LETTER TO THE EDITOR

HIDDEN VARIABLES AND WAVES IN RELATIVISTIC THERMO-VISCOUS FLUIDDYNAMICS

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(*Received September 21, 1979*)

Within the framework of general relativistic thermodynamics, the note delivers an account of viscosity and heat conduction in fluids, compatible with wave propagation, via a hidden variable approach. Detailed results are exhibited.

In the last decade the attempt to achieve more realistic astrophysical models has determined an increasing interest in topics concerning viscosity in General Relativity. In this connection we cite, for example, Matzner and Misner's account of neutrino viscosity in anisotropic homogeneous cosmologies [1], Weinberg's work on the role of viscosity in the survival of protogalaxies [2], and the researches on viscosity effects in Friedman cosmology carried out by Belinskii and Khalatnikov [3] and by Grabińska et al. [4]. Yet, it is an unpleasant feature of Navier-Stokes' law of viscosity that mechanical disturbances would propagate at infinite speed. The same is true for temperature disturbances in Fourier's theory. Although solutions to the latter paradox are known, we still need a proper scheme of both phenomena, compatible with thermodynamics and accounting for finite wave speed. This note investigates such a topic through a relativistic hidden variable approach which mirrors some points of the classical procedure [5–8].

Consider a heat-conducting viscous fluid described by the usual energy-momentum tensor T of the form

$$T_{\alpha\beta} = r(1+e)u_\alpha u_\beta - S_{\alpha\beta} + 2q_{(\alpha}u_{\beta)}, \tag{1}$$

where S is the stress, $S_{\alpha\beta}u^\beta = 0$, q the heat flux, $q_\alpha u^\alpha = 0$, and $u_\alpha u^\alpha = -1$. The balance equations are summarized by

$$(ru^\alpha)_{;\alpha} = 0, \qquad T^{\alpha\beta}{}_{;\alpha} = 0. \tag{2}$$

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Introduce now the absolute temperature ϑ, the specific entropy s, and $\lambda_\alpha = h_\alpha{}^\beta(\vartheta_{;\beta}+\vartheta\dot{u}_\beta)$ where $h_{\alpha\beta} = g_{\alpha\beta}+u_\alpha u_\beta$ and $\dot{u}_\beta = u_{\beta;\alpha}u^\alpha$. Then, on accounting for the second law of thermodynamics through the Clausius–Duhem inequality, we get

$$-r(\dot{\psi}+s\dot{\vartheta})+S^{\alpha\beta}\sigma_{\alpha\beta}+\tfrac{1}{3}\,S^\alpha{}_\alpha\theta-\frac{1}{\vartheta}\,q^\alpha\lambda_\alpha \geqslant 0, \tag{3}$$

where $\psi = e-\vartheta s$, $\theta = u^\alpha{}_{;\alpha}$ and $\sigma_{\alpha\beta} = h_\alpha{}^\mu h_\beta{}^\nu u_{(\mu;\nu)}-\tfrac{1}{3}\,\theta h_{\alpha\beta}$.

Strictly speaking, a material with hidden variables consists of a suitable set of response functions — here ψ, s, S, q — and of differential equations governing the evolution of the hidden variables via the real (physical) variables. Besides, as shown in [5, 6] — see also [8] — compatibility with wave propagation in fluids requires that the response functions can depend only on r, ϑ, and on the hidden variables; such variables are here represented by a spatial symmetric traceless tensor Σ, a scalar Θ, and a spatial vector Λ. To avoid an unduly cumbersome theory and to get a model with an immediate physical interpretation, the relaxation effects associated with viscosity and heat conduction are described through the differential equations

$$h_\alpha{}^\mu h_\beta{}^\nu(\dot{\Sigma}_{\mu\nu}+\tfrac{1}{3}\,\dot{\Theta}h_{\mu\nu}) = \frac{1}{\tau_v}\,\{h_\alpha{}^\mu h_\beta{}^\nu u_{(\mu;\nu)}-(\Sigma_{\alpha\beta}+\tfrac{1}{3}\,\Theta h_{\alpha\beta})\}, \tag{4}$$

$$h_\alpha{}^\beta\dot{\Lambda}_\beta = \frac{1}{\tau_c}\,\{\lambda_\alpha-\Lambda_\alpha\}. \tag{5}$$

So, the parameters τ_v, τ_c represent relaxation times for viscosity and heat conduction. From a formal viewpoint equations (4), (5) are closely related to Matzner and Misner's ansatz concerning neutrino viscosity in a Bianchi type I spacetime [1]; there the evolution equation governs the growth of the distribution function by means of the shear σ and of a relaxation time τ calculable from microphysics. We remark that sometimes, on the basis of arguments concerning objectivity, the dot derivative is replaced by other derivatives [9, 10]; however, as the physical behaviour is qualitatively unaffected by the particular choice of the derivative, we use the dot derivative which allows a simpler treatment. Within the outlined framework, the inequality (3) becomes

$$-r(\psi_\vartheta+s)\dot{\vartheta}+\left(S^{\alpha\beta}-\frac{1}{\tau_v}\,\psi_{\Sigma\alpha\beta}\right)\sigma_{\alpha\beta}+\left(r^2\psi_r+\tfrac{1}{3}\,S^\alpha{}_\alpha-\frac{r}{\tau_v}\,\psi_\Theta\right)\theta$$
$$-\left(\frac{1}{\vartheta}\,q^\alpha+\frac{r}{\tau_c}\,\psi_{\Lambda_\alpha}\right)\lambda_\alpha+r\left(\frac{1}{\tau_v}\,\psi_{\Sigma\alpha\beta}\Sigma_{\alpha\beta}+\frac{1}{\tau_v}\,\psi_\Theta\Theta+\frac{1}{\tau_c}\,\psi_{\Lambda\alpha}\Lambda_\alpha\right)\geqslant 0. \tag{6}$$

As it happens in the classical case [5, 8], the hidden variables $\Sigma(t)$, $\Theta(t)$, $\Lambda(t)$ at the proper time t are independent of the present values $\sigma(t)$. $\theta(t)$, $\lambda(t)$ because, in view of (4), (5) Σ, Θ, Λ turn out to be expressed by suitable integrals of the past histories of σ, θ, λ, respectively. Accordingly, the assumed validity of (6) for every thermodynamic process leads to a reduced dissipation inequality and to relations yielding s, S, q in terms of the free energy ψ. In the particular case

$$\psi = \Psi(\vartheta, r) + \frac{1}{r}\left(\eta\tau_v \Sigma_{\alpha\beta}\Sigma^{\alpha\beta} + \tfrac{1}{2}\zeta\tau_v\Theta^2 + \frac{\kappa\tau_c}{2}\Lambda_\alpha\Lambda^\alpha\right), \tag{7}$$

η, ζ, κ being suitable parameters, the reduced dissipation inequality provides $\eta \geqslant 0$, $\zeta \geqslant 0$ $\kappa \geqslant 0$ while s, S, q are given by

$$s = -\Psi_\vartheta + \frac{\kappa\tau_c}{2r}\Lambda_\alpha\Lambda^\alpha, \qquad q_\alpha = -\kappa\Lambda_\alpha, \tag{8}$$

$$S_{\alpha\beta} = -ph_{\alpha\beta} + 2\eta\Sigma_{\alpha\beta} + \zeta\Theta h_{\alpha\beta}, \tag{9}$$

where $p = r^2\psi_r$. Of course, Σ, Θ, Λ in (8), (9) stand for the solutions of (4), (5) so that the hidden variables are entirely eliminated at the outcome and (8), (9) involve only real variables.

Two points stress the value of the present approach. First, the constitutive equations (8), (9) reduce to the customary relativistic laws of viscosity and heat conduction when stationary processes are involved; in fact $\dot{\Sigma} = 0$, $\dot{\Theta} = 0$, $\dot{\Lambda} = 0$ imply that $\Sigma = \sigma$, $\Theta = \theta$ $\Lambda = \lambda$. So, η, ζ, κ may be viewed as the coefficients of shear viscosity, bulk viscosity and heat conduction, respectively. Second, equations (8), (9) are consistent with wave propagation. This claim can be ascertained through direct calculations. Letting $\phi(x^\alpha) = 0$ be a discontinuity surface, the spatial normal speed of propagation U, relative to the comoving frame of the fluid, is defined as $U = -u^\alpha N_\alpha$, being $N_\alpha = \phi_{,\alpha}(\phi_{,\beta}\phi^{,\beta})^{-\frac{1}{2}}$. Denoting by[ ] the jump across the surface, for any tensor ξ such that $[\xi] = 0$ we have the compatibility relations

$$[\xi_{;\alpha}] = [\nabla_N\xi]N_\alpha, \qquad [\dot{\xi}] = -U[\nabla_N\xi], \tag{10}$$

where $\nabla_N\xi = N^\alpha\xi_{;\alpha}$. Setting $n^\alpha = h^\alpha{}_\beta N^\beta$ we find that

$$[\nabla_n\xi] = (1-U^2)[\nabla_N\xi]. \tag{11}$$

In the case of acceleration waves — jump discontinuities of $\nabla_N r$, $\nabla_N\vartheta$, $\nabla_N u$ — propagating into a constant state, application of the standard procedure to the balance equations (2) yields

$$-U[\nabla_N r] + r[\nabla_N u]_n = 0, \tag{12}$$

$$\left(s_r + \frac{\kappa}{r^2\tau_c}\right)U[\nabla_N r] + \left(s_\vartheta U - \frac{\kappa(1-U^2)}{r\vartheta U\tau_c}\right)[\nabla_N\vartheta] = 0, \tag{13}$$

$$\left(p_r + \frac{\eta+3\zeta}{3r\tau_v}\right)[\nabla_N r]n^\alpha + \left(p_\vartheta - \frac{\kappa}{\tau_c}\right)[\nabla_N\vartheta]n^\alpha + \left(\frac{\eta(1-U^2)}{U\tau_v} - (r+re)U + \frac{\kappa\vartheta U}{\tau_c}\right)[\nabla_N u^\alpha] = 0, \tag{14}$$

where $[\nabla_N u]_n = [\nabla_N u^\alpha] n_\alpha$. Two types of waves can occur. (*i*) Transverse waves, namely $[\nabla_N u]_n = 0$. In this case the system (12)–(14) provides

$$[\nabla_N r] = 0, \quad [\nabla_N \vartheta] = 0, \quad U = \left\{1 + \frac{\tau_v}{\eta}\left(r + re - \frac{\kappa\vartheta}{\tau_c}\right)\right\}^{-1/2}. \tag{15}$$

Obviously $\tau_c > \kappa\vartheta/(r+re)$ guarantees that $U^2 < 1$. (*ii*) Longitudinal waves, namely $[\nabla_N u^\alpha] = [\nabla_N u]_n n^\alpha$. The corresponding determinantal equation follows straightforwardly from (12)–(14). It turns out that, within the approximation $\kappa \cong 0$, the condition $\tau_v > (\eta + 3\zeta)/\{3r(1+e-p_r+s_r p_\vartheta/s_\vartheta)\}$ ensures that $U^2 < 1$.

Further applications of the present approach will be given later.

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