COMMENT ON DISCONTINUOUS SOLUTIONS TO RELATIVISTIC EQUATIONS FOR TWO SPIN-1/2 PARTICLES*

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The aim of this comment is twofold. Firstly, it is pointed out that the bag-like solution conjectured recently by H. Suura in the case of the relativistic Breit equation (in its primary form consistent with the single-particle theory) with an infinitely rising central potential is in fact a solution to a related inhomogeneous equation. One may speculate on a possible physical meaning of the source term there. Secondly, discontinuous solutions to the original homogeneous equation are constructed. In particular, they can be confined to the inner region $r \leqslant r_0$ or restricted to the outer region $r \geqslant r_0$, where $V(r_0) = E$, but both kinds of solutions can be mixed by an additional interaction. The new solutions are normalized to the Dirac δ-function and so belong to the continuous-energy spectrum, in contrast with the Suura solution which is normalizable in the usual sense. One may wonder if the new solutions have only a formal meaning.

1. *Inhomogeneous Breit equation*

In a recent paper H. Suura [1] (cf. also Ref. [2]) has found a normalized bag-like solution to the *inhomogeneous* Breit equation

$$[E-(\vec{\alpha}^{(1)}\cdot\vec{p}+\beta^{(1)}m^{(1)})-(-\vec{\alpha}^{(2)}\cdot\vec{p}+\beta^{(2)}m^{(2)})-V(\vec{r})]\psi(\vec{r}) = S\delta(r-r_0), \tag{1}$$

with a central static potential $V(\vec{r}) = V(r)$, where $V(r) \to \infty$ at $r \to \infty$, $V(0) = 0$ and $V(r_0) = E$ (for an $r_0 > 0$) as is the case e.g. for $V(r) = \mu^2 r$. The operator S appearing in the source term in Eq. (1) is a constant matrix built from Dirac matrices of both particles. The source term, which introduces an important physical and mathematical difference between Eq. (1) and the familiar homogeneous Breit equation [3, 4] with a potential $V(\vec{r})$ [5], was in fact not considered in Refs. [1] and [2], where the homogeneous Breit equation (in its primary form consistent with the single-particle theory) was discussed. To this equation a solution was conjectured (in the case of $m^{(1)} = m^{(2)}$) which was normali-

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zed and equal to zero outside the sphere $r = r_0$, and contained a component that vanished as $(r-r_0)^2$ when $r \to r_0$ (from inside) and another component which *jumped* as $\theta(r_0-r)$. The latter, when differentiated in Eq. (1), gives rise to a *source* term proportional to $\delta(r-r_0)$ in the corresponding component of Eq. (1). Thus the Suura solution is a solution to the inhomogeneous rather than to the homogeneous Breit equation, the latter being a relativistic wave equation for a closed system of two spin-1/2 particles.

In Refs. [1] and [2] an argument was given that the invented solution represents properly the physical situation in the quarkonium $q\bar{q}$. In this argument, a crucial role was played by the assumption that at $r \to \infty$ the proper solution should contain only outgoing waves (if any), since incoming waves would imply the existence of sources or reflecting walls at infinity. This assumption, excluding standing waves, led to a zero solution for $r > r_0$, because non-zero outgoing waves could not be consistent with the wave function which contained a component vanishing as $(r-r_0)^2$ at $r \to r_0$, the last property being necessary for the normalizability of the solution. The crucial assumption of the absence of incoming waves was supported by the observation that the "confining" potential $V(r)$ which rises to infinity at $r \to \infty$, when inserted into the relativistic Breit equation, gives effectively an infinitely growing repulsion rather than a reflecting wall (though this property might be unphysical, being connected with the Klein paradox).

While this crucial assumption does seem to be intuitively justified, it leads to an inhomogeneous Breit equation (1) which implies that the quarkonium $q\bar{q}$ *is not* a closed system. On the other hand, the argument that the "confining" potential $V(r)$ produces an infinitely growing repulsion at $r \to \infty$ is true *only* for the primary, one-particle version of the Breit equation (which suffers from the Klein paradox). In the modified, hole-theory version of the Breit equation (called usually the Salpeter equation) [6, 4], where the potential $V(r)$ is multiplied from the left by the projector

$$P(\vec{p}) = \Lambda_+^{(1)}(\vec{p})\Lambda_+^{(2)}(-\vec{p})-\Lambda_-^{(1)}(\vec{p})\Lambda_-^{(2)}(-\vec{p})$$

$$= \frac{1}{2}\left(\frac{\vec{\alpha}^{(1)}\cdot\vec{p}+\beta^{(1)}m^{(1)}}{\sqrt{\vec{p}^2+m^{(1)2}}}+\frac{-\vec{\alpha}^{(2)}\cdot\vec{p}+\beta^{(2)}m^{(2)}}{\sqrt{\vec{p}^2+m^{(2)2}}}\right), \tag{2}$$

the situation is changed, since a part of the interaction $P(\vec{p})V(r)$ produces terms which are additive to the masses $m^{(1)}$ and $m^{(2)}$ rather than to the energy E. In the case of the "confining" potential $V(r)$, such terms lead effectively to an infinitely growing attraction at $r \to \infty$ i.e., to a reflecting wall at infinity. Approximately, one can get such terms if one expands the projector (2) into powers of velocity $\vec{v} = \vec{p}/m$,

$$P(\vec{p}) = \frac{1}{2}(\beta^{(1)}+\beta^{(2)})\left(1-\frac{1}{2}\frac{\vec{p}^2}{m^2}\right)+\frac{1}{2}(\vec{\alpha}^{(1)}-\vec{\alpha}^{(2)})\frac{\vec{p}}{m}+O(\vec{v}^4) = \frac{1}{2}(\beta^{(1)}+\beta^{(2)})+O(\vec{v}^2), \tag{3}$$

where $m^{(1)} = m^{(2)} = m$.

In this situation, we would like only to mention that the Suura bag-like solution, which is implied necessarily by the inhomogeneous Breit equation (1) (with a proper source term) might be justified by a justification of this equation (or possibly its modified form consistent with the hole theory). We have in mind an attractive possibility that the existence of a *source shell* at $r = r_0$, as appearing in Eq. (1), might be physically reasonable. If it was true, such a source shell would represent a surface accumulation of quark-antiquark pairs which could be treated as external surroundings of the quark-antiquark pair under consideration. At this point let us notice that the *inhomogeneous* wave equation for a given particle configuration (e.g. for a $q\bar{q}$ pair as in our case) follows in general from the field theory [4] if one eliminates all other particle configurations (e.g. those containing additional $q\bar{q}$ pairs). Then the source term is provided by those particle configurations which can transit into the distinguished one (or vice versa) and are present in the Fock space at the moment.

2. *Homogeneous Breit equation*

In the remainder of this comment we would like to answer another question which is opposite in a way to the problem aroused by the Suura solution. The question is whether the familiar homogeneous Breit equation given by formula (1) with

$$S\delta(r-r_0) \equiv 0 \tag{4}$$

(and without the projector (2)) has some solutions with *discontinuities* at $r = r_0$, as it is the case for the inhomogeneous Breit equation (1) having the Suura solution which is discontinuous at $r = r_0$. The answer is in the affirmative, but it turns out that such solutions can be normalized *only* to the Dirac δ-function, in contrast with the Suura solution which is normalizable in the usual sense.

To this end let us write the system of radial equations following from Eqs. (1) and (4), where $V(\vec{r}) = V(r)$ [5]. In the case of $j = 0$ this system reduces to two separate subsystems of 4 equations which read

$$\frac{d}{dr}f_2^{\pm} \pm \frac{m^{(1)}\mp m^{(2)}}{2} f_4^{\mp} + \frac{E-V}{2} f_3^{\mp} = 0, \qquad \pm\frac{m^{(1)}\mp m^{(2)}}{2} f_3^{\mp} + \frac{E-V}{2} f_4^{\mp} = 0,$$

$$\pm\frac{m^{(1)}\pm m^{(2)}}{2} f_2^{\pm} + \frac{E-V}{2} f_1^{\pm} = 0,$$

$$-\left(\frac{d}{dr}+\frac{2}{r}\right)f_3^{\mp} \pm \frac{m^{(1)}\pm m^{(2)}}{2} f_1^{\pm} + \frac{E-V}{2} f_2^{\pm} = 0. \tag{5}$$

Here, the components f_1 and f_2 correspond to $l = 0$ and $s = 0$, while f_3 and f_4 to $l = 1$ and $s = 1$. The superscripts "+" and "−" refer to the intrinsic parities $\pi = +\eta$ and $-\eta$ described by the matrix $\eta\beta^{(1)}\beta^{(2)}$, where $\eta^2 = 1$. The total parity is $P = +\eta$ and $-\eta$ for upper and lower superscripts in Eq. (5), respectively.

The general solution of two algebraic equations in the system (5) (in terms of $f_3^{\mp}$ and $f_2^{\pm}$) is

$$f_4^{\mp} = \mp\frac{m^{(1)}\mp m^{(2)}}{E-V} f_3^{\mp} - a^{\pm}\delta(E-V), \qquad f_1^{\pm} = \mp\frac{m^{(1)}\pm m^{(2)}}{E-V} f_2^{\pm} - b^{\mp}\delta(E-V), \tag{6}$$

where the δ-terms are relevant if there is an $r_0 > 0$ such that $V(r_0) = E$. If it is the case and if $a^{\pm} \neq 0$ and/or $b^{\mp} \neq 0$, then it is evident that the corresponding wave function $\psi(\vec{r})$ may be normalized only to the Dirac δ-function[1]. It can be done for the continuous energy spectrum (if it exists). For the discrete energy spectrum (if it exists) the wave function $\psi(\vec{r})$ following from the solution (6) (with $a^{\pm} \neq 0$ and/or $b^{\mp} \neq 0$) cannot be normalized, being, therefore, excluded by the probability interpretation of the wave function.

Inserting now the solutions (6) into two differential equations in the system (5) one gets the following inhomogeneous equations:

$$\frac{d}{dr} f_2^{\pm} + \tfrac{1}{2}\left[E - V - \frac{(m^{(1)} \mp m^{(2)})^2}{E-V}\right] f_3^{\mp} = \pm \frac{m^{(1)} \mp m^{(2)}}{2}\, a^{\pm} \delta(E-V),$$

$$-\left(\frac{d}{dr} + \frac{2}{r}\right) f_3^{\mp} + \tfrac{1}{2}\left[E - V - \frac{(m^{(1)} \pm m^{(2)})^2}{E-V}\right] f_2^{\pm} = \pm \frac{m^{(1)} \pm m^{(2)}}{2}\, b^{\mp} \delta(E-V), \quad (7)$$

where the source terms are proportional to

$$\delta(E-V) = \frac{1}{\left|\dfrac{dV}{dr}(r_0)\right|}\,\delta(r-r_0) \quad (8)$$

since $V(r_0) = E$. We would like to stress that the inhomogeneous equations (7) have been derived from the homogeneous equations (5) i.e., from the homogeneous Breit equation. In the case of the Suura solution (to the inhomogeneous equation (1)), the inhomogeneous equations of the form (7) are also valid, but the δ-terms in Eq. (6) are zero. The last property enables the Suura solution to be normalized in the usual sense.

In the case of equal masses, $m^{(1)} = m^{(2)}$ $(= m)$, Eq. (7) with upper superscripts shows that for $r \to r_0$

$$f_2^{+} \simeq [\theta(r_0 - r) f_2^{+\prime\prime}(r_0 - 0) + \theta(r - r_0) f_2^{+\prime\prime}(r_0 + 0)] \frac{(r-r_0)^2}{2},$$

$$f_3^{-} \simeq \theta(r_0 - r) f_3^{-}(r_0 - 0) + \theta(r - r_0) f_3^{-}(r_0 + 0). \quad (9)$$

Otherwise the wave function $\psi(\vec{r})$ cannot be normalized unless to the Dirac δ-function even if $a^{+} = 0$ and $b^{-} = 0$[2]. Here

$$f_3^{-}(r_0 + 0) - f_3^{-}(r_0 - 0) = -m b^{-} \Big/ \left|\frac{dV}{dr}(r_0)\right|. \quad (10)$$

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[1] Since the relativistic barrier at $r = r_0$ is moving with energy, $r_0 = r_0(E)$, two discontinuous solutions to Eq. (5) corresponding to different E *are not* orthogonal (in general), even if they satisfy the same boundary conditions at $r = r_0$. Nevertheless, their scalar product may be normalized to the Dirac δ-function plus finite non-zero terms spoiling the orthogonality, and this is the only possibility if $a^{\pm} \neq 0$ and/or $b^{\mp} \neq 0$.

[2] In fact, even if $a^{+} = 0$ and $b^{-} = 0$, the component f_1^{+} of the second independent solution (for which $f_2^{+} \simeq \text{const} \neq 0$ at $r \to r_0$) cannot be normalized unless to the Dirac δ-function (cf. Eq. (6)). This singular solution, however, must collaborate with the regular solution (9) in order to satisfy the regularity condition at $r = 0$. So the wave function $\psi(\vec{r})$ is discontinuous at $r = r_0$ even if $a^{+} = 0$ and $b^{-} = 0$, unless its singular part can vanish giving us the discrete spectrum.

We can see from (9) and (10) that f_2^+ and its first derivative are then continuous and equal to zero at $r = r_0$, while f_3^- jumps at $r = r_0$ if $b^- \neq 0$.

If $a^+ = 0$ and $b^- = 0$, one gets the "conventional" solutions to Eq. (5) (with upper superscripts). These "conventional" solutions may a priori belong to continuous or discrete energy spectrum and be normalizable in both cases (to the Dirac δ-function in the first case and in the usual sense in the second).

Thus, we can conclude that *all* discontinuous solutions to Eq. (5) (with upper superscripts) can be normalized and may have a physical interpretation *only* in the case when they are normalizable to the Dirac δ-function and so belong to the *continuous* energy spectrum. In the other case they are excluded by the probability interpretation of the wave function. In contrast, the Suura solution to the inhomogeneous Breit equation, though also discontinuous, is normalizable in the usual sense.

In particular, if $f_3^-(r_0+0) = 0$ but $f_3^-(r_0-0) \neq 0$, all components $f_1^+, f_2^+, f_3^-, f_4^-$ are zero for $r > r_0$ but non-zero for $r < r_0$ and one gets solutions *confined* to the *inner* region $r \leqslant r_0$. If the situation is opposite i.e., $f_3^-(r_0-0) = 0$ but $f_3^-(r_0+0) \neq 0$, one gets solutions *restricted* to the *outer* region $r \geqslant r_0$. Both solutions are discontinuous at $r = r_0$. The difference of the inner and outer solutions corresponding to the same a^+ and b^- as well as E represents the "conventional" solution. It follows that the formal existence of these "conventional" solutions is a *necessary* and *sufficient* condition for the formal existence of the inner and outer solutions. In fact, we can write

$$f_i^{\text{inner}} = \theta(r_0-r)f_i^{\text{conv}} \quad (i = 2, 3), \quad f_i^{\text{outer}} = -\theta(r-r_0)f_i^{\text{conv}} \quad (i = 2, 3), \tag{11}$$

where f_i^{inner}, f_i^{outer} and f_i^{conv} $(i = 1, 2, 3, 4)$ have the obvious meaning (here all f's correspond to upper signs in Eq. (5)). Notice that $f_4^{\text{conv}} \equiv 0$ (cf. Eq. (6), where $a^+ = 0$ and $b^- = 0$).

Since the inner and outer solutions can be normalized only to the Dirac δ-function, we conclude that they are normalizable *if* and *only if* the corresponding "conventional" solutions can be normalized to the Dirac δ-function and so belong to the continuous energy spectrum. In particular, the inner solutions, if expressed through Eq. (11) by the "conventional" continuous-energy solutions, belong to the continuous energy spectrum, in spite of their vanishing outside the sphere $r = r_0$ (and their regularity at $r = 0$ when $V(0) = 0$). If one discussed the inner solutions exclusively in the inner region $r \leqslant r_0$, one would ignore discontinuities (and the δ-source term in Eq. (7)), in spite of their having an important influence not only at $r = r_0$. Their influence in fact provides the link (11) between f_i^{inner} and f_i^{conv}.

Thus, we turn now to the problem of existence of the "conventional" continuous--energy solutions to Eq. (5) (with upper superscripts). We will consider for example two potentials, the Coulombian repulsive potential $V(r) = \alpha/r$ and the popular "confining" potential $V(r) = \mu^2 r$. In these cases $r_0 = \alpha/E$ and $r_0 = E/\mu^2$, respectively.

In the first case there are no doubts that the scattering solutions exist. Here they are the "conventional" continuous-energy solutions. So, in this case the inner and outer solutions given by Eq. (11) exist (for this conclusion the absence of projector (2) in the

Breit equation is irrelevant). Notice that the "conventional" solutions have here a direct physical meaning and the same may be true for inner and outer solutions.

In the second case the "conventional" solutions satisfy the conditions

$$f_i^{\text{conv}} \sim \frac{1}{r} \exp\left[\mp i\left(\frac{\mu^2}{4} r^2 - \frac{E}{2} r\right)\right] \text{ at } r \to \infty, \quad f_i^{\text{conv}} \sim r^l \text{ at } r \to 0 \tag{12}$$

and for their regular part

$$f_2^{\text{conv}} \sim (r-r_0)^2, \quad f_3^{\text{conv}} \sim (r-r_0)^0, \quad f_4^{\text{conv}} \equiv 0, \quad f_1^{\text{conv}} \sim r-r_0 \text{ at } r \to r_0 \tag{13}$$

(cf. Eqs. (9) and (6), where $a^+ = 0$ and $b^- = 0$). We can see already from Eq. (13) that *not all* components f_i^{conv} are zero at $r = r_0$, so one gets for the wave function $\psi^{\text{conv}}(\vec{r})$ a *leakage* through the relativistic barrier at $r = r_0$. Thus $\psi^{\text{conv}}(\vec{r})$ is non-zero for $r > r_0$ if it is such for $r < r_0$ and vice versa. The non-zero oscillatory behaviour (12) of f_i^{conv} at $r \to \infty$ (caused by the infinitely growing repulsion) implies that $\psi^{\text{conv}}(\vec{r})$ can be normalized only to the Dirac δ-function and so belongs to the continuous energy spectrum. Thus, also in this case there exist the inner and outer solutions given by Eq. (11) (for this conclusion the absence of projector (2) in the Breit equation is relevant). Notice, however, that due to the Klein paradox at $r \to \infty$, implied in this case by the infinitely growing repulsion, would exclude any non-zero $\psi^{\text{conv}}(\vec{r})$ both for $r > r_0$ as for $r < r_0$. So the "conventional" continuous-energy solutions, though existing, might have here no direct physical meaning. The same could be said about the outer solutions, but *not* necessarily about the inner solutions. At this point one should keep in mind that the projector (2), if introduced into the Breit equation with a "confining" potential $V(r)$, seems to lead effectively to the infinitely growing attraction, excluding thereby all continuous-energy solutions.

One may wonder if the continuous-energy inner solutions could not be realized in Nature as some intermediate states produced in highly localized interaction, e.g. in high energy pp collisions. They would have some resemblance to such phenomenological notions as fire-balls or clusters.

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