$$m_{u_N} \simeq 4m_{d_N} \quad (N > 1). \tag{21}$$

So, in particular for $N = 2$ and $N = 3$

$$m_c \simeq 4m_s, \quad m_t \simeq 4m_b. \tag{22}$$

These relations, obtained on a different ground already in Ref. [7], are consistent with our estimation of quark masses both for $\lambda = 4$ and 3.5. The second relation (22) suggests that the mass of toponium $t\bar{t}$ should be about 36–40 MeV as the mass of bottomonium $\Upsilon = b\bar{b}$ is 9.4 MeV.

## 4. *Termination of spectra*

In conclusion of this paper we should like to mention *the possibility of termination of the lepton and quark pair sequences* $(\nu_N, l_N)$ and $(u_N, d_N)$ at some $N_{max}$, over which the particle's Compton wave length becomes smaller than its Schwarzschild gravitational radius. Then, leptons and quarks start to play the role of *elementary black holes with spin 1/2*. The condition for $N_{max}$ is

$$\frac{\hbar}{m_{max}c} \gtrsim \frac{2Gm_{max}}{c^2} \tag{23}$$

or

$$m_{\max} \lesssim \sqrt{\frac{\hbar c}{2G}} \equiv \frac{m_{\text{Planck}}}{\sqrt{2}} = 8.6345 \times 10^{18}\ \text{GeV}/c^2, \tag{24}$$

where $m_{\max} = m_{l_{N\max}^-}$ for leptons and $m_{\max} = m_{u_{N\max}}$ for quarks are given by Eqs (9) and (10), respectively. Using the values (8) and (13) of $\lambda$ and $\varepsilon$ we obtain for leptons

$$N_{\max} = 18, \tag{25}$$

i.e. 18 pairs $(\nu_N, l_N)$ with $N \leqslant N_{\max}$. Of course, it is tempting to expect also for quarks 18 pairs $(u_N, d_N)$ with $N \leqslant N_{\max}$. In fact, using the value (17) of $\varepsilon$ and taking $\lambda = 3.5, 3.55, 3.6$ and 4 one gets for quarks $N_{\max} = 19$, 19, 18 and 17, respectively. For $\lambda = 3.6$ Eq. (16) gives $m_b \gtrsim 5.2$ GeV and $m_t \simeq 21$ GeV. We can see in this way that the possibility of a common gravitational upper bound $N_{\max}$ for the lepton and quark generations is reasonably consistent with the determined values of $\lambda$ and $\varepsilon$ in our spectral formulae (9) and (10).

It is interesting to notice that in the Weinberg–Salam $SU(2)\times U(1)$ standard model with $N_{\max}$ lepton pairs and also $N_{\max}$ quark pairs one gets in the leading-log approximation the following formula for the Weinberg angle [8]:

$$\tan^2 \theta_W = \tfrac{3}{5}\left(1 - \frac{11}{2N_{\max}}\right). \tag{26}$$

It gives $\sin^2 \theta_W = 0.19, 0.25, 0.30$ for $N_{\max} = 9, 12, 18$, respectively. We can see from Eq. (26) that in this approximation certainly $N_{\max} \geqslant 6$ since otherwise $\tan^2 \theta_W < 0$. So, the number of quark flavours is $2N_{\max} \geqslant 12$. The number of lepton flavours is in this case also $2N_{\max} \geqslant 12$.

It seems now rather obvious that for very heavy leptons and quarks, *if they exist*, the simple perturbative approach to their electroweak interactions must fail to work [9]. In fact, this is true already for $m_{l_N^-} \gtrsim 550$ GeV [10], i.e. in our theory for $N \geqslant 6$ (cf. Eq. (15)). Then, the very heavy elementary fermions may behave in their electroweak interactions as the so-called *sthenons* [11] whose electroweak interactions (with other sthenons) become effectively strong.

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