THE RENORMALIZATION GROUP AND DEEP INELASTIC SCATTERING

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The renormalization group is applied to analyze deep inelastic electron-proton scattering. The connection between renormalization group techniques and parton-model is also discussed.

Introduction

In these lectures I shall be talking about the renormalization group, with specific emphasis on its application to deep inelastic electron–proton scattering. I shall also demonstrate, where possible, the connection between renormalization group techniques and parton-model ideas.

I shall start from the beginning of the subject assuming only a minimum knowledge of quantum field theory and some familiarity with the idea of renormalization. I shall therefore not have time to discuss in detail the latest developments in this subject.

The plan of these notes is as follows: in Section 1, the concept of an effective coupling is discussed and the β function is defined. In Section 2 deep inelastic scattering processes are described, first in the free parton model, and subsequently using renormalization group techniques to describe corrections to the free parton model results. In Section 3 the calculation of the anomalous dimensions of the operators occurring in the Wilson operator product expansion is described using techniques of dimensional regularization. This section is rather technical and may be omitted by the pedestrian reader, without loss of continuity. In Section 4 the solutions of the renormalization group equations are applied to deep inelastic processes and the comparison of the theoretical predictions with experimental data is discussed. Finally in Section 5 some of the corrections to the lowest order renormalization group results are briefly discussed.

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1. The effective coupling

It is well known that if we consider any quantum field theory, e. g. QED, and look at the higher order corrections to the coupling between the photon and fermions, we obtain a result which is not constant but depends on the momenta of the external legs. For the one-loop correction shown in Fig. 1, we obtain

$$e(Q^2) = e+(e^3/48\pi^2)\ln Q^2+\cdots, \tag{1.1}$$

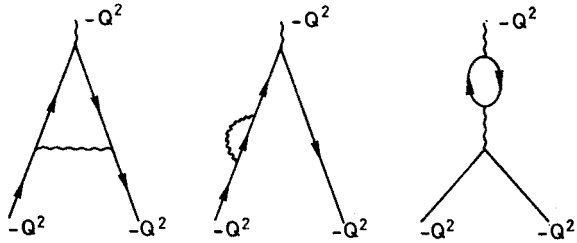

Fig. 1

where we have put the invariant mass squared of each external leg equal to $-Q^2$ (we choose a negative value simply to avoid complications arising from mass thresholds or infrared divergences). For a general field theory with coupling constant g we have a similar equation

$$g(Q^2) = g+\frac{g^3}{16\pi^2}\frac{\beta_0}{2}\ln Q^2+O(g^5). \tag{1.2}$$

We may write this as an implicit equation for $g(Q^2)$

$$g(Q^2) = g+\frac{g^3(Q^2)}{16\pi^2}\frac{\beta_0}{2}\ln Q^2+O(g^5(Q^2)). \tag{1.3}$$

From this equation we may define the β function

$$\beta(g(Q^2)) = 2\frac{\partial}{\partial \ln Q^2}g(Q^2). \tag{1.4}$$

For small values of $g(Q^2)$, β may be expanded in a power series

$$\beta(g) = \frac{\beta_0}{16\pi^2}g^3+\frac{\beta_1}{(16\pi^2)^2}g^5+\cdots. \tag{1.5}$$

The function β may in general have any of the forms shown in Fig. 2, where $\beta(g)$ is plotted against g. In Figs. 2a and 2b, β_0 is positive, whereas in Fig. 2c it is negative. In Fig. 2a β is positive near $g = 0$ so that as Q^2 (i. e. $\ln Q^2$) increases $g(Q^2)$ increases until it reaches the value g_0, for which β is zero so that as we further increase $\ln Q^2$, $g(Q^2)$ remains at g_0. g_0 is called an ultraviolet stable fixed point. In Fig. 2b no such fixed point

exists, so that $g(Q^2)$ increases indefinitely as Q^2 increases. Figure 2c has an ultraviolet stable fixed point at the origin, i. e. β is negative for small values of g so that as we increase Q^2, $g(Q^2)$ decreases becoming closer and closer to a free field theory as Q^2 tends to infinity. Such a theory is called an asymptotically free field theory.

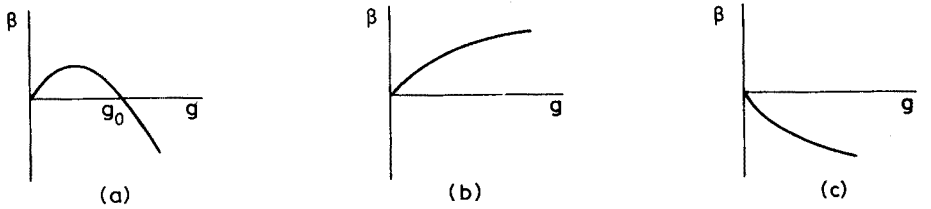

Fig. 2

What field theories are asymptotically free?

Coleman and Politzer have shown that no theory which does not contain a non--Abelian gauge symmetry can be asymptotically free[1].

A strong candidate for the field theory of strong interactions in QCD. This is an SU(3) colour gauge theory with fermions arranged in colour triplets. The Lagrangian for such a theory is

$$L = -\tfrac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} + i\bar{\psi}_\alpha(\gamma^\mu D^\mu + im_\alpha)\psi_\alpha, \tag{1.6}$$

where

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu, \qquad D_\mu = \partial_\mu - ig\tau^a A^a_\mu.$$

The τ^a are the fundamental representations of the generators of colour SU(3) and obey the commutation relations

$$[\tau^a, \tau^b] = i f^{abc} \tau^c, \tag{1.7}$$

where f^{abc} are the structure constants of SU(3). The index α of the fermions runs over the possible flavours (i. e. up, down, strange, charm, etc.) which the fermions (quarks) may have.

The Lagrangian of Eq. (1.6) gives rise to three-point and four-point interactions. between the A^a_μ fields (gluons), which are absent in an Abelian theory for which $f^{abc} = 0$

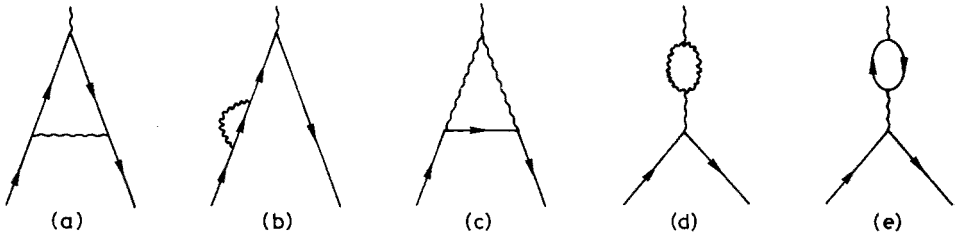

Fig. 3

[1] One exception to this is a $\lambda\phi^4$ theory with negative coupling constant, but such theories are usually rejected on the grounds that the energy is not bounded from below as $\phi \ldots \infty$.

There is also an interaction between the quarks and gluons, which is similar to the QED vertex between electrons and protons but carries a group theory factor τ. The diagrams required to calculate β_0 in such a theory are shown in Fig. 3. In the Feynman gauge we get the following contributions: $-(C_F - C_A/2)$ from Fig. 3a, C_F from Fig. 3b, $-\frac{3}{2}C_A$ from Fig. 3c, $-\frac{5}{3}C_A$ from Fig. 3d, $\frac{4}{3}T_R$ from Fig. 3e, where C_F, C_A, T_R are functions of the group generators defined by

$$\delta_{ij}C_F = \tau^a_{ik}\tau^a_{kj}, \tag{1.8a}$$

$$\delta_{ab}C_A = f^{acd}f^{bcd}, \tag{1.8b}$$

$$\delta_{ab}T_R = \tau^a_{ik}\tau^b_{kj} \times \text{no. of flavours.} \tag{1.8c}$$

For SU(3) with quark triplets, $C_F = \frac{4}{3}$, $C_A = 3$, $T_R = \frac{1}{2} \times$ no. of flavours. Summing these contributions we get

$$\beta_0 = \left(\frac{-11C_A + 4T_R}{3}\right).$$

Provided we do not have more than 16 flavours this is negative. We note immediately that for an Abelian gauge theory $C_A = 0$, so that β_0 would be positive.

2. Deep inelastic electron-proton scattering

The purpose of performing deep inelastic electron-proton scattering experiments is to examine the internal structure of the proton. A schematic diagram of the parton picture of this internal structure is shown in Fig. 4. There are three principle (valence)

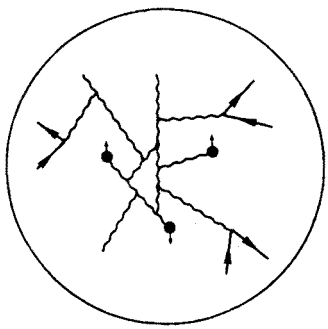

Fig. 4

quarks which carry flavour (2 up and 1 down) and colour. By changing colour they exchange gluons with each other and this is the mechanism which binds them together in the same way that the exchange of photons inside an atom binds the electrons and nucleus together. However, unlike QED these gluons can interact with each other. Furthermore, the gluons can produce quark antiquark pairs which is known as the sea (this can also happen inside an atom, but since QED is a weak coupling theory the amplitude for such a pair production is extremely small).

When we hit a proton with a hard off-shell photon from an electron current, the photon hits one of the charged constituents; either a valence quark or a quark or anti-quark from the sea (see Fig. 5). The parton model assumes that the initial and final momenta of the struct quarks are on their mass shells and that in the "infinite momentum frame" (where the three-momentum of the proton, $|\boldsymbol{p}|$, is much larger than the proton mass)

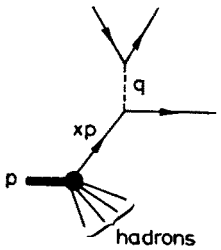

Fig. 5

the struck quark comes a fraction x of the longitudinal momentum of the proton (for the moment we neglect the transverse momentum of the struck quark). The initial momentum of the struck quark is

$$p_i = ((m_q^2 + x^2 p^2)^{1/2}, 0, 0, xp), \tag{2.1}$$

and the final momentum is $p_i + q$, where q is the momentum transferred to the quark mass. The requirement that the final quark is also on its mass shell gives

$$x = -q^2/2p \cdot q. \tag{2.2}$$

It is easy to show that the kinematic limits on x are $0 \leqslant x \leqslant 1$. If we define $F(x)$ as the probability of finding a quark inside the proton with longitudinal momentum xp, then the parton model tells us that the total cross-section is given by

$$\sigma_{ep \to eX} = \int_0^1 F(x) \sigma_{eq}^{\text{elastic}}(x) dx. \tag{2.3}$$

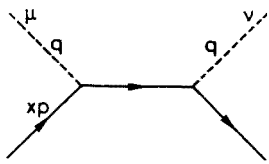

Fig. 6

The elastic electron quark cross-section is easily calculated in lowest order. We require the imaginary part of the Compton scattering diagram shown in Fig. 6. This is

$$\frac{l_{\mu\nu}}{q^4} \tfrac{1}{2} \operatorname{Tr} [\gamma_\mu (x\gamma \cdot p + \gamma \cdot q) \gamma_\nu x\gamma \cdot p] \delta(q^2 + 2p \cdot qx), \tag{2.4}$$

neglecting quark masses. Here $l_{\mu\nu}$ comes from the lepton currents and the $1/q^4$ terms is from the two photon propagators and the delta function is the imaginary part of the internal quark propagator. Performing the trace we get

$$\frac{2l_{\mu\nu}}{q^4}\left[\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)x+\frac{2x^2}{p\cdot q}\left(p_\mu-q_\mu\frac{p\cdot q}{q^2}\right)\left(p_\nu-q_\nu\frac{p\cdot q}{q^2}\right)\right]\delta(q^2+2p\cdot qx), \quad (2.5)$$

where we have obtained the form of Eq. (2.5) by exploiting the fact that the quark current is conserved. Putting this into Eq. (2.3) we obtain

$$\sigma_{ep\to eX}=\frac{2l_{\mu\nu}}{q^4}\left[\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)xF(x)+\frac{2x^2F(x)}{p\cdot q}\left(p_\mu-q_\mu\frac{p\cdot q}{q^2}\right)\left(p_\nu-q_\nu\frac{p\cdot q}{q^2}\right)\right]. \quad (2.6)$$

We define the structure functions F_1 and F_2 to be the coefficients of $-g_{\mu\nu}$, and $p_\mu p_\nu/p\cdot q$, respectively, and obtain the relations

$$F_1 = xF(x), \quad (2.7a)$$

$$F_2 = 2x^2F(x), \quad (2.b)$$

where $x = -q^2/2p\cdot q$. We see that to this order F_1 and F_2 depend only on the ratio x of the two kinematic invariants q^2 and $2p\cdot q$. This is called Bjorken scaling.

In the next order in perturbation theory we get diagrams shown in Fig. 7. In general we would expect such diagrams to give us terms proportional to $\ln(-q^2)$ which would violate exact Bjorken scaling; calculating such violations is not easy. First of all QCD is a strong interaction theory, so we cannot simply perform perturbation theory in the

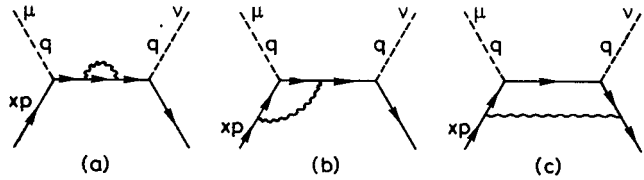

Fig. 7

strong coupling constant and expect to get reasonable results. Alternatively, for sufficiently large values of $-q^2$ we could say that it is the effective coupling constant which is important and attempt to perform the perturbation expansion in $g(-q)^2$. This also fails since the perturbation expansion generates, in each order of perturbation theory, powers of the product $g^2(-q^2)\ln(-q^2/p^2)$, where p is the momentum of the external leg ($p^2 = m_p^2$ for electron-proton scattering). Thus, although $g^2(-q^2)$ decreases with $\ln(-q^2)$ as $-q^2$ increases this product remains of order one. The only possibility is to try to factorize the cross-section into a term which depends only on p^2, multiplied by terms which depend only on q^2 and $2p\cdot q$. If this can be achieved then (except possibly at the kinematic end points $x = 0$ and $x = 1$) the terms depending only on q^2 and $2p\cdot q$ may be calculated in perturbation theory using the effective coupling constant $g(-q^2)$.

The Wilson operator product expansion

If we look again at the three diagrams of Fig. 6 we see that in coordinate space this may be written

$$\frac{\mathrm{Im}}{2\pi}\int d^4y e^{iq\cdot y}\langle p|J_\mu(0)J_\nu(y)\,|p\rangle = \int d^4y\langle p|T\bar{\psi}(0)\gamma_\mu\psi(0)\bar{\psi}(y)\gamma_\nu\psi(y)\,|p\rangle \tag{2.8}$$

to lowest order in perturbation (we have dropped the lepton current and photon propagator factors).

We observe that $T\psi(0)\bar{\psi}(y)$ is the fermion propagator to lowest order and this may be written

$$T\psi(0)\bar{\psi}(y) = S_f(y) \simeq \frac{-i}{(2\pi)^2}\frac{\gamma^\varrho y^\varrho}{(y-i\varepsilon)^2}. \tag{2.9}$$

$\psi(y)$ may be expanded as a Taylor series

$$\psi(y) = \sum_{N=0}^{\infty} y_{\mu_1}\cdots y_{\mu_N}\partial^{\mu_1}\cdots\partial^{\mu_N}\psi(0), \tag{2.10}$$

and $\gamma^\mu\gamma^\varrho\gamma^\nu$ may be written

$$\gamma^\mu\gamma^\varrho\gamma^\nu = 2g^{\mu\varrho}\gamma^\nu - 2\gamma^\varrho g^{\mu\nu}, \tag{2.11}$$

where we have thrown away a term antisymmetric in μ, ν (which vanishes when multiplied by the lepton currents).

Combining Eqs. (2.8) to (2.11) we have

$$\langle p|TJ_\mu(0)J_\nu(y)\,|p\rangle = \frac{-i}{\pi^2}\sum_{N=0}^{\infty}\left[\frac{g_{\mu\nu}y_{\mu_1}y_{\mu_2}\cdots y_{\mu_N}}{(y^2-i\varepsilon)^2} - \frac{y_\mu g_{\nu\mu_1}y_{\mu_2}\cdots y_{\mu_N}}{(y^2-i\varepsilon)^2}\right]$$
$$\times\langle p|\bar{\psi}(0)\gamma^{\mu_1}\partial^{\mu_2}\cdots\partial^{\mu_N}\psi(0)\,|p\rangle. \tag{2.12}$$

The next step is to arrange the operators of Eq. (2.12) into representations of the Lorentz group (by symmetrizing and removing traces) and then to make the operators gauge invariant by replacing the derivatives ∂^μ by covariant derivatives $D_\mu = \partial_\mu - ig\tau^a A_\mu^a$. The term $\langle p|\psi(0)\gamma^{\mu_1}\partial^{\mu_2}\ldots\partial^{\mu_N}\psi|p\rangle$ becomes

$$S\langle p|\bar{\psi}(0)\gamma^{\mu_1}D^{\mu_2}\cdots D^{\mu_N}\psi(0) - \text{traces}|p\rangle + S\langle p|\bar{\psi}(0)\gamma^{\mu_1}D^2D^{\mu_4}\cdots D^{\mu_N}\psi(0)$$
$$-\text{traces}|p\rangle + \cdots, \tag{2.13}$$

where S means symmetrized over all Lorentz indices.

We now make the Fourier transform of Eq. (2.8) and obtain

$$\frac{i^{N-2}}{\pi^2}\sum_{N=0}^{\infty}\left(g_{\mu\nu}-\frac{2p_\mu p_\nu}{p\cdot q}\right)\left\{\frac{q_{\mu_1}\cdots q_{\mu_N}}{(q^2)^N}S\langle p|\bar{\psi}(0)\gamma^{\mu_1}D^{\mu_2}\cdots D^{\mu_N}\psi - \text{traces}|p\rangle\right.$$
$$\left. + \frac{q_{\mu_1}\cdots q_{\mu_N}}{(q^2)^{N+1}}S\langle p|\bar{\psi}D^2\gamma^{\mu_1}D^{\mu_2}\cdots D^{\mu_N}\psi - \text{traces}|p\rangle + \cdots\right\}. \tag{2.14}$$

Now by dimensional counting and the fact that in Eq. (2.14) we have matrix elements of operators between single particle states which can therefore only depend on the momentum of that single particle we have

$$\langle p|\bar{\psi}\gamma^{\mu_1}D^{\mu_2}\cdots D^{\mu_N}\psi - \text{traces}|p\rangle = A_N(p_{\mu_1}\cdots p_{\mu_N} - \text{traces}), \tag{2.15a}$$

$$\langle p|\bar{\psi}\gamma^{\mu_1}D^2D^{\mu_2}\cdots D^{\mu_N}\psi - \text{traces}|p\rangle = A'_N p^2(p_{\mu_1}\cdots p_{\mu_N} - \text{traces}), \tag{2.15b}$$

where A_N and A'_N are dimensionless. We see that the second term in Eq. (2.14) is smaller than the first term by $O(p^2/q^2)$, which can be neglected if q^2 is large and $p^2 = m_p^2$. Similarly we may throw away the trace terms. In general, an operator of spin N and dimension D has a single particle matrix element.

$$\langle p|O^{\mu_1\cdots\mu_N}|p\rangle = B_N(p_{\mu_1}\cdots p_{\mu_N} - \text{traces})\,(p^2)^{T-2}, \tag{2.16}$$

where $T = D - N$ is called the twist of the operator. In the limit of large q^2, therefore, we need to keep only operators of twist 2. The above derivation has been conducted in lowest order of perturbation theory. If perturbation theory is valid a more general relation can be derived

$$\int d^4y e^{iq\cdot y}\langle p|TJ_\mu(0)J_\nu(y)\,|p\rangle = \sum_{N=0}^{\infty}\sum_j\left(-g_{\mu\nu}C^N_{1,j}(q^2) + \frac{p_\mu q_\alpha + p_\nu q_\mu}{2p\cdot q}\,C^N_{2,j}(q^2)\right)$$

$$\times\,\frac{q_{\mu_1}\cdots q_{\mu_N}}{(q^2)^N}\,\langle p|O^{\mu_1\cdots\mu_N}|p\rangle + \text{higher twist terms}, \tag{2.17}$$

where the sum j runs over all twist 2 operators of spin N, which are

$$O^{\text{NS}}_N = S(\bar{\psi}_\alpha\lambda^l_{\alpha\beta}\gamma^{\mu_1}D^{\mu_2}\cdots D^{\mu_N}\psi_\beta - \text{traces}), \tag{2.18a}$$

$$O^{\text{f}}_N = S(\bar{\psi}_\alpha\gamma^{\mu_1}D^{\mu_2}\cdots D^{\mu_N}\psi_\alpha - \text{traces}), \tag{2.18b}$$

$$O^{\text{g}}_N = S(G^{\mu_1\nu}D^{\mu_2}\cdots D^{\mu_{N-1}}G^{\mu_N\nu} - \text{traces}). \tag{2.18c}$$

The first operator is a flavour non-singlet operator and λ is a generator of the flavour group. The second and third operators are the fermionic and gluonic parts of the flavour singlet operators. Using Eq. (2.16) and the fact that $p\cdot q/q^2 = -1/2x$, Eq. (2.17) becomes

$$\int d^4y e^{iq\cdot y}\langle p|TJ_\mu(0)J_\nu(y)\,|p\rangle = -g_{\mu\nu}T_1(x,q^2) + \frac{(p_\mu q_\alpha + p_\nu q_\mu)}{2p\cdot q}\,T_2(x,q^2), \tag{2.19}$$

where

$$T_{1(2)}(x,q^2) = \sum_{N=0}^{\infty}\sum_j C^N_{1(2),j}(q^2)\,\frac{(-1)^N}{(2x)^N}\,B^N_j. \tag{2.20}$$

We now have to project out a particular value of N. Firstly we note that $T_{1\,(2)}(x,q^2)$ is symmetric under $x\leftrightarrow -x$. This can be seen from Eq. (2.19), where on the LHS $x\leftrightarrow -x$ is equivalent to $q\leftrightarrow -q$ or $y\leftrightarrow -y$. Using the translation invariance of operator matrix elements and the symmetry of the RHS under $\mu\leftrightarrow\nu$, we see that both sides of Eq. (2.19)

are symmetric under this interchange. The structure functions are $\frac{1}{2}\pi \times$ the imaginary parts of T_1 and T_2. Since the physical region of x is $0 < x < 1$ and T_1 and T_2 are symmetric functions of x, they can only be analytic in an x plane which is cut along the

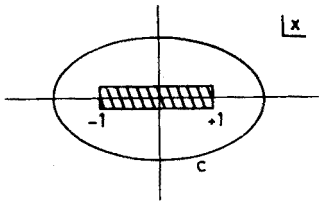

Fig. 8

real axis from -1 to $+1$ (see Fig. 8). Now the expansion of Eq. (2.20) is valid on any contour for which $|x| > 1$, and the coefficient of $1/x^N$ may be obtained from the integral

$$\frac{1}{2\pi i}\oint_c x^{N-1}T_{1(2)}(x, q^2) = \frac{1}{2^N}\sum_j C^N_{1,j}(q^2)B^N_j \quad (N \text{ even}) \tag{2.21}$$

around such a contour. Since T_1 and T_2 have no singularities outside the cut, the contour may be shrunk until it surrounds the cut. We now have

$$\frac{1}{2\pi}\int_{-1}^{1} dx x^{N-1} 2 \operatorname{Im} T_{1(2)}(x, q^2) = \frac{1}{2^N} C^N_{1(2),j}(q^2)B^N_j \quad (N \text{ even}), \tag{2.22}$$

or

$$\int_0^1 dx x^{N-1}F_{1(2)}(x, q^2) = \frac{1}{2^{N+2}}\sum_j C^N_{1(2),j}(q^2)B^N_j. \tag{2.23}$$

Thus we have arrived at an expression for the N^{th} moment of the structure functions with only three terms in it, each of which is a product of a coefficient function which depends only on q^2 and a factor which is related to the matrix element of a twist 2 operator between single proton states.

We cannot calculate the quantities B^N_j for operator matrix elements between proton states. We can, however, calculate them in perturbation theory between single particle quark or gluon states. For example, if we consider the non-singlet operator

$$B^N_{\text{NS}}p_{\mu_1}\cdots p_{\mu_N} = \langle p|\bar{\psi}\lambda\gamma^{\mu_1}D^{\mu_2}\cdots D^{\mu_N}\psi|p\rangle, \tag{2.24}$$

where $|p\rangle$ is now a single particle quark state of momentum p. To lowest order D^{μ_N} may be replaced by the ordinary derivative, ∂_{μ_N} and we can expand the fields in creation and annihilation operators

$$\begin{aligned}\partial_{\mu_2}\cdots\partial_{\mu_N}\psi(0) &= \lim_{x\to 0}\int\frac{d^3p}{2p_0}\partial_{\mu_2}\cdots\partial_{\mu_N}[u(p)a(p)e^{-ip\cdot x}+v(p)b^\dagger(p)e^{ip\cdot x}] \\ &= (-i)^N\int\frac{d^3p}{2p_0}p_{\mu_2}\cdots p_{\mu_N}[u(p)a(p)e^{-ip\cdot x}+(-1)^N v(p)b^\dagger(p)e^{ip\cdot x}].\end{aligned} \tag{2.25}$$

Thus we have

$$\tfrac{1}{2}\sum_{\text{spins}}\langle p|\bar{\psi}(0)\gamma^{\mu_1}\partial^{\mu_2}\cdots\partial^{\mu_N}\psi|p\rangle = \tfrac{1}{2}\,\mathrm{Tr}\,(\gamma^{\mu_1}\gamma\cdot p)p_{\mu_2}\cdots p_{\mu_N} = 2p_{\mu_1}\cdots p_{\mu_N}, \quad (2.26)$$

so that to this order $B^N_{\mathrm{NS}} = 2$. In the next order in perturbation theory we find infinities (the details of how to calculate these quantities in one loop are given in Section 3) so we must renormalize the matrix elements by subtracting at some value of p^2, say $p^2 = -\mu^2$ (again we choose a negative value of p^2 to avoid possible infrared and mass threshold problems). To one loop, therefore, we have

$$B^N_{\mathrm{NS}} = 2\left[1 + \frac{g}{16\pi^2}\,\frac{\gamma^N_0}{2}\ln(p^2/\mu^2)\right]. \quad (2.27)$$

The quantity $C^N_{1(2),\,\mathrm{NS}}(q^2)B_{\mathrm{NS}}$ is a physical quantity (although it contains only the non-singlet operators it could be related to the differences of electron-proton and electron-neutron scattering structure functions which are purely non-singlet). All physical quantities are independent of the renormalization point μ. This is the fundamental axiom of the renormalization point μ. It permits us to calculate the μ dependence of $C^N_{1(2),\,\mathrm{NS}}(q^2)$ from the μ dependence of B^N_{NS}

$$\frac{d}{d\mu}C^N_{1(2),\mathrm{NS}}(q^2) = \frac{-C^N_{1(2),\mathrm{NS}}(q^2)}{B^N_{\mathrm{NS}}(p^2)}\,\frac{d}{d\mu}B^N_{\mathrm{NS}}(p^2). \quad (2.28)$$

The quantity $(d/d\mu)B^N_{\mathrm{NS}}$ comes from the infinites of the operator matrix elements and, as we shall see in the next section, is a property of the operator only, and not the states between which the matrix elements are calculated. In other words, for any matrix element we may write

$$B^N_{\mathrm{NS}}(p^2 = -\mu^2)|_{\text{all orders}} = Z^N_{\mathrm{NS}}(\mu)B^N_{\mathrm{NS}}(p^2 = -\mu^2)|_{\text{lowest order}}, \quad (2.29)$$

where Z^N_{NS} is a property of the operator. We now define a quantity γ^N_{NS}, the anomalous dimension of the operator O^N_{NS},

$$\gamma^N_{\mathrm{NS}}(g) = \frac{-1}{Z^N_{\mathrm{NS}}}\,\frac{d}{d\ln\mu}Z^N_{\mathrm{NS}}(\mu) = \frac{g^2\gamma^N_0}{16\pi^2} + O(g^4). \quad (2.30)$$

Now the coefficient functions $C^N_{1(2),\,\mathrm{NS}}(q^2)$ are dimensionless quantities so that they can only depend on q^2 through the ratios $-q^2/\mu^2$.

$$\frac{d}{d\ln(-q^2)}C^N_{1(2),\mathrm{NS}}(q^2) = -\tfrac{1}{2}\,\frac{d}{d\ln\mu}C^N_{1(2),\mathrm{NS}}(-q^2/\mu^2) = \frac{-\gamma^N_{\mathrm{NS}}}{2}\,C^N_{1(2),\mathrm{NS}}(-q^2/\mu^2). \quad (2.31)$$

If we now expand $C^N_{1(2),\,\mathrm{NS}}(q^2/-\mu^2)$ in a power series in the effective coupling constant, $g(-q^2)$, and use the chain rule

$$\frac{d}{d\ln(-q^2)} = \frac{\partial}{\partial\ln(-q^2)} + \tfrac{1}{2}\beta(g(-q^2))\,\frac{\partial}{\partial g(-q^2)}, \quad (2.32)$$

we arrive at the renormalization group equation for the q^2 dependence of coefficient functions $C^N_{1(2),\,\mathrm{NS}}(q^2/-\mu^2)$

$$\left(\frac{\partial}{\partial \ln(-q^2)} + \tfrac{1}{2}\beta(g(-q^2))\frac{\partial}{\partial g(-q^2)} + \gamma^N_{\mathrm{NS}}/2\right) C^N_{1(2),\mathrm{NS}}(q^2) = 0. \tag{2.33}$$

This equation has a solution

$$C^N_{1(2),\mathrm{NS}}(q^2) = C^N_{1(2),\mathrm{NS}}(q^2 = -\mu^2, g(-q^2)) \exp - \int_{g(\mu^2)}^{g(-q^2)} \gamma^N_{\mathrm{NS}}(g')/\beta(g')dg', \tag{2.34}$$

where the first factor is the coefficient function calculated at $q^2 = -\mu^2$ in ordinary perturbation theory but using the effective coupling constant $g(-q^2)$. Now if we expand γ^N_{NS} and β to lowest order (one loop) we have

$$\gamma^N_{\mathrm{NS}}(g') = \frac{g'^2}{16\pi^2}\gamma^0_N, \tag{2.35a}$$

$$\beta(g') = -\frac{g'^3}{16\pi^2}\beta_0, \tag{2.35b}$$

and Eq. (2.35) can be used to give us (to leading order) the coefficient functions at q^2 in terms of its value at some fixed value of q^2 $(= q_0^2)$

$$C^N_{1(2),\mathrm{NS}}(q^2) = C^N_{1(2)}(q_0^2)\,[g^2(-q^2)/g^2(-q_0^2)]^{\gamma_N{}^0/2\beta_0}. \tag{2.36}$$

The approximation is valid provided both $g^2(-q^2)/16\pi^2$ and $g^2(-q_0^2)/16\pi^2 \ll 1$.

Now solving Eq. (2.35b) we have

$$g(-q^2) = 1/\beta_0 \ln(-q^2/\Lambda^2), \tag{2.37}$$

where Λ is the one theoretical free parameter of the theory which sets the scale of the strength of the coupling constant (it may be considered to be the one constant of integration of the first-order differential equation (2.35b)).

Thus for the moments of a structure function which depend only on the non-singlet operators we have the following equation for the q^2 development:

$$M_N(q^2) = \int_0^1 x^{N-1}F_1(x, q^2) = M_N(q_0^2)\,[\ln(-q^2/\Lambda^2)/\ln(-q_0^2/\Lambda^2)]^{-\gamma_N{}^0/2\beta_0} + O(1/\ln(q^2/-\Lambda^2)). \tag{2.38}$$

For the coefficient functions of the singlet operators the situation is a little more complicated. This is because the two singlet operators of spin N have the same quantum numbers and can mix in higher orders. This means that one can begin with one operator (say O^{f}_N of Eq. (2.18)) and through higher order interactions generate O^{g}_N and vice versa. The situation is shown schematically in Fig. 9. All the diagrams in Fig. 9 are divergent so that the operator renormalization constant of Eq. (2.29) becomes a 2×2 matrix

$$\begin{pmatrix} Z^N_{\mathrm{gg}} & Z^N_{\mathrm{gf}} \\ Z^N_{\mathrm{fg}} & Z^N_{\mathrm{ff}} \end{pmatrix}$$

and the anomalous dimension is a 2×2 matrix. The coefficient functions $C^N_{1(2),\mathrm{f}}$ and $C^N_{1(2),\mathrm{g}}$ obey the matrix equation

$$\left(\left(\frac{\partial}{\partial \ln(-q^2)} + \tfrac{1}{2}\beta(g(-q^2))\frac{\partial}{\partial g(-q^2)}\right)\delta_{ab} + \gamma^N_{ab}\right) C^N_b = 0, \quad a, b = \mathrm{f} \text{ or } \mathrm{g}. \tag{2.39}$$

The matrix γ^N can be diagonalized

$$A_N^{-1}\gamma^N A_N = \begin{pmatrix} \gamma^N_+ & 0 \\ 0 & \gamma^N_- \end{pmatrix} \tag{2.40}$$

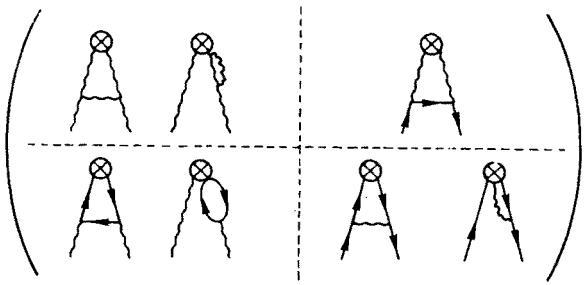

Fig. 9

and to leading order A_N is independent of g, so that we have

$$C^N_b(q^2)\,(A^N_{b+(-)})^{-1} = C^N_b(q^2 = -\mu^2, g(-q^2))\,(A^N_{b+(-)})^{-1}[g^2(-q^2)/g^2(-q_0^2)]^{\gamma_{+(-)}^N/2\beta_0} \tag{2.41}$$

or

$$C^N_b(q^2) = \sum_a C^N_a(q^2 = -\mu^2, g(-q^2))\,(A^N_{a+})^{-1}(g(-q^2)/g(\mu^2))^{\gamma_+^N/2\beta_0} A^N_{+b} + \sum_c C^N_c(q^2 = -\mu^2, g(-q^2))\,(A^N_{c-})^{-1}(g(-q^2)/g(\mu^2))^{\gamma_-^N/2\beta_0} A^N_{-b}. \tag{2.42}$$

In Eqs. (2.41) and (2.42) we have assumed that $g^2(\mu^2)/16\pi^2 \ll 1$. We are always free to make an arbitrary choice of μ, but remember that the operator matrix elements must be considered to be calculated as a perturbation series in $g(\mu)$ such that the product of the coefficient function and the operator matrix element are μ independent.

We shall return to this problem of operator mixing at the end of the next section.

3. Calculation of anomalous dimensions

3.1. Incorporation of operators

Operators can be incorporated into the theory of QCD simply by defining an effective Lagrangian which is the usual QCD Lagrangian to which an operator multiplied by a source term is added

$$L^{\mathrm{eff}} = -\tfrac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} + i\bar{\psi}_\alpha(\gamma^\mu D_\mu + im_\alpha)\psi_\alpha + J^i_{\mu_1 \ldots \mu_N} O_i^{\mu_1 \ldots \mu_N}, \tag{3.1}$$

where the source of the operator O_i may be written

$$J^i_{\mu_1 \ldots \mu_N} = \Delta_{\mu_1}\Delta_{\mu_2} \cdots \Delta_{\mu_N}, \quad \text{where} \quad \Delta^2 = 0. \tag{3.2}$$

Such a source term automatically projects out the traceless symmetric parts of the operator so that we have operators which are irreducible representations of the Lorentz group. For example for the flavour non-singlet operator we would add to the Lagrangian a term

$$\Delta_{\mu_1}\Delta_{\mu_2} \cdots \Delta_{\mu_N}\bar{\psi}_\alpha \lambda^a_{\alpha\beta}\gamma^{\mu_1}D^{\mu_2} \cdots D^{\mu_N}\psi. \tag{3.3}$$

Such a term gives rise to a series of effective vertices, some of which are shown in Fig. 10. It we replace all the covariant derivatives by ordinary derivatives in Eq. (3.3) we obtain the vertex of Fig. 10a whose Feynman rule is $\gamma \cdot \Delta\,(\Delta \cdot k)^{N-1}$. If we replace all but one of

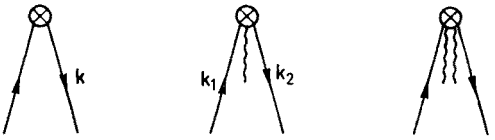

Fig. 10

the covariant derivatives by ordinary derivatives and pick up a gluon from the other one we obtain the vertex of Fig. 10b whose Feynman rule is

$$\sum_{j=0}^{N-2} \gamma \cdot \Delta(\Delta \cdot k_1)^{N-j-2}(\Delta \cdot k_2)^j \Delta_\mu.$$

Similarly we have vertices with 2, 3, ..., $N-1$ gluons coming out.

The diagrams which contribute to higher order corrections of the effective vertex of Fig. 10a are shown in Fig. 11. They give infinite contributions so that we must define

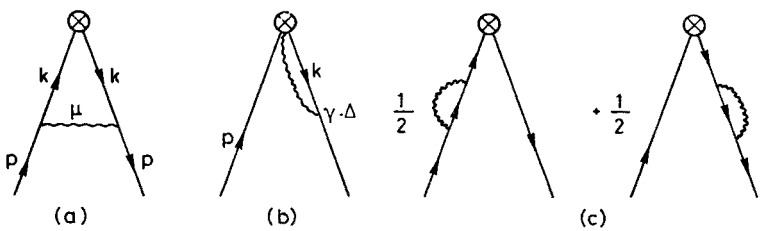

Fig. 11

a bare Lagrangian in which the operators are multiplied by their renormalization constants (or matrices in the case of operator mixing)

$$L_0^{\text{eff}} = -\tfrac{1}{4}\,Z_A G^a_{\mu\nu}G^a_{\mu\nu} + Z_\psi i\bar{\psi}_\alpha(\gamma^\mu D^0_\mu - m^0_\alpha)\psi_\alpha + Z^i_N \Delta_{\mu_1} \cdots \Delta_{\mu_N} O^{\mu_1 \cdots \mu_N}, \tag{3.4}$$

so that Fig. 10a, for example, has a counter term $(Z_N - 1)\,\gamma \cdot \Delta(\Delta \cdot k)^{N-1}$ which cancels the infinities of the diagrams in Fig. 10, but which depend on the renormalization point, μ.

3.2. The method of dimensional regularization

If we have an integral

$$I_4 = \int \frac{d^4k}{(2\pi)^4} \frac{g^2}{(k^2+p^2)^2} \tag{3.5}$$

we know that this diverges logarithmically, However, if we calculate it in 3, 2, or 1 dimensions, we would have convergent integrals

$$I_3 = \int \frac{d^3k}{(2\pi)^3} \frac{g^2}{(k^2+p^2)^2} = \frac{g^2}{2\pi\sqrt{p^2}}, \tag{3.6}$$

$$I_2 = \int \frac{d^2k}{(2\pi)^2} \frac{g^2}{(k^2+p^2)^2} = \frac{g^2}{4\pi p^2}, \tag{3.7}$$

$$I_1 = \int \frac{dk}{(2\pi)} \frac{g^2}{(k^2+p^2)^2} = \frac{g^2}{4(p^2)^{3/2}}. \tag{3.8}$$

The analytic function

$$I_d = g^2 \frac{\Gamma(2-d/2)}{(4\pi)^{d/2}} (p^2)^{d/2-2} \tag{3.9}$$

reproduces Eqs. (3.6), (3.7), and (3.8) for $d = 3, 2, 1$, respectively. We see that this analytic function has a pole at $d = 4$. Since gauge invariance (the Ward identities) holds in all dimensions it must hold separately for each coefficient of $(d-4)^n$ or $(d-4)^{-n}$ so that if we remove the pole of Eq. (3.9) at $d = 4$ we are guaranteed that the part we have removed is gauge invariant, so that what remains is also gauge invariant (provided, of course, that we were calculating a gauge invariant quantity in the first place). Doing this and expanding Eq. (3.9) about $d = 4$ we have

$$I_4^{\text{Reg}} = \frac{g^2}{16\pi^2} (\ln 4\pi - \gamma - \ln (p^2/\mu^2)) + O(d-4), \tag{3.10}$$

where γ is the Euler constant. The $\ln \mu^2$ term arose from the fact that in d dimensions the coupling constant g has dimensions $4-d$

$$g = h\mu^{4-d}, \tag{3.11}$$

where h is dimensionless and expanding g^2 around $d = 4$ gives us $\ln \mu^2$.

Note that this method of regularization is not the same as the renormalization prescriptions of subtracting at $p^2 = \mu^2$, because of the presence of the terms $\ln 4\pi$ and γ. However, to this order, the $\ln \mu^2$ coefficient is the same.

The method of dimensional regularization is very useful for defining the anomalous dimensions (to any order in perturbation theory) remember the definition of γ is

$$\gamma = (-1/Z(\mu)) \frac{d}{d \ln \mu} Z(\mu). \tag{3.12}$$

To order $g^4(\mu^2)$ we can write $Z(\mu)$ as

$$Z(\mu) = 1+g(\mu^2)\,(a_0+b_0 \ln \mu^2)+g^4(\mu^2)\,(a_1+b_1 \ln \mu^2+c_1 \ln^2 \mu^2)+\ldots,$$

so that $Z(\mu)$ has both an explicit and implicit (through $g^2(\mu^2)$) dependence on μ. This clearly gets very complicated. However, in $d = 4-\varepsilon$ dimensions we have for the coupling constant

$$g(\mu^2) = \mu^{\varepsilon/2} Z_g(\mu) h_0, \tag{3.13}$$

where h_0 is a bare dimensionless quantity. $Z_g(\mu)$ to any finite order in perturbation theory (order r) can be expanded in powers of $1/\varepsilon$ up to order r

$$Z_g(\mu) = 1+\sum_{k=1}^{r} \frac{a_k(\mu)}{\varepsilon^k}. \tag{3.14}$$

The definitions of β is such that from Eq. (3.13) we have

$$\beta = \frac{\varepsilon}{2}\, g(\mu^2)+(g(\mu^2)/Z_g(\mu))\,\frac{d}{d \ln \mu} Z_g(\mu). \tag{3.15}$$

Using Eq. (3.14) this gives

$$\left(1+\sum_{k=1}^{r} a_k(\mu)/\varepsilon^k\right)\beta = \frac{\varepsilon}{2}\, g(\mu^2)\left(1+\sum_{k=1}^{r} a_k(\mu)/\varepsilon^k\right)+g(\mu^2)\sum_{k=1}^{r} \frac{\partial}{\partial \ln \mu} a_k(\mu)/\varepsilon^k. \tag{3.16}$$

β is finite as $\varepsilon \to 0$ and the highest power of ε in the RHS of Eq. (3.16) is ε^1, so that β must have the form $A+B\varepsilon$ and comparing coefficients of ε in Eq. (3.16) we have

$$\beta = A+\frac{\varepsilon}{2}\, g(\mu^2). \tag{3.17}$$

Now the renormalization constant of an operator up to r^{th} order in perturbation may also be expanded in a power series in $1/\varepsilon$,

$$Z_N(\mu) = 1+\sum_{k=1}^{r} \frac{1}{\varepsilon^k}\, b_k(g(\mu^2)), \tag{3.18}$$

where we have written the μ dependence of the coefficients b_k implicitly by expressing them as functions of $g(\mu^2)$ only (it is always possible to do this). Now from the definitions of γ_N(Eq. (3.12)) we have

$$\begin{aligned} Z_N\gamma_N &= \left(1+\sum_{k=1}^{r} b_k(g(\mu^2))/\varepsilon^k\right)\gamma_N = -\sum_{k=1}^{r} \frac{1}{\varepsilon^k}\,\beta\,\frac{\partial}{\partial g(\mu)}\, b_k(g(\mu^2)) \\ &= -\sum_{k=1}^{r} \frac{1}{\varepsilon^k}\,(A+g(\mu^2)\varepsilon/2)\,\frac{\partial}{\partial g(\mu^2)}\, b_k(g(\mu^2)), \end{aligned} \tag{3.19}$$

where we have used Eq. (3.17) and the fact that

$$\frac{d}{d\ln\mu} b_k(g(\mu^2)) = \beta \frac{d}{dg(\mu)} b_k(g(\mu^2)). \tag{3.20}$$

Now again γ_N is finite as $\varepsilon \to 0$ and the highest power of ε on the left-hand side is ε^0 whose coefficient is γ_N. Comparing this with the coefficient of ε^0 on the right-hand side of Eq. (3.20) we have

$$\gamma_N = -(g(\mu^2)/2) \frac{d}{dg(\mu^2)} b_1(g(\mu^2)). \tag{3.21}$$

b_1 is the residue of the simple pole part of Z_N and Eq. (3.21) is valid to all orders. This greatly simplifies the problem of finding γ_N from Z_N, particularly when one goes beyond one loop.

We now return to the diagrams of Fig. 11. The contributions from the first diagram in $d = 4-\varepsilon$ dimensions is (in the Feynman gauge)

$$-ig^2 \int \frac{d^{4-\varepsilon}k}{(2\pi)^{4-\varepsilon}} \frac{\gamma^\mu \gamma \cdot k \gamma \cdot \Delta \gamma \cdot k \gamma_\mu (\Delta \cdot k)^{N-1}}{k^4 (k-p)^2}. \tag{3.22}$$

In $4-\varepsilon$ dimensions $\gamma^\mu \gamma \cdot A \gamma_\mu = (-2+\varepsilon)\gamma \cdot A$ so this becomes

$$ig^2(2-\varepsilon) \int \frac{d^{4-\varepsilon}k}{(2\pi)^{4-\varepsilon}} \frac{2\gamma \cdot k(\Delta \cdot k)^N - k^2 \gamma \cdot \Delta (\Delta \cdot k)^{N-1}}{k^4 (k-p)^2}. \tag{3.23}$$

This has the general form $A\gamma \cdot \Delta + B\gamma \cdot p$. But the second term is not proportional to the original vertex, so it is finite and therefore uninteresting (it cannot depend on μ). We can get rid of this term and convert the numerator of Eq. (3.23) into a scalar quantity rather than a Dirac matrix by multiplying by $\gamma \cdot p$, taking the trace and putting $p^2 = 0$ (we must finally multiply by $\gamma \cdot \Delta / 4\Delta \cdot p$ to project out the term $A\gamma \cdot \Delta$). We find it also useful to use the relation $2k \cdot p = k^2 - (k-p)^2$ (with $p^2 = 0$). Now in $4-\varepsilon$ dimensions

$$\int \frac{d^{4-\varepsilon}k}{(2\pi)^{4-\varepsilon}} \frac{1}{(k^2-m^2)^2} \sim \Gamma\left(\frac{\varepsilon}{2}\right)(m^2)^{-\varepsilon/2} \xrightarrow[m\to 0]{} 0 \quad \text{for} \quad \varepsilon < 0, \tag{3.24}$$

so that

$$\int \frac{d^{4-\varepsilon}k}{(2\pi)^{4-\varepsilon}} \frac{1}{k^4} = 0 \tag{3.25}$$

and we are left with

$$i(2-\varepsilon)g^2 \frac{\gamma \cdot \Delta}{\Delta \cdot p} \int \frac{d^{4-\varepsilon}k}{(2\pi)^{4-\varepsilon}} \frac{((\Delta \cdot k)^N - (\Delta \cdot k)^{N-1} \Delta \cdot p)}{k^2 (k-p)^2}. \tag{3.26}$$

Performing the Feynman parametrization this may be written

$$i(2-\varepsilon)g^2 \frac{\gamma \cdot \Delta}{\Delta \cdot p} \int \frac{d^{4-\varepsilon}k}{(2\pi)^{4-\varepsilon}} \int_0^1 d\alpha \frac{((\Delta \cdot k)^N - (\Delta \cdot k)^{N-1} \Delta \cdot p)}{[k^2 - 2p \cdot k\alpha + p^2\alpha]^2}. \tag{3.27}$$

Now we symmetrize the denominator by making the shift $k_\mu \to k_\mu + \alpha p_\mu \cdot (\Delta \cdot k)^N$ becomes

$$(\Delta \cdot k)^N \to \sum_{j \text{ even}} \frac{N!}{j!(N-j)!} (\Delta \cdot k)^j (\Delta \cdot p)^{N-j} \alpha^{N-j}, \tag{3.28}$$

where we sum over even j only because the odd terms vanish by symmetric integration in k. But all but the $j = 0$ terms also vanish because $\Delta^2 = 0$ and

$$\Delta_\mu \Delta_\nu \int \frac{d^{4-\varepsilon}k}{(2\pi)^{4-\varepsilon}} \frac{k^\mu k^\nu}{[k^2 + p^2\alpha(1-\alpha)]^2} \sim \Delta^2. \tag{3.29}$$

Thus we have

$$i(2-\varepsilon)(\Delta \cdot p)^{N-1} \gamma \cdot \Delta g^2 \int \frac{d^{4-\varepsilon}k}{(2\pi)^{4-\varepsilon}} \int_0^1 d\alpha \frac{(\alpha^N - \alpha^{N-1})}{[k^2 + p^2\alpha(1-\alpha)]^2}$$

$$= -\frac{(2-\varepsilon)}{(16\pi^2)} g^2 (\Delta \cdot p)^{N-1} \gamma \cdot \Delta \Gamma\left(\frac{\varepsilon}{2}\right) (4\pi p^2)^{-\varepsilon/2} \int_0^1 d\alpha \frac{(\alpha^N - \alpha^{N-1})}{\alpha^{\varepsilon/2}(1-\alpha)^{\varepsilon/2}}$$

$$= \frac{-(2-\varepsilon)}{16\pi^2} g^2 (\Delta \cdot p)^{N-1} \gamma \cdot \Delta \Gamma\left(\frac{\varepsilon}{2}\right) (4\pi p^2)^{-\varepsilon/2} \left[\frac{\Gamma(N+1-\varepsilon/2)\Gamma(1-\varepsilon/2)}{\Gamma(N+2-\varepsilon)}\right.$$

$$\left. - \frac{\Gamma(N-\varepsilon/2)\Gamma(1-\varepsilon/2)}{\Gamma(N+1-\varepsilon)}\right] = \frac{-4}{16\pi^2} g^2 (\Delta \cdot p)^{N-1} \gamma \cdot \Delta \frac{1}{\varepsilon} [1/(N+1) - 1/N]$$

$$+ \text{finite terms as } \varepsilon \to 0. \tag{3.30}$$

Therefore the contribution to γ_{NS}^N from this diagram is

$$\frac{-4g^2}{(16\pi^2)} \frac{1}{N(N+1)} C_F, \tag{3.31}$$

where the factor C_F comes from the group theory (i. e. form summing over all the possible gluons which can contribute to the diagram). Repeating the procedure for the second diagram in Fig. 11 we have

$$2i \sum_{j=0}^{N-2} \int \frac{d^{4-\varepsilon}k}{(2\pi)^{4-\varepsilon}} \frac{\gamma \cdot \Delta \gamma \cdot k \gamma \cdot \Delta (\Delta \cdot k)^j (\Delta \cdot p)^{N-2-j}}{k^2(k-p)^2}$$

$$= 4i \sum_{j=0}^{N-2} \int \frac{d^{4-\varepsilon}k}{(2\pi)^{4-\varepsilon}} \frac{\gamma \cdot \Delta (\Delta \cdot k)^{j+1} (\Delta \cdot p)^{N-2-j}}{k^2(k-p)^2} + O(\Delta^2), \tag{3.32}$$

where we have thrown away a term proportional to Δ^2. Performing the Feynman parametrization, shifting the variable of integration, and keeping only $(\Delta \cdot p)^{j+1} a^{j+1}$ from the binomial expansion of $(\Delta \cdot k + \Delta \cdot p\alpha)^{j+1}$, we have

$$4ig^2 \sum_{j=0}^{N-2} \int \frac{d^{4-\varepsilon}k}{(2\pi)^{4-\varepsilon}} \int_0^1 d\alpha \frac{(\Delta \cdot p)^{N-1} \gamma \cdot \Delta \alpha^{j+1}}{[k^2 + p^2\alpha(1-\alpha)]^2}$$

$$= \frac{-4g^2}{16\pi^2} (\Delta \cdot p)^{N-1} \gamma \cdot \Delta \Gamma(\varepsilon/2) \sum_{j=0}^{N-2} \int_0^1 d\alpha \frac{\alpha^{j+1} (4\pi p^2)^{-\varepsilon/2}}{\alpha^{\varepsilon/2} (1-\alpha)^{\varepsilon/2}}$$

$$= \frac{-4g^2}{16\pi^2} (\Delta \cdot p)^{N-1} \gamma \cdot \Delta \Gamma(\varepsilon/2) \sum_{j=0}^{N-2} \frac{\Gamma(j+2-\varepsilon/2)\Gamma(1-\varepsilon/2)(4\pi p^2)^{-\varepsilon/2}}{\Gamma(j+3-\varepsilon/2)}, \tag{3.33}$$

so that the contribution to γ_{NS}^N is

$$\frac{8g^2}{16\pi^2} \sum_{j=0}^{N-2} \frac{1}{j+2} C_F. \tag{3.34}$$

Finally the contribution from the third diagram of Fig. 11 is $(g^2/16\pi^2) 2C_F$ so that we have

$$\gamma_{\mathrm{NS}}^N = \frac{2g^2}{16\pi^2} C_F \left[1 - \frac{2}{N(N+1)} + \sum_{j=0}^{N-2} \frac{4}{(j+2)} \right]. \tag{3.35}$$

Note that this is zero for $N = 1$. This is because the operator for $N = 1$ is $\bar{\psi}^\alpha \gamma^\mu \lambda^a_{\alpha\beta} \psi_\beta$ which is the flavour current operator. This is conserved (to all orders) so that its anomalous dimension is zero. This is a restatement of the Ward-Takahashi identity of QED, $Z_1 = Z_2$.

The anomalous dimensions of the singlet operators can be calculated in the same way. To this order γ_{ff}^N is the same as γ_{NS}^N and the other anomalous dimensions are

$$\gamma_{\mathrm{gg}}^N = \frac{g^2}{16\pi^2} \left[2C_A \left(\tfrac{1}{3} - \frac{4}{N(N-1)} + \frac{4}{(N+1)(N+2)} + \sum_{j=0}^{N-2} \frac{4}{j+2} \right) + \frac{8T_R}{3} \right], \tag{3.36a}$$

$$\gamma_{\mathrm{fg}}^N = \frac{g^2}{16\pi^2} 4C_F \left(\frac{1}{(N+1)} + \frac{2}{N(N-1)} \right), \tag{3.36b}$$

$$\gamma_{\mathrm{gf}}^N = \frac{g^2}{16\pi^2} T_R \left(\frac{8}{(N+2)} + \frac{16}{N(N+1)(N+2)} \right). \tag{3.36c}$$

We note that for $N = 2$ we have the relations

$$\gamma_{\mathrm{ff}}^2 - \gamma_{\mathrm{fg}}^2 = \gamma_{\mathrm{gg}}^2 - \gamma_{\mathrm{gf}}^2 = 0. \tag{3.37}$$

This is because the operator combination

$$O_f^2 - O_g^2 = \bar{\psi}\gamma^{\mu_1} D^{\mu_2}\psi - G^{\mu_1\alpha}G^{\mu_2\alpha} \tag{3.38}$$

is the energy momentum tensor which is also conserved to all orders.

We now return to the problem of the calculation of the coefficient function in perturbation theory. The full solution to the renormalization group equations for the moments of structure functions which depend both on singlet and non-singlet operators is

$$\int_0^1 x^{N-1} F_{1(2)}(x, q^2)dx = \frac{1}{2^{N+2}} [C^N_{1(2),NS}(q^2) \langle p|O^N_{NS}|p\rangle + C^N_{1(2),f}(q^2) \langle p|O^N_f|p\rangle$$

$$+ C^N_{1(2),g}(q^2) \langle p|O^N_g|p\rangle], \tag{3.39}$$

where

$$\langle p|O^N_i|p\rangle p_{\mu_1} \cdots p_{\mu_N} = \langle p|O^{\mu_1 \cdots \mu_N}_i|p\rangle \tag{3.40}$$

and $C^N_{1(2),NS}(q^2)$ obeys the equation

$$C^N_{1(2),NS}(q^2) = C^N_{1(2),NS}(q^2 = \mu^2, g(q^2)) [g^2(-q^2)/g^2(\mu^2)]^{\gamma_{NS}{}^N/2\beta_0}. \tag{3.41}$$

The first factor must be calculated in perturbation theory. Since the coefficient functions do not depend on the states between which the operators are sandwiched we may calculate them for scattering off an external quark of momentum p. The tree diagram approximation has been calculated in Section 2 and the diagrams contributing to the next order are shown

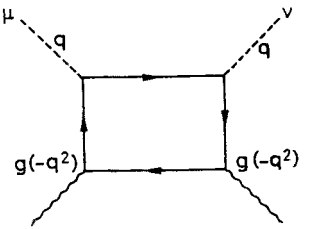

Fig. 12

in Fig. 7. These are calculated using the effective coupling constant $g(-q^2)$ at each vertex and it is necessary to take the moments to calculate $C^N_{1(2),NS}$ or $C^N_{1(2),f}$. In order to calculate $C^N_{1(2),g}$ we must consider electron-gluon scattering. The leading order non-trivial diagram for such a process is shown in Fig. 12. We see that it begins in order $g^2(-q^2)$ and may therefore be neglected to leading order. However the $C^N_{1(2),g}$ term may not be simply thrown away since $C^N_{1(2),f}$ and $C^N_{1(2),g}$ mix through the mixing of the operators in higher order. What we can do is to neglect $C^N_{1(2),g}(q^2 = -\mu^2)$. Doing this and using the elements of the matrix A^N which diagonalizes the matrix γ^N (giving diagonal components γ^N_+ and

γ_-^N) we have

$$C^N_{1(2),f}(q^2) = [(A^N_{f+})^{-1} A^N_{+f}[g^2(-q^2)/g^2(\mu^2)]^{\gamma_+ N/2\beta_0}$$
$$+(A^N_{f-})^{-1} A^N_{-f}[g^2(-q^2)/g^2(\mu^2)]^{\gamma_- N/2\beta_0}]C^N_{1(2),f}(q^2 = -\mu^2), \tag{3.42a}$$
$$C^N_{1(2),g}(q^2) = [(A^N_{f+})^{-1} A^N_{+g}[g^2(-q^2)/g^2(\mu^2)]^{\gamma_+ N/2\beta_0}$$
$$+(A^N_{f-})^{-1} A^N_{-g}[g^2(-q^2)/g^2(\mu^2)]^{\gamma_- N/2\beta_0}]C^N_{1(2),f}(q^2 = -\mu^2). \tag{3.42b}$$

Now if we define

$$\alpha_N = (A^N_{f+})^{-1} A^N_{+f}, \tag{3.43a}$$
$$\beta_N = (A^N_{f+})^{-1} A^N_{+g}, \tag{3.43b}$$

then using the fact that $(A^N)^{-1} A^N = 1$ we have

$$(A^N_{f-})^{-1} A^N_{-f} = 1 - \alpha_N, \tag{3.44a}$$
$$(A^N_{f-})^{-1} A^N_{-g} = -\beta_N, \tag{3.44b}$$

so that combining Eqs. (3.39) to (3.44) we have

$$\int_0^1 x^{N-1} F_{1(2)}(x, q^2)dx = \frac{1}{2^{N+2}} [C^N_{1(2),NS}(q^2 = -\mu^2) \langle p|O^N_{NS}|p\rangle [g^2(-q^2)/g^2(\mu^2)]^{\gamma_{NS} N/2\beta_0}$$
$$+ C^N_{1(2),f}(q^2 = -\mu^2) \langle p|O^N_f|p\rangle \{\alpha_N[g^2(-q^2)/g^2(\mu^2)]^{\gamma_+ N/2\beta_0}$$
$$+(1-\alpha_N)[g^2(-q^2)/g^2(\mu^2)]^{\gamma_- N/2\beta_0}\} + C^N_{1(2),f}(q^2 = -\mu^2) \langle p|O^N_g|p\rangle$$
$$\times \{\beta_N[g^2(-q^2)/g^2(\mu^2)]^{\gamma_+ N/2\beta_0} - \beta_N[g^2(-q^2)/g^2(\mu^2)]^{\gamma_- N/2\beta_0}\}]. \tag{3.45}$$

We interpret the matrix element of the flavour non-singlet operators as the valence quark distribution at $q^2 = -\mu^2$. This is because to a very good approximation the sea is a flavour singlet (i. e. a non-singlet operator has zero matrix elements between states consisting of just the sea). The fermion singlet operator matrix elements are interpreted as the total quark (valence and sea) plus antiquark distributions at $q^2 = -\mu^2$ and the matrix elements of the gluon operators as the gluon distribution at $q^2 = -\mu^2$. We note that the last term of Eq. (3.45) vanishes at $q^2 = -\mu^2$. This is because one cannot directly detect a gluon with a photon. The presence of the last term away from $q^2 = -\mu^2$ arises from the fact that gluons can be indirectly detected in higher orders due to sea pair production.

4. Comparison with experiment

In the previous two sections we have derived formulae relating the moments of structure functions at one value of q^2 to their values at a fixed value q_0^2 (which must be obtained from experiment). Before discussing how to invert these moments to obtain the structure functions we compare the renormalization group predictions with a recent

neutrino scattering experiment at CERN (BEBC) for which the moments of the structure functions have been calculated directly from the experiment. In neutrino scattering it is an off-shell W^+ or W^- which is exchanged and not a photon. This gives rise to parity violating interactions and hence an extra structure function $F_3(x, Q^2)$. It can be shown that for an isoscalar target F_3 is an odd function of x and so depends on the anomalous dimensions of odd spin operators. Gluon operators have only even spin and so there is no mixing between gluon and fermion operators, and the moments of F_3 are governed by the anomalous dimensions $\gamma_{NS}^N = \gamma_{ff}^N$. Thus

$$M_3^N = \int x^{N-1} F_3(x, q^2) dx \sim [\ln(-q^2)]^{\gamma_{NS}^N/2\beta_0}. \tag{4.1}$$

Taking logarithms of both sides we have for two values of N (N_1 and N_2):

$$\ln(M_3^{N_1}) = C_1 + (\gamma_{NS}^{N_1}/2\beta_0) \ln\ln(-q^2), \tag{4.2a}$$

$$\ln(M_3^{N_2}) = C_2 + (\gamma_{NS}^{N_2}/2\beta_0) \ln\ln(-q^2). \tag{4.2b}$$

So that if we plot $\ln(M_3^{N_1})$ against $\ln(M_3^{N_2})$ we expect a straight line of slope $\gamma_{NS}^{N_1}/\gamma_{NS}^{N_2}$. This has been plotted in Fig. 13, where the straight lines are the theoretical values. The success is spectacular.

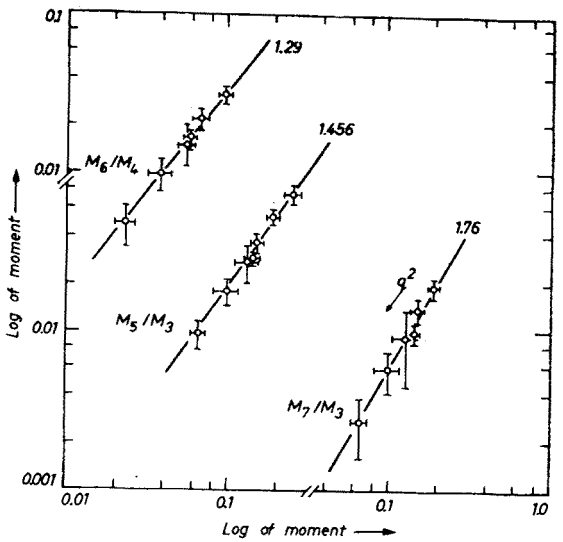

Fig. 13

Inverting moments

In principle, one can obtain the structure functions from their moments by performing an inverse Mellin transform. That means that if we can take all the moments, M^N, of a structure function and make an analytic continuation in N

$$M^N = M(N). \tag{4.3}$$

Then the structure functions may be obtained from the integral

$$F(x) = \int_{-i\infty}^{i\infty} x^N M(N) dN. \tag{4.4}$$

In practice, however, it is impossible to perform the analytic continuation and a much simpler approximate method was developed by Buras and Gaemers. This method essentially consists of guessing the form of the valence quark, sea, and gluon distributions with some parameters and then fitting the parameters to the solution of the renormalization group equation (Eq. (3.45)). Thus they take for the valence quarks

$$V(x) = \frac{3\Gamma(\alpha_1+\alpha_2+1)}{\Gamma(\alpha_1)\Gamma(\alpha_2+1)} x^{\alpha_1-1}(1-x)^{\alpha_2}. \tag{4.5}$$

The normalization factor has been chosen to satisfy the criterion that the total number of valence quarks is three, i. e.

$$\int_0^1 V(x) dx = 3. \tag{4.6}$$

For the sea quark distributions they take

$$S(x) = A(1-x)^{\alpha_3}/x, \tag{4.7}$$

and for gluons

$$G(x) = B(1-x)^{\alpha_4}/x. \tag{4.8}$$

They assume that the q^2 dependence is all contained in the α's and has the form

$$\alpha_i = \alpha_i(q^2 = q_0^2) + \alpha_i' \ln(\ln(-q^2/\Lambda^2)/\ln(-q_0^2/\Lambda^2)). \tag{4.9}$$

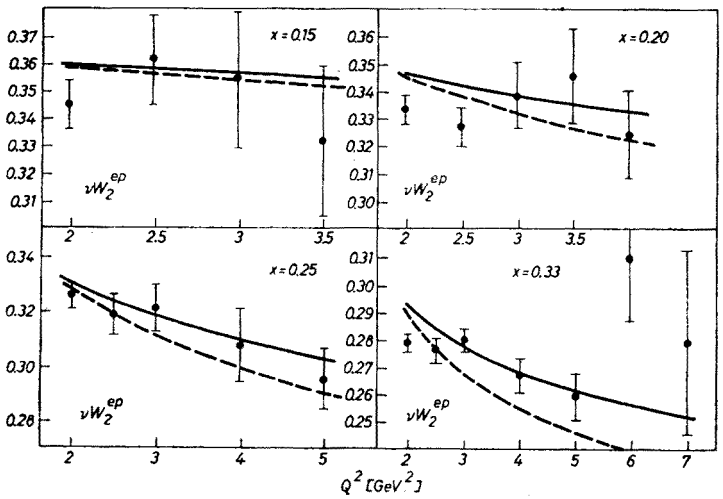

Fig. 14

Now the quantities A, B and $\alpha_i(q^2 = q_0^2)$ are obtained by fitting Eqs. (4.5), (4.7) and (4.8) to electroproduction at $q^2 = q_0^2$ (they take $-q_0^2 = 1.8$ GeV^2 although there is no data at this point for all values of x — what they actually do is to use the renormalization group results to extrapolate the data for all x down to $-q^2 = 1.8$ GeV^2). Having obtained these quantities the slopes α_i' are obtained by fitting to Eq. (3.45) (note that this second step is not a fit to data but a parametrization of the theoretical prediction of the q^2 dependence of the moments).

Figure 14 shows the results of this fitting for low values of x (up to $x = 0.33$). The dotted lines are with $\Lambda = 0.5$ GeV and the solid lines (which give a better fit) are for $\Lambda = 0.3$ GeV. We defer the comparison of theory and experiment for larger values of x until the next sections where we discuss the effects of target masses.

5. Higher order corrections

5.1. Target mass effects

We recall that in Section 2, the operators were classified into representations of the Lorentz group by removing the traces and symmetrizing over Lorentz indices (such a classification is necessary in the renormalization group analysis since it is only operators of the same spin and dimension which mix in higher orders — in other words the index N, in γ^N refers to the spin of an operator and thus assumes that the operators have well defined spin). Later the trace terms were dropped since they gave rise to terms of order m_p^2/q^2, where m_p is the proton mass. Georgi and Politzer have performed an analysis in which these trace terms have been taken into account. By looking at the parton model Barbieri et al. have obtained the same results. They assume that the parton which is struck (see Fig. 5) carries a fraction ξ (not $x = -q^2/2p \cdot q$) of the momentum of the proton. Now in the infinite momentum frame ($|\boldsymbol{p}| \gg m_p$) the proton has momentum

$$p = ((|\boldsymbol{p}|^2 + m_p^2)^{1/2}, 0, 0, |\boldsymbol{p}|), \tag{5.1}$$

and the struck quark has momentum

$$p_i = (\xi|\boldsymbol{p}|, 0, 0, \xi|\boldsymbol{p}|). \tag{5.2}$$

For simplicity we continue to neglect quark masses although both Georgi and Politzer and Barbieri et al. have extended their analyses to include quark masses. The momentum transfer, in this frame, is

$$\begin{aligned} q = (((|\boldsymbol{p}|^2 + m_p^2)^{1/2} p \cdot q - m_p((p \cdot q)^2 - m_p^2 q^2)^{1/2})/m_p^2, 0, 0, (|\boldsymbol{p}| p \cdot q \\ - (|\boldsymbol{p}|^2 + m_p^2)^{1/2}((p \cdot q)^2 - m_p^2 q^2)^{1/2})/m_p^2). \end{aligned} \tag{5.3}$$

As $|\boldsymbol{p}| \to \infty$ we may rewrite this

$$q = \left(\frac{|\boldsymbol{p}|\,(p \cdot q - ((p \cdot q)^2 - m_p^2 q^2)^{1/2})}{m_p^2}, 0, 0, p \cdot q - ((p \cdot q)^2 - m_p^2 q^2)^{1/2} \left(\frac{|\boldsymbol{p}|}{m_p^2} - \frac{1}{2|\boldsymbol{p}|} \right) \right). \tag{5.4}$$

Now the requirement that the final struck quark momentum must be on the mass shell

$$(p_i+q)^2 = 0 \tag{5.5}$$

gives us for ξ

$$\xi = [(p\cdot q)-(p\cdot q-((p\cdot q)^2-m_p^2q^2)^{1/2})]/m_p^2 = \frac{2x}{1+(1+4m_p^2x^2/(-q^2))^{1/2}}\,. \tag{5.6}$$

We notice first that as $m_p^2/q^2 \to 0$, ξ approaches x so that this is a low q^2 correction. Secondly we notice that for fixed q^2, ξ approaches x as $x \to 0$, so that these corrections are also more important at large x.

What this means is that we should write $F_{1(2)}(x, q^2)$ as a function $\tilde{F}_{1(2)}(\xi, q^2)$ and it is this function $\tilde{F}_{1(2)}$ which scales in lowest order and whose moments have scaling violations which go as powers of logarithms (owing to higher order corrections). Actually of the electromagnetic currents and the fact that F_1 and F_2 and the coefficients of specific Lorentz structures. For examples, the full result for F_2 is

$$F_2(x, q^2) = \frac{x}{[1+4x^2m_p^2/(-q^2)]^{3/2}\xi^2}\tilde{F}_2(\xi, q^2)+(6m_p^2/(-q^2))\frac{x^2}{[1+4x^2m_p^2/(-q^2)]^2}$$

$$\int_\xi^1 \frac{d\xi'}{\xi'^2}\tilde{F}_2(\xi', q^2)+(12m_p^4/(-q^2)^2)\frac{x^3}{[1+4x^2m_p^2/(-q^2)]^{5/2}}\int_\xi^1 d\xi'\int_{\xi'}^1 \frac{d\xi''}{\xi''^2}\tilde{F}_2(\xi'', q^2) \tag{5.7}$$

with a similarly complicated expression for F_1.

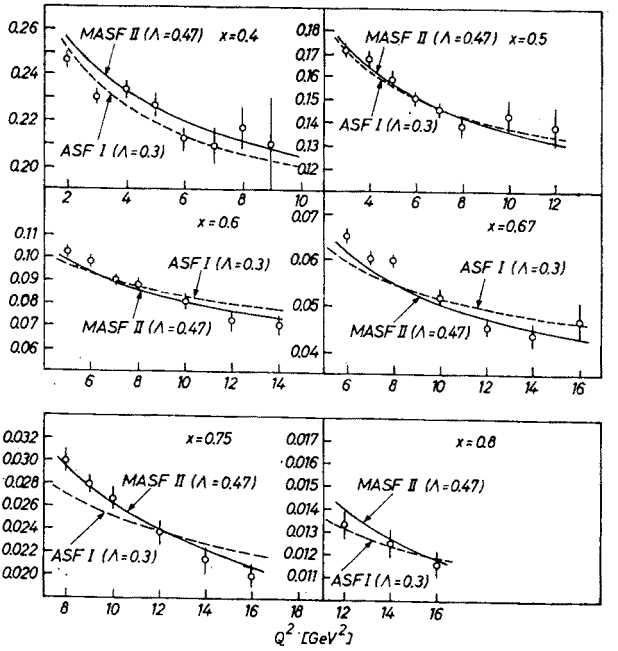

Fig. 15

One should note at this point that it is not strictly consistent to consider the target mass effects without also considering the effects of operators of higher twist, which are also $O(m_p^2/q^2)$. This is still an unsolved problem; however it is believed that these higher twist operators only became relevant for very high $x(> 0.8)$ and that target mass effects alone should be sufficient below $x = 0.8$. Figure 15 shows the comparison of theory and data for $0.4 < x < 0.8$. The broken line is the theoretical curve without target mass effects and the solid contains both target mass effects and the next order in the solution to the renormalization group equations (discussed at the end of this section). The improvement in the agreement with data is mainly due to the target mass effects.

5.2. Violation of the Callan-Gross relation

In Section 3 we derived in the tree diagram approximation the relation between F_1 and F_2 (see Eq. (2.7))

$$F_2 = 2xF_1 . \tag{5.8}$$

This is known as the Gallan-Gross relation.

In the parton model this could be avoided if the struck parton has a component of momentum transverse to the momentum of the proton, as well as a component $x|\boldsymbol{p}|$, along it (for the moment we neglect target mass effects). The momentum of the struck quark is now

$$p_i = ((x^2|\boldsymbol{p}|^2 + p_T^2)^{1/2}, 0, p_T, x|\boldsymbol{p}|) \tag{5.9}$$

and Eq. (2.4) becomes

$$\frac{l_{\mu\nu}}{q^4} \tfrac{1}{2} \operatorname{Tr} [\gamma_\mu(x\gamma \cdot p + \gamma \cdot p_T + \gamma \cdot q)\gamma_\nu(x\gamma \cdot p + \gamma \cdot p_T)]\delta(q^2 + 2p \cdot qx), \tag{5.10}$$

where $p \cdot p_T = q \cdot p_T = 0$. This leads to the relation

$$\frac{2xF_1 - F_2}{2xF_1} = \frac{4\langle p_T \rangle^2}{-q^2} , \tag{5.11}$$

where $\langle p_T^2 \rangle$ is the average square transverse momentum of the quarks.

The relation (5.8) is also obeyed when leading order scaling violations are taken into account. This is because, to leading order, the coefficient functions at $q^2 = -\mu^2$ are simply those obtained from the tree diagram approximation, whereas the operators whose anomalous dimensions govern the q^2 dependence of the moments of the structure functions are the same for both F_1 and F_2. If we go to subleading order then we must calculate the coefficient function to one loop. The diagrams for this are shown in Fig. 7. The diagram of Fig. 7c gives a different contributions to F_1 and F_2. This can be related to the notion of transverse momentum in the parton by observing that even if the external quark has zero transverse momentum, before it is struck by the photon it emits a gluon which may in general have a large transverse momentum so that the quark also carries a large (equal and opposite) transverse momentum when it interacts with the photon. It is not necessary

to take moments of this diagram. If we write the contribution from this diagram at $q^2 = -\mu^2$ as

$$(-g_{\mu\nu}+q_\mu q_\nu/q^2)\frac{g^2(-q^2)}{16\pi^2}\Delta_1(x)+\frac{2x}{p\cdot q}\left(p_\mu-\frac{p\cdot q}{q^2}q_\mu\right)\left(p_\nu-\frac{p\cdot q}{q^2}q_\nu\right)\frac{g^2(-q^2)}{16\pi^2}\Delta_2(x) \tag{5.12}$$

and

$$\Delta_{12}(x) = \Delta_1(x)-\Delta_2(x). \tag{5.13}$$

Then to leading order we have

$$2F_1(x,q^2)-\frac{F_2(x,q^2)}{x} = \int_x^1 \frac{dy}{y} F_2(y,q^2)\,(g^2(-q^2)/16\pi^2)\Delta_{12}(x/y). \tag{5.14}$$

This can be proved by taking the N^{th} moment of both sides of Eq. (5.14)

$$2M_1^N(q^2)-M_2^N(q^2) = M_2^N(q^2)\int_0^1 z^{N-1}(g^2(-q^2)/16\pi^2)\Delta_{12}(z), \tag{5.15}$$

where we have put $z = x/y$. The second factor on the RHS of Eq. (5.15) is simply $C_1^N-C_2^N$, the difference in the coefficient functions of the spin N operators for the Wilson expansions for the structure functions F_1 and F_2 (to leading non-trivial order).

Target mass effects will also increase the quantity R, where

$$R = \frac{2xF_1-F_2}{2xF_1}. \tag{5.16}$$

The experimental data on R is not very good. Nevertheless we can see from Fig. 16 that the theoretical prediction for this quantity is consistently too low, even when the increases due to target mass effects (solid line) are included. It is possible that the inclusion of higher twist operators may improve the situation.

5.3. Other higher order corrections

The solution to the renormalization group equation for the coefficient function of the spin N non-singlet operator is

$$C_{\text{NS}}^N(q^2) = C_{\text{NS}}^N(q^2=\mu^2, g(-q^2))\exp-\int_{g(\mu^2)}^{g(-q^2)}\gamma_{\text{NS}}^N(g')/\beta(g')dg'. \tag{5.17}$$

$C_{\text{NS}}^N(q^2 = -\mu^2, g(-q^2))$ may be written

$$\overset{(0)}{C}{}_{\text{NS}}^N[1+f_{\text{NS}}^N g^2(-q^2)/(16\pi^2)], \tag{5.18}$$

where the coefficients f_{NS}^{N} come from the one loop correction calculated in ordinary perturbation theory. If we expand γ_{NS}^{N} and β to two loops we get

$$\gamma_{NS}^{N}(g') = \gamma_0^N g'^2/(16\pi^2)+\gamma_1^N g'^4/(16\pi^2), \tag{5.19a}$$

$$\beta(g') = -\beta_0 g'^3/(16\pi^2)^2-\beta_1 g'^5/(16\pi^2)^2. \tag{5.19b}$$

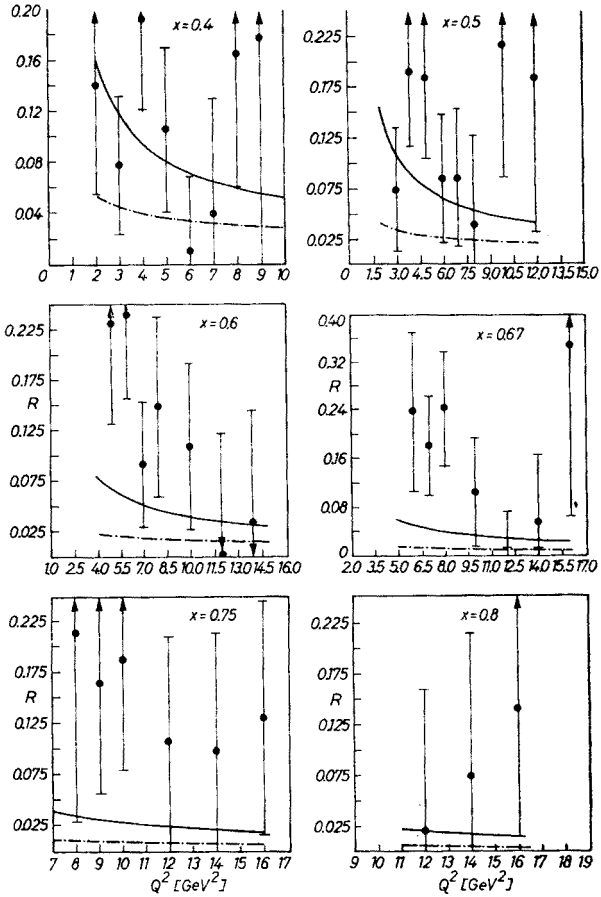

Fig. 16

Now the exponential in Eq. (5.17) may be written as the product of two factors

$$\exp-\int_{g(\mu^2)}^{g(-q^2)} dg'\gamma_{NS}^{N}(g')/\beta(g')$$

$$= \exp\int_{g(\mu^2)}^{g(-q^2)} \frac{dg'}{g'}(\gamma_0^N/\beta_0)\left\{1+(\gamma_1^N/\beta_0-\gamma_0^N\beta_1/\beta_0^2)\int_{g(\mu^2)}^{g(-q^2)} dg'g'/(16\pi^2)\right\}, \tag{5.20}$$

where in the second factor we have expanded the exponential to first order. We note that for the quantity R discussed above this factor is the same for the moments of both F_1 and F_2 so that it was only necessary to calculate the differences of the quantities $f_{\rm NS}^N$ of Eq. (5.18). Using Eqs (5.18) and (5.20) in Eq. (5.17) we have

$$C_{\rm NS}^N(q^2) = \overset{(0)}{C}{}_{\rm NS}^N[g^2(-q^2)/g^2(\mu^2)]^{\gamma_0^N/2\beta_0}\{1+f_{\rm NS}^N g^2(-q^2)/(16\pi^2)$$

$$+(\gamma_1^N/2\beta_0-\gamma_0^N\beta_1/2\beta_0^2)\,[g^2(-q^2)/16\pi^2-g^2(\mu^2)/(16\pi^2)]\}. \tag{5.21}$$

If we are working consistently to two loops for $\gamma_{\rm NS}^N$ and β then the coupling constant $g^2(-q)^2$ must be calculated from the solutions to the β-function equation also taken to two loops, i. e.

$$2\,\frac{\partial g(-q^2)}{\partial \ln(-q^2)} = -\beta_0 g^3(-q^2)/(16\pi^2)-\beta_1 g^5(-q^2)/(16\pi^2)^2 \tag{5.22}$$

which gives us

$$g^2(-q^2) = 16\pi^2/\beta_0 \ln(-q^2/\Lambda^2)-(\beta_1/\beta_0^3)16\pi^2 \ln(\ln(-q^2/\Lambda^2))/\ln^2(-q^2/\Lambda^2). \tag{5.23}$$

The quantity β has been calculated by Jones and Caswell (independently). For the non-singlet operators the quantities γ_1^N have also been calculated (by my collaborators and myself). They are found to be renormalization prescription dependent. That means they are different if you use the method of dimensional regularization and perform the subtraction by removing the pole parts (as discussed in Section 3) from those obtained by subtracting the operator matrix elements at a given value of the external momentum ($p^2 = -\mu^2$). However, Eq. (5.21) is clearly independent of the renormalization prescription since it describes a physical quantity and it turns out that the quantities $f_{\rm NS}^N$ also have a renormalization prescription dependence which exactly cancels those of γ_1^N.

For the singlet operators the situation is more complicated because of the mixing of the fermion and gluon operators. In higher orders the matrix which diagonalizes the anomalous dimension itself depends on the coupling constant and this must be taken into account when solving the renormalization group equations. Furthermore, the calculation of the anomalous dimensions of the singlet operators in two loops has not yet been completed. However, we believe that for large x ($x > 0.4$) it is sufficient to consider the non-singlet operators only since it is found that the sea and gluons have distributions which are large at very low x and fall extremely rapidly as x grows (i. e. the indices α_3 and α_4 of Eqs. (4.7) and (4.8) are much larger than α_2 for Eq. (4.5). This leaves only the valence quarks at high x, and as shown in Section 3 the valence-quark distribution is controlled by the non-singlet operators. We find that although the individual corrections in Eq. (5.21) are quite large, when we taken them all together (and include Eq. (5.23)) the total change due to higher order corrections is rather small, due to cancellations between various components.

BIBLIOGRAPHY

I have not given reference numbers in the text. I present here a list of references under various subheadings discussed in the lectures. It is not my intention that this list should be exhaustive but merely some of the more important contributions to the subject, which I recommend for further reading. Furthermore, some of the papers listed under one subheading contain material relevant to other subheadings.

The renormalization group

C. G. Callan, *Phys. Rev.* **D2**, 1541 (1970).
K. Symanzik, *Commun. Math. Phys.* **18**, 227 (1970).
M. Gell-Mann, F. Low, *Phys. Rev.* **95**, 1300 (1954).
E. G. G. Stuekelberg, A. Peterman, *Helv. Phys. Acta* **26**, 499 (1953).

Asymptotic freedom

H. D. Politzer, *Phys. Rev. Lett.* **30**, 1346 (1973).
H. D. Politzer, *Phys. Report* **14**, 129 (1974).
D. J. Gross, F. Wilczek, *Phys. Rev. Lett.* **30**, 1323 (1973).
D. J. Gross, F. Wilczek, *Phys. Rev.* **D8**, 3633 (1973).
D. J. Gross, F. Wilczek, *Phys. Rev.* **D9**, 980 (1973).

Application to deep inelastic scattering

D. Bailin, A. Love, D. V. Nanopoulos, *Lett. Nuovo Cimento* **9**, 501 (1974).
D. Nanopoulos, G. G. Ross, *Phys. Lett.* **58B**, 105 (1975).
G. Parisi, R. Petronzio, *Phys. Lett.* **62B**, 331 (1976).
G. Altarelli, R. Petronzio, G. Parisi, *Phys. Lett.* **63B**, 183 (1976).
M. Glück, E. Reya, *Phys. Lett.* **64B**, 169 (1976).
M. Glück, E. Reya, *Phys. Rev.* **D14**, 3034 (1976).
R. K. Ellis, R. Petronzio, G. Parisi, *Phys. Lett.* **64B**, 97 (1976).
R. M. Barnett, H. Georgi, H. D. Politzer, *Phys. Rev. Lett.* **37**, 313 (1976).
G. Fox, *Nucl. Phys.* **B131**, 107 (1977).
G. Fox, *Nucl. Phys.* **B134**, 269 (1978).
I. Hinchcliffe, C. Llewellyn Smith, *Phys. Lett.* **66B**, 281 (1977).
I. Hinchcliffe, C. Llewellyn Smith, *Nucl. Phys.* **B128**, 93 (1977).
V. Baluni, E. Eichten, *Phys. Rev. Lett.* **37**, 1181 (1976).
A. J. Buras, *Nucl. Phys.* **B128**, 125 (1977).
A. J. Buras, K. J. K. Gaemers, *Nucl. Phys.* **B132**, 249 (1978).

Experimental data

E. M. Riordan et al., SLAC Report No. SLAC-PUB-1634 (1975).
Y. Watnabe et al., *Phys. Rev. Lett.* **35**, 898 (1975).
Y. Watnabe et al., *Phys. Rev. Lett.* **35**, 901 (1975).
H. L. Anderson et al., *Phys. Rev. Lett.* **38**, 1450 (1977).
H. C. Barish et al., *Phys. Rev. Lett.* **39**, 1595 (1977).
B. C. Barish et al., *Phys. Rev. Lett.* **40**, 1414 (1977).
R. E. Taylor, *Proc. Internat. Symposium on Leptons and Photo Interactions at High Energies*, Stanford 1975, ed. W. T. Kirk, Stanford University 1975, p. 679.
P. C. Bosetti et al., Oxford University preprint No. 16/78 (1978).

Mass effects

O. Nachtman, *Nucl. Phys.* **B63**, 237 (1973).
H. Georgi, H. D. Politzer, *Phys. Rev.* **D14**, 1829 (1976).
S. Wandzura, *Nucl. Phys.* **B42**, 412 (1977).
R. G. Barbieri, J. Ellis, M. K. Gaillard, G. G. Ross, *Phys. Lett.* **64B**, 178 (1976).
R. G. Barbieri, J. Ellis, M. K. Gaillard, G. G. Ross, *Nucl. Phys.* **B117**, 50 (1976).

Higher order corrections

D. R. T. Jones, *Nucl. Phys.* **B75**, 531 (1974).
W. Caswell, *Phys. Rev. Lett.* **33**, 224 (1974).
M. Calvo, *Phys. Rev.* **D15**, 730 (1977).
A. De Rujula, H. Georgi, H. D. Politzer, *Ann. Phys.* (USA) **103**, 315 (1977).
E. G. Floratos, D. A. Ross, C. T. Sachrajda, *Nucl. Phys.* **B124**, 66 (1977).
A. J. Buras, E. G. Floratos, D. A. Ross, C. T. Sachrajda, *Nucl. Phys.* **B131**, 308 (1977).
W. A. Bardeen, A. J. Buras, D. N. Duke, T. Muta, FERMILAB PUB 78/42 THY (1978).