EQUATORIAL GEODESIC MOTION OF TEST PARTICLES IN THE KERR AND FIRST TOMIMATSU–SATO METRICS

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Four crucial effects of General Relativity in the Kerr and first Tomimatsu–Sato metrics are calculated by a new method. A possible time-delay test of General Relativity in the gravitational field of a rotating mass is considered.

1. *Introduction*

From the standpoint of possible astrophysical applications it is of great interest to investigate the geodesic motion and observable effects in those axi-symmetric space-times which are regarded as representing the gravitational fields of spinning masses. Not long ago the Kerr metric [1] has been the only known example of this type of solutions of the Einstein equations. It is generally accepted that this metric describes space-time outside a gravitational mass m with specific angular momentum a related to the total angular momentum by $L = ma$, and to the quadrupole momentum by $Q = ma^2$. It is still not clear what is the physical nature of the Kerr metric's source (see, for example, [2, 3]). Therefore, the a^2 and higher order terms are dubious to some extent (in the linear approximation the Kerr metric coincides with the Lense–Thirring solution, being interpreted as a gravitational field of a slowly rotating mass).

For this reason it is interesting to calculate the crucial effects of General Relativity with greater accuracy by including higher powers of m and a. An experimental test of these effects may help to decide whether the Kerr metic is suitable for describing the gravitational fields of real celestial objects. Such an investigation is necessary because the opinion is sometimes expressed, that the Kerr metric does not describe the gravitational field outside the Sun [4]. In fact, if $a = L_\odot/m_\odot$ the quadrupole momentum corresponding to the Kerr metric is equal to $Q = 5 \cdot 10^{-19} r_\odot^3$ ($r_\odot$ being the Sun radius), that is much less than the observed upper limit $Q_\odot \lesssim 10^{-10} r_\odot$ [4]. (Of course, such a situation may be changed

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after more accurate measurements of $L_\odot$ and $Q_\odot$ through SOREL or "solar probe" programs [5, 6].)

In view of the above mentioned deficiencies of the Kerr metric, it is desirable to investigate the geodesic motion in recently discovered Tomimatsu–Sato (TS) family of solutions [7, 8]. The remarkable feature of these space-times is that the quadrupole momentum $Q_{TS\delta}$ for TS metrics increases with the growth of the dimensionless parameter δ (the parameter which classifies a member of the TS series) related to the quadrupole momentum of the source. The Kerr metric is the simplest member of the TS series and corresponds to $\delta = 1$. The geodesic motion in the cases $\delta = 2, 3$ has already been investigated by some authors but primarily by numerical methods. Besides, these results have only a qualitative character (see, for example, [9, 10]).

This article deals with the geodesic motion of test particles and four crucial effects of General Relativity in the equatorial planes of $\delta = 1$ and $\delta = 2$ TS space-times, which correspond to the Kerr and the first Tomimatsu–Sato metrics respectively. In Section 2 we find the line elements for these metrics with an accuracy up to $(m/r)^4$. In Sections 3–6 the expressions for perihelion precession, deflection of the light ray, time-delay and redshift are obtained with an accuracy up to $(m/r)^3$. In Sections 3–5 we also discuss a new method for solving the trajectory differential equation and calculating the crucial effects. Section 7 deals with an analysis of the obtained results.

2. *TS metrics for $\delta = 1$ and $\delta = 2$*

The most general line element for a stationary axi-symmetric space-time may be written in the Weyl–Papapetrou form

$$ds^2 = f(dx^0 - \omega d\varphi)^2 - f^{-1}[e^{2\gamma}(d\varrho^2 + dz^2) + \varrho^2 d\varphi^2], \tag{1}$$

by use of canonical cylindrical coordinates ϱ and z. Here f, ω and γ are functions of ϱ and z only. The TS family of space-times is defined by the following expressions:

$$f = \frac{A}{B}, \quad \omega = \frac{2mq}{A}(1-y^2)C, \quad e^{2\gamma} = \frac{A}{p^{2\delta}(x^2-y^2)^{\delta^2}}, \tag{2}$$

where p and q are dimensionless parameters related to each other by $p^2+q^2 = 1$, and the polynomials A, B and C are as follows:

$$\delta = 1: \quad A = p^2(x^2-1) - q^2(1-y^2), \quad B = (px+1)^2 + q^2y^2, \quad C = -(px+1), \tag{3a}$$

$$\begin{aligned}\delta = 2: \quad A &= p^4(x^2-1)^4 + q^4(1-y^2)^4 - 2p^2y^2(x^2-1)(1-y^2)[2(x^2-1)^2 \\ &\quad + 2(1-y^2)^2 + 3(x^2-1)(1-y^2)], \\ B &= [p^2(x^4-1) - q^2(1-y^4) + 2px(x^2-1)]^2 + 4q^2y^2[px(x^2-1) \\ &\quad + (px+1)(1-y^2)]^2, \\ C &= -p^3x(x^2-1)[2(x^4-1) + (x^2+3)(1-y^2)] - p^2(x^2-1)[4x^2(x^2-1) \\ &\quad + (3x^2+1)(1-y^2)] + q^2(px+1)(1-y^2)^3.\end{aligned} \tag{3b}$$

The prolate spheroidal coordinates (x, y) may be related to cylindrical coordinates (ϱ, z) and Schwarzschild-like coordinates (r, θ) through the relations

$$\varrho^2 = D^2(x^2-1)(1-y^2) = (r^2-2mr+m^2q^2)\sin^2\theta,$$
$$z = Dxy = (r-m)\cos\theta, \quad D = mp\delta^{-1}. \tag{4}$$

For futher investigations it is more convenient to use (r, θ). After substituting Eq. (4) into (1) the line element transforms as follows

$$ds^2 = f(dx^0-\omega d\varphi)^2 - f^{-1}e^{2\gamma}E(dr^2+\Delta d\theta^2) - f^{-1}\Delta\sin^2\theta d\varphi^2, \tag{5}$$

where

$$E = 1+m^2p^2\Delta^{-1}\sin^2\theta, \quad \Delta = r^2-2mr+m^2q^2.$$

By calculating the Christoffel symbols $\Gamma^{\lambda}{}_{\mu\nu}$ it is not difficult to conclude that the equatorial geodesic lines are plane curves. In fact, the geodesic equation (the dot means differentiation with respect to the affine parameter)

$$\ddot{x}^{\lambda} + \Gamma^{\lambda}{}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0$$

for a test particle with the initial values $\theta = \pi/2$ and $\dot{\theta} = 0$ gives $d^{(n)}\theta/ds^{(n)} = 0$ for an arbitrary $n \geqslant 2$. So, there is no departure of the test particle from the equatorial plane, which is present, for example, in the Newman–Unti–Tamburino space-time [11]. Therefore, we shall restrict ourselves to geodesic motion in the equatorial plane $y = 0$.

In the equatorial plane polynomials A, B and C read:

$$\delta = 1: \quad A = p^2(x^2-1)-q^2, \quad B = (px+1)^2, \quad C = -(px+1), \tag{6a}$$

$$\delta = 2: \quad A = p^4(x^2-1)^4+q^4-2p^2q^2(x^2-1)[2(x^2-1)^2+3x^2-1],$$
$$B = [p^2(x^4-1)-q^2+2px(x^2-1)]^2,$$
$$C = -p^3x(x^2-1)[2(x^4-1)+x^2+3]-p^2(x^2-1)[4x^2(x^2-1)$$
$$+3x^2+1]+q^2(px+1). \tag{6b}$$

By taking into account $y = 0$ we have from Eq. (4)

$$x^2-1 = \Delta D^{-2}.$$

Substituting this to Eqs. (6a), (6b) we find the polynomials A, B and C in the next form:

$$\delta = 1: \quad A = \frac{r^2}{m^2}\left(1-\frac{2m}{r}\right), \quad B = \frac{r^2}{m^2}, \quad C = -\frac{r}{m}, \tag{7a}$$

and with necessary accuracy:

$$\delta = 2: \quad A = \frac{p^4r^8}{D^8}\left[1-\frac{8m}{r}+\frac{3m^2(8+q^2)}{r^2}-\frac{2m^3(16+9q^2)}{r^3}+\frac{m^4(128+285q^2+27q^4)}{8r^4}\right],$$

$$B = \frac{p^4r^8}{D^8}\left[1-\frac{6m}{r}+\frac{3m^2(4+q^2)}{r^2}-\frac{m^3(31+49q^2)}{4r^3}+\frac{3m^4(-2+33q^2+9q^4)}{8r^4}\right],$$

$$C = -\frac{mp^4r^7}{D^8}\left[1-\frac{6m}{r}+\frac{m^2(49+11q^2)}{4r^2}-\frac{m^3(41+179q^2)}{16r^3}\right]. \tag{7b}$$

Therefore, for the functions f, ω and $e^{2\gamma}$ defined by Eq. (2) one finds

$$\delta = 1: \quad f = 1 - \frac{2m}{r}; \quad \omega = -\frac{2m^2 q}{r}\left(1 - \frac{2m}{r}\right)^{-1}; \quad e^{2\gamma} = \left(1 - \frac{2m}{r}\right)\left(1 - \frac{m}{r}\right)^{-2}, \tag{8a}$$

$$\delta = 2: \quad f = 1 - \frac{2m}{r} + \frac{m^3(q^2-1)}{4r^3} + \frac{m^4(q^2-1)}{4r^4},$$

$$\omega = -\frac{2m^2 q}{r}\left[1 + \frac{2m}{r} - \frac{m^2(q^2-17)}{4r^2}\right] \tag{8b}$$

and

$$e^{2\gamma} = 1 - \frac{m^2}{r^2} - \frac{2m^3}{r^3} + \frac{3m^4(q^2-9)}{8r^4}.$$

The line elements for $\delta = 1$ and $\delta = 2$ TS space-times, which are called the TS1 and TS2 metrics hereafter, are given by the following expressions:

$$ds^2_{\mathrm{TS1}} = \left(1 - \frac{2m}{r}\right)\left[dx^0 + \frac{2m^2 q}{r}\left(1 - \frac{2m}{r}\right)^{-1} d\varphi\right]^2$$

$$- \left[1 + \frac{m^2(1-q^2)}{\Delta}\right]\left(1 - \frac{m}{r}\right)^{-2} dr^2 - \Delta\left(1 - \frac{2m}{r}\right)^{-1} d\varphi^2,$$

$$ds^2_{\mathrm{TS2}} = \left[1 - \frac{2m}{r} + \frac{m^3(q^2-1)}{4r^3} + \frac{m^4(q^2-1)}{4r^4}\right]\left\{dx^0 + \frac{2m^2 q}{r}\left[1 + \frac{2m}{r} - \frac{m^2(q^2-17)}{4r^2}\right] d\varphi\right\}^2$$

$$- \left[1 + \frac{m^2(1-q^2)}{\Delta}\right]\left[1 - \frac{m^2}{r^2} - \frac{2m^3}{r^3} + \frac{3m^4(q^2-9)}{8r^4}\right]\left[1 - \frac{2m}{r} + \frac{m^3(q^2-1)}{4r^3}\right.$$

$$\left. + \frac{m^4(q^2-1)}{4r^4}\right]^{-1} dr^2 - \Delta\left[1 - \frac{2m}{r} + \frac{m^3(q^2-1)}{4r^3} + \frac{m^4(q^2-1)}{4r^4}\right]^{-1} d\varphi^2.$$

Putting the quadrupole momentum for TS series as follows (see [7, 12])

$$Q_{\mathrm{TS}\delta} = m^3\left(\frac{\delta^2 - 1}{3\delta^2} p^2 + q^2\right), \tag{9}$$

we obtain

$$Q_{\mathrm{TS1}} = m^3 q^2, \quad Q_{\mathrm{TS2}} = m^3 q^2 + \tfrac{1}{4} m^3(1-q^2) = Q_{\mathrm{TS1}} + Q,$$

where Q is an additional quadrupole momentum associated with the growth of the value of δ. Then, one can easily find that

$$f_{\mathrm{TS2}} = f_{\mathrm{TS1}} + \frac{Q}{r^3}\left(1 + \frac{m}{r}\right), \quad \omega_{\mathrm{TS2}} = \omega_{\mathrm{TS1}} - \frac{2mQq}{r^3}, \quad e^{2\gamma}_{\mathrm{TS2}} = e^{2\gamma}_{\mathrm{TS1}} + \frac{3mQ}{2r^4}.$$

These expressions are suitable to compare the line elements of investigated metrics.

3. *Calculation of time-like geodesics*

Let us consider the geodesic motion of test particles in the equatorial planes of TS1 and TS2 gravitational fields. Putting the Lagrangian of test particle in TS metric (5) into the following form, $L = -\frac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$, we obtain a set of first integrals of motion

$$1 = f(\dot{x}^0 - \omega\dot{\varphi})^2 - f^{-1}e^{2\gamma}E\dot{r}^2 - f^{-1}\Delta\dot{\varphi}^2, \quad (\Delta f^{-1} - f\omega^2)\dot{\varphi} + \omega f\dot{x}^0 = h,$$

$$f(\dot{x}^0 - \omega\dot{\varphi}) = k, \tag{10}$$

where the constants k and h denote respectively the energy and z-component of the particle's angular momentum divided by the particle's proper mass. As a consequence of the elimination of time from Eqs. (10) and the introduction of a new variable $u = 1/r$ the trajectory differential equation for an arbitrary value of δ is derived

$$\left(\frac{du}{d\varphi}\right)^2 = \frac{\Delta}{Ee^{2\gamma}}\left[\frac{\Delta(k^2 - f)}{(h - \omega k)^2 f^2} - 1\right]u^4. \tag{11}$$

In particular cases of TS1 and TS2 metrics and in assumption that $\omega k/h < 1$ the last equation transforms to

$$\left(\frac{du}{d\varphi}\right)^2_{\mathrm{TS1}} = \frac{k^2-1}{h^2} + \frac{2m}{h^2}\left[1 - \frac{2mqk(k^2-1)}{h}\right]u - \left[1 - \frac{3m^2q^2(k^2-1)}{h^2} + \frac{8m^3qk^3}{h^3} - \frac{12m^4q^2k^2(k^2-1)}{h^4}\right]u^2 + 2m\left[1 + \frac{3m^2q^2k^2}{h^2} - \frac{8m^3qk}{h^3} + \frac{12m^4q^2k^2}{h^4} - \frac{2m^3q(4+3q^2)k(k^2-1)}{h^3}\right]u^3 - 2m^2q^2\left(1 - \frac{6m^2}{h^2}\right)u^4, \tag{12a}$$

$$\left(\frac{du}{d\varphi}\right)^2_{\mathrm{TS2}} = \frac{k^2-1}{h^2} + \frac{2m}{h^2}\left[1 - \frac{2mqk(k^2-1)}{h}\right]u - \left[1 - \frac{3m^2q^2(k^2-1)}{h^2} + \frac{8m^3qk^3}{h^3} - \frac{12m^4q^2k^2(k^2-1)}{h^4}\right]u^2 + 2m\left[1 + \frac{m^2(q^2-1)}{8h^2} + \frac{m^2(11q^2+1)k^2}{4h^2} - \frac{8m^3qk}{h^3} + \frac{12m^4q^2k^2}{h^4} - \frac{m^3q(17+11q^2)k(k^2-1)}{2h^3}\right]u^3 - 2m^2\left[q^2 - \frac{m^2(43q^2+5)}{8h^2}\right]u^4 \tag{12b}$$

by taking into account Eqs. (8a), (8b).

We shall utilize a simple method which has recently been developed in [13, 14] to solve Eqs. (12a), (12b) approximately. It is based on Darwin's idea [15], that is to decompose equations of this type into two algebraic equations which give the relationship between (k, h) and the source and trajectory parameters, and a differential equation in order to find the relativistic anomaly. Some realizations of this idea were proposed in [16, 17] Here we shall follow the alternative approach from [13, 14].

Let us express the solution of Eqs. (12a), (12b) as a quasiconical section

$$u = \frac{1}{r} = \frac{1}{p}(1+e\cos\psi), \tag{13}$$

where p and e are respectively the focal parameter and the eccentricity. The relativistic anomaly $\psi = \psi(\varphi)$ is an unknown function of the classical anomaly φ. By substituting Eq. (13) into (12a) (or (12b)) and setting in the resultant equation $\cos\psi = +1$ and $\cos\psi = -1$ we obtain two independent algebraic equations:

$$0 = \frac{k^2-1}{h^2} + \frac{2m}{h^2}\left[1 - \frac{2mqk(k^2-1)}{h}\right]\frac{(1\pm e)}{p} - \left[1 - \frac{3m^2q^2(k^2-1)}{h^2} + \frac{8m^3qk^3}{h^3} - \frac{12m^4q^2k^2(k^2-1)}{h^4}\right]\frac{(1\pm e)^2}{p^2} + 2m\left[1 + \frac{3m^2q^2k^2}{h^2} - \frac{8m^3qk}{h^3} + \frac{12m^4q^2k^2}{h^4} - \frac{2m^3q(4+3q^2)k(k^2-1)}{h^3}\right]\frac{(1\pm e)^3}{p^3} - 2m^2q^2\left(1 - \frac{6m^2}{h^2}\right)\frac{(1\pm e)^4}{p^4},$$

which after the addition and subtraction may be rewritten as following system to determine k, h through p, e and vice versa:

$$0 = \frac{k^2-1}{h^2} + \frac{2m}{ph^2}\left[1 - \frac{2mqk(k^2-1)}{h}\right] - \left[1 - \frac{3m^2q^2(k^2-1)}{h^2} + \frac{8m^3qk^3}{h^3} - \frac{12m^4q^2k^2(k^2-1)}{h^4}\right]\frac{1+e^2}{p^2} + \frac{2m(1+3e^2)}{p^3}\left[1 + \frac{3m^2q^2k^2}{h^2} - \frac{8m^3qk}{h^3} + \frac{12m^4q^2k^2}{h^4} - \frac{2m^3q(4+3q^2)k(k^2-1)}{h^3}\right] - \frac{2m^2q^2}{p^4}\left(1 - \frac{6m^2}{h^2}\right)(1+6e^2+e^4),$$

$$0 = \frac{m}{h^2}\left[1 - \frac{2mqk(k^2-1)}{h}\right] - \frac{1}{p}\left[1 - \frac{3m^2q^2(k^2-1)}{h^2} + \frac{8m^3qk^3}{h^3} - \frac{12m^4q^2k^2(k^2-1)}{h^4}\right] + \frac{m(3+e^2)}{p^2}\left[1 + \frac{3m^2q^2k^2}{h^2} - \frac{8m^3qk}{h^3} + \frac{12m^4q^2k^2}{h^4} - \frac{2m^3q(4+3q^2)k(k^2-1)}{h^3}\right] - \frac{4m^2q^2}{p^3}\left(1 - \frac{6m^2}{h^2}\right)(1+e^2). \tag{14}$$

(For the sake of simplicity only the equations for TS1 are written down.) The solution of this system is found to be

$$k^2_{\mathrm{TS1}} = k^2_{\mathrm{TS2}} = 1 - \frac{m(1-e^2)}{p} + \frac{m^2(1-e^2)^2}{p^2},$$

$$h^2_{\mathrm{TS1}} = h^2_{\mathrm{TS2}} = mp\left[1 + \frac{m(3+e^2)}{p} - \frac{2m^{3/2}q(3+e^2)}{p^{3/2}}\right], \tag{15}$$

with reasonable accuracy. After substitution of Eqs (13), (15) into (12a) and (12b) the differential equations with respect to ψ follow:

$$\left(\frac{d\psi}{d\varphi}\right)^2_{\mathrm{TS1}} = 1 - \frac{3m^2q^2(k^2-1)}{h^2} + \frac{8m^3qk^3}{h^3} - \frac{12m^4q^2k^2(k^2-1)}{h^4}$$

$$-2m\left[1+\frac{3m^2q^2k^2}{h^2} - \frac{8m^3qk}{h^3} + \frac{12m^4q^2k^2}{h^4} - \frac{2m^3q(4+3q^2)k(k^2-1)}{h^3}\right]\frac{3+e\cos\psi}{p}$$

$$+2m^2q^2\left(1-\frac{6m^2}{h^2}\right)\frac{6+4e\cos\psi+e^2+e^2\cos^2\psi}{p^2} = 1+N_{\mathrm{TS1}}; \qquad (16a)$$

$$\left(\frac{d\psi}{d\varphi}\right)^2_{\mathrm{TS2}} = 1 - \frac{3m^2q^2(k^2-1)}{h^2} + \frac{8m^3qk^3}{h^3} - \frac{12m^4q^2k^2(k^2-1)}{h^4} - 2m\left[1+\frac{m^2(q^2-1)}{8h^2}\right.$$

$$\left.+\frac{m^2(11q^2+1)k^2}{4h^2} - \frac{8m^3qk}{h^3} + \frac{12m^4q^2k^2}{h^4} - \frac{m^3q(17+11q^2)k(k^2-1)}{2h^3}\right]\frac{3+e\cos\psi}{p}$$

$$+2m^2\left[q^2-\frac{m^2(43q^2+5)}{8h^2}\right]\frac{6+4e\cos\psi+e^2+e^2\cos^2\psi}{p^2} = 1+N_{\mathrm{TS2}}. \qquad (16b)$$

Let us integrate these equations according to the scheme

$$\int_0^{2\pi+\Delta\varphi} d\varphi = \int_0^{2\pi} d\psi\left(1-\frac{N}{2}+\frac{3N^2}{8}-\frac{5N^3}{16}+\dots\right)$$

taking into account $N < 1$. Then, using Eq. (15) for the perihelion precession in TS1 and TS2 we find

$$\left(\frac{\Delta\varphi}{2\pi}\right)_{\mathrm{TS1}} = \frac{3m}{p} - \frac{4m^{3/2}q}{p^{3/2}}\left(1+\frac{9m}{p}\right) + \frac{3m^2}{2p^2}\left(9+q^2+\frac{e^2}{2}\right)$$

$$+\frac{3m^3}{2p^3}\left(45+25q^2-3q^2e^2+\tfrac{15}{2}e^2\right), \qquad (17a)$$

$$\left(\frac{\Delta\varphi}{2\pi}\right)_{\mathrm{TS2}} = \frac{3m}{p} - \frac{4m^{3/2}q}{p^{3/2}}\left(1+\frac{9m}{p}\right) + \frac{3m^2}{2p^2}\left(\frac{37+3e^2}{4}+\frac{e^2}{2}\right)$$

$$+\frac{3m^3}{2p^3}\left(\frac{97+43q^2}{2}-4q^2e^2+\tfrac{17}{2}e^2\right). \qquad (17b)$$

It is interesting to see that

$$\left(\frac{\Delta\varphi}{2\pi}\right)_{\mathrm{TS2}} = \left(\frac{\Delta\varphi}{2\pi}\right)_{\mathrm{TS1}} + \frac{3Q}{2mp^2} + \frac{3Q(7+2e^2)}{p^3} \qquad (18)$$

as a consequence of the growth of parameter δ.

4. Deflection of null geodesics

The differential trajectory equation for the massless particles (null geodesics) may be obtained from Eq. (11) or (12a), (12b) by standard procedures in the limit k, $h \to \infty$ and $(k^2-1)h^{-2} = b^{-2}$ [18], where b is an impact parameter. Then equations for the time-like geodesics lead to the following differential equations:

$$\left(\frac{du}{d\varphi}\right)^2_{TS1} = b^{-2} - \frac{4m^2q}{b^3}u - \left[1 - \frac{3m^2q^2}{b^2} + \frac{8m^3q}{b^3}\left(1 - \frac{3mq}{2b}\right)\right]u^2$$
$$+2m\left[1 + \frac{3m^2q^2}{b^2} - \frac{2m^3q(3q^2+4)}{b^3}\right]u^3 - 2m^2q^2u^4, \tag{19a}$$

$$\left(\frac{du}{d\varphi}\right)^2_{TS2} = b^{-2} - \frac{4m^2q}{b^3}u - \left[1 - \frac{3m^2q^2}{b^2} + \frac{8m^3q}{b^3}\left(1 - \frac{3mq}{2b}\right)\right]u^2$$
$$+2m\left[1 + \frac{m^2(11q^2+1)}{4b^2} - \frac{m^3q(17+11q^2)}{2b^3}\right]u^3 - 2m^2q^2u^4. \tag{19b}$$

Similarly as in Section 3 we put the solution of Eqs. (19a), (19b) into the form of (13). After repeating the procedure described above we may represent the focal parameter and the eccentricity for null geodesics as functions of the source's characteristics and impact parameter:

$$p_{TS1} = \frac{b^2}{m}\left[1 + \frac{2mq}{b} - \frac{m^2(8-q^2)}{b^2} + \frac{16m^3q}{b^3}\right],$$
$$e_{TS1} = \frac{b}{m}\left[1 + \frac{2mq}{b} - \frac{m^2(11-3q^2)}{b^2} + \frac{m^3q(q^2+11)}{b^3}\right], \tag{20a}$$

$$p_{TS2} = \frac{b^2}{m}\left[1 + \frac{2mq}{b} - \frac{m^2(33-5q^2)}{4b^2} + \frac{m^3q(q^2+31)}{2b^3}\right],$$
$$e_{TS2} = \frac{b}{m}\left[1 + \frac{2mq}{b} - \frac{m^2(23-7q^2)}{4b^2} + \frac{3m^2q(q^2+7)}{2b^3}\right]. \tag{20b}$$

The differential equations for relativistic anomalies are given as

$$\left(\frac{d\psi}{d\varphi}\right)^2_{TS1} = 1 - \frac{3m^2q^2}{b^2} + \frac{8m^3q}{b^3}\left(1 - \frac{3mq}{2b}\right) - 2m\left[1 + \frac{3m^2q^2}{b^2} - \frac{2m^3q(3q^2+4)}{b^3}\right]\frac{3+e\cos\psi}{p}$$
$$+ \frac{2m^2q^2}{p^2}(6+4e\cos\psi+e^2+e^2\cos^2\psi) = 1+M_{TS1}, \tag{21a}$$

$$\left(\frac{d\psi}{d\varphi}\right)^2_{TS2} = 1 - \frac{3m^2q^2}{b^2} + \frac{8m^3q}{b^3}\left(1 - \frac{3mq}{2b}\right) - 2m\left[1 + \frac{m^2(11q^2+1)}{4b^2}\right.$$
$$\left.- \frac{m^3q(17+11q^2)}{2b^3}\right]\frac{3+e\cos\psi}{p} + \frac{2m^2q^2}{p^2}(6+4e\cos\psi+e^2+e^2\cos^2\psi) = 1+M_{TS2}. \tag{21b}$$

In the developed approach the angle of null ray deflection from a straight line may be composited of two parts. The first is the angle between the asymptotes of the hyperbola in the coordinate system (r, ψ). Then from Eq. (13) we find

$$\theta_0 = 2 \operatorname{arctg} \frac{1}{\sqrt{e^2-1}} \approx \frac{2}{e} + \frac{1}{3e^3},$$

which for TS1 and TS2 gives

$$(\theta_0)_{\mathrm{TS1}} = \frac{2m}{b} - \frac{4m^2 q}{b^2} + \frac{m^3}{b^3}\left(\tfrac{34}{3} + 5q^2\right), \tag{22a}$$

$$(\theta_0)_{\mathrm{TS2}} = \frac{2m}{b} - \frac{4m^2 q}{b^2} + \frac{m^3}{b^3}\left(\tfrac{71}{6} + \frac{9q^2}{2}\right) \tag{22b}$$

in the case of equatorial null geodesics. The second part of the total deflection is determined by intergrating Eqs. (21a) ans 21b) and corresponds to the rotation of coordinate system (r, ψ) with respect to (r, φ). Therefore, the last part of deflection may be found in accordance with the following rule:

$$\int_{-\pi/2}^{+\pi/2+\Delta\theta} d\varphi = \int_{\psi_1}^{\psi_2} d\psi \left(-\frac{M}{2} + \frac{3M^2}{8} - \frac{5M^3}{16} + \ldots \right),$$

where $\psi_1 = -\pi/2 - e^{-1}$ and $\psi_2 = \pi/2 + e^{-1}$ are the approximate expressions for the roots of the equation $1 + e\cos\psi = 0$ which corresponds to infinite values for the radial coordinate r. After approximate integration by taking into consideration Eqs. (20a), (20b) we obtain

$$\Delta\theta_{\mathrm{TS1}} = \frac{2m}{b} + \frac{15\pi m^2}{4b^2} + \frac{m^3}{b^3}\left(\tfrac{94}{3} - 10\pi q - q^2\right), \tag{23a}$$

$$\Delta\theta_{\mathrm{TS2}} = \frac{2m}{b} + \frac{15\pi m^2}{4b^2} + \frac{m^3}{b^3}\left(\tfrac{94}{3} - 10\pi q - \frac{3q^2-1}{2}\right). \tag{23b}$$

The total angles of deflection in TS1 and TS2 are easily given as

$$\theta_{\mathrm{TS1}} = (\theta_0 + \Delta\theta)_{\mathrm{TS1}} = \frac{4m}{b} + \frac{m^2}{b^2}\left(\frac{15\pi}{4} - 4q\right) + \frac{m^3}{b^3}\left(\tfrac{128}{3} - 10\pi q + 4q^2\right), \tag{24a}$$

$$\theta_{\mathrm{TS2}} = (\theta_0 + \Delta\theta)_{\mathrm{TS2}} = \frac{4m}{b} + \frac{m^2}{b^2}\left(\frac{15\pi}{4} - 4q\right) + \frac{m^3}{b^3}\left(\tfrac{131}{3} - 10\pi q + 3q^2\right). \tag{24b}$$

Thus, deflections in the Kerr and first TS metrics are connected by the following relation

$$\theta_{\mathrm{TS2}} = \theta_{\mathrm{TS1}} + \frac{4Q}{b^3} \tag{25}$$

as a consequence of the difference in δ's.

5. *Retardation of radar signal*

The time-delay formula for equatorial null geodesics may be deduced by using the expression $ds^2 = 0$ which gives the system of first integrals

$$0 = f^2\left(1-\omega\frac{d\varphi}{dx^0}\right)^2 - Ee^{2\gamma}\left(\frac{dr}{dx^0}\right)^2 - \Delta\left(\frac{d\varphi}{dx^0}\right)^2,$$

$$\left[(\Delta f^{-1}-f\omega^2)\frac{d\varphi}{dx^0}+\omega f\right]\dot{x}^0 = h, \quad f\left(1-\omega\frac{d\varphi}{dx^0}\right)\dot{x}^0 = k$$

in place of Eqs. (10). Using these equations we obtain

$$\left(\frac{dr}{dx^0}\right)^2 = \frac{\Delta f^2[\Delta - f^2(b-\omega)^2]}{Ee^{2\gamma}[\Delta+\omega f^2(b-\omega)]^2}, \tag{26}$$

$$\frac{d\varphi}{dx^0} = f^2(b-\omega)\,[\Delta+f^2\omega(b-\omega)]^{-1}. \tag{27}$$

Having in mind the really observed situation when the impact parameter b is unknown, we must introduce the minimal distance r_0 between the ray and the source of the TS field. It may be found from Eq. (26) that

$$b = \omega_0 + f_0^{-1}\sqrt{\Delta_0} \tag{28}$$

by taking into account

$$(dr/dx^0)_{r=r_0} = 0.$$

Here f_0, ω_0 and Δ_0 are functions of r_0 only. The substitution of Eqs. (8a), (8b) into (28) gives the approximate expressions

$$b_{\mathrm{TS}1} = r_0\left[1+\frac{m}{r_0}+\frac{m^2(q^2+3)}{2r_0^2}-\frac{2m^2q}{r_0^2}\left(1+\frac{2m}{r_0}\right)+\frac{m^3(5+3q^2)}{2r_0^3}\right], \tag{29a}$$

$$b_{\mathrm{TS}2} = r_0\left[1+\frac{m}{r_0}+\frac{m^2(q^2+3)}{2r_0^2}-\frac{2m^2q}{r_0^2}\left(1+\frac{2m}{r_0}\right)+\frac{m^3(11+5q^2)}{4r_0^3}\right]. \tag{29b}$$

Now the time of the signal's propagation may be determined from Eq. (27). Substituting (28) into (27) we obtain the relation

$$dx^0 = \left[\omega+\frac{\Delta}{f^2}\left(\omega_0-\omega+\frac{\sqrt{\Delta_0}}{f_0}\right)^{-1}\right]\frac{d\psi}{\sqrt{1+M}}, \tag{30}$$

where the value M is represented by Eq. (21a) or (21b). The angular coordinates ψ_1 and ψ_2 of the source of radiation at the point $(r_1, \psi_1, \theta_1 = \pi/2)$ and the observer at the point $(r_2, \psi_2, \theta_2 = \pi/2)$ may be found as follows:

$$\psi_1 = -\frac{\pi}{2}+\frac{1}{e}\left(\frac{p}{r_1}-1\right), \quad \psi_2 = \frac{\pi}{2}-\frac{1}{e}\left(\frac{p}{r_2}-1\right).$$

In view of the above remark it is necessary to make use of the quantities e and p, expressed in the form

$$e_{TS1} = \frac{r_0}{m}\left[1+\frac{m}{r_0}+\frac{2mq}{r_0}\left(1-\frac{m}{r_0}\right)-\frac{2m^2(2-q^2)}{r_0^2}+\frac{m^3q}{r_0^3}(7+q^2)+\frac{8m^3}{r_0^3}\right],$$

$$p_{TS1} = \frac{r_0^2}{m}\left[1+\frac{2m}{r_0}+\frac{2mq}{r_0}\left(1-\frac{m}{r_0}\right)-\frac{2m^2(2-q^2)}{r_0^2}+\frac{m^3q(7+q^2)}{r_0^3}+\frac{8m^3}{r_0^3}\right], \quad (31a)$$

$$e_{TS2} = \frac{r_0}{m}\left[1+\frac{m}{r_0}+\frac{2mq}{r_0}\left(1-\frac{m}{r_0}\right)-\frac{m^2(17-9q^2)}{4r_0^2}+\frac{m^3q(13+3q^2)}{2r_0^3}+\frac{m^3(17-q^2)}{2r_0^3}\right],$$

$$p_{TS2} = \frac{r_0^2}{m}\left[1+\frac{2m}{r_0}+\frac{2mq}{r_0}\left(1-\frac{m}{r_0}\right)-\frac{m^2(17-9q^2)}{4r_0^2}+\frac{m^3q(13+3q^2)}{2r_0^3}+\frac{m^3(17-q^2)}{2r_0^3}\right], \quad (31b)$$

which follows from Eqs. (20a), (20b) after the substitution of (29a), (29b). Substituting Eqs. (13), (31a), (31b) into (30) we obtain

$$dx^0_{TS1} = d\psi\left\{\frac{r^2}{r_0}\left[1-\frac{m}{r_0}+\frac{2m^2q}{r_0^2}\left(1-\frac{5m}{r_0}\right)+\frac{5m^2}{2r_0^2}-\frac{m^3(19+4q^2)}{2r_0^3}\right]\right.$$
$$+\frac{mr^2\cos\psi}{r_0^2}\left[1-\frac{2m}{r_0}+\frac{4m^2q}{r_0^2}+\frac{23m^2}{2r_0^2}\right]+\frac{m^2r^2(3-2q^2)\cos^2\psi}{2r_0^3}\left(1-\frac{3m}{r_0}\right)$$
$$+\frac{m^3r^2(5-6q^2)\cos^3\psi}{2r_0^4}+\frac{2mr}{r_0}\left[1-\frac{m}{r_0}(1+q)+\frac{5m^2}{2r_0^2}\left(1+\frac{8q}{5}\right)\right]+\frac{2m^2r\cos\psi}{r_0^2}$$
$$\times\left[1-\frac{m}{r_0}(2+q)\right]+\frac{m^3r(3-2q^2)\cos^2\psi}{r_0^3}+\frac{m^2(4+q^2)}{r_0}\left(1-\frac{m}{r_0}\right)-\frac{10m^3q}{r_0^2}$$
$$\left.-\left[\frac{2m^2q}{r_0}-\frac{m^3(4+q^2)}{r_0^2}-\frac{4m^3(2+q^2)}{r_0^2}\right]\cos\psi-\frac{4m^3q}{r_0^2}\cos^2\psi\right\},$$

$$dx^0_{TS2} = d\psi\left\{\frac{r^2}{r_0}\left[1-\frac{m}{r_0}+\frac{2m^2q}{r_0^2}\left(1-\frac{5m}{r_0}\right)+\frac{5m^2}{2r_0^2}-\frac{m^3(39+7q^2)}{4r_0^3}\right]\right.$$
$$+\frac{mr^2\cos\psi}{r_0^2}\left[1-\frac{2m}{r_0}+\frac{4m^2q}{r_0^2}+\frac{m^2(47-q^2)}{4r_0^2}\right]+\frac{m^2r^2(3-2q^2)\cos^2\psi}{2r_0^3}\left(1-\frac{3m}{r_0}\right)$$
$$+\frac{m^3r^2(5-6q^2)\cos^3\psi}{2r_0^4}+\frac{2mr}{r_0}\left[1-\frac{m}{r_0}(1+q)+\frac{5m^2}{2r_0^2}\left(1+\frac{8q}{5}\right)\right]+\frac{2m^2r\cos\psi}{r_0^2}$$
$$\times\left[1-\frac{m}{r_0}(2+q)\right]+\frac{m^3r(3-2q^2)\cos^2\psi}{r_0^3}+\frac{m^2(4+q^2)}{r_0}\left(1-\frac{m}{r_0}\right)-\frac{10m^3q}{r_0^2}$$
$$\left.-\left[\frac{2m^2q}{r_0^0}-\frac{m^3(4+q^2)}{r_0^2}-\frac{m^3(17+7q^2)}{2r_0^2}\right]\cos\psi-\frac{4m^3q}{r_0^2}\cos^2\psi\right\},$$

which gives after integration the final expressions for the time of propagation

$$x^0_{\mathrm{TS1}} = x^0_{\mathrm{TS2}} = r_1 + r_2 - \frac{r_0^2}{2}\left(\frac{1}{r_1}+\frac{1}{r_2}\right) + 2m\left(1+\ln\frac{4r_1r_2}{r_0^2}\right) - mr_0\left(\frac{1}{r_1}+\frac{1}{r_2}\right) - \frac{4m^2}{r_0}\left(1+2q-\frac{15\pi}{8}\right). \tag{32}$$

It must be noted that the relations

$$\frac{m}{r_0} \sim \frac{r_0}{r_1} \sim \frac{r_0}{r_2}, \quad \frac{m}{r_1} \sim \frac{m}{r_2} \sim \frac{m^2}{r_0^2}$$

were used. The first two terms in Eq. (32) have a Newtonian nature and must be neglected in the time-delay. Therefore, for general relativistic retardation of the null signal in the TS fields we find

$$\Delta x^0_{\mathrm{TS1}} = \Delta x^0_{\mathrm{TS2}} = 2m\left(1+\ln\frac{4r_1r_2}{r_0^2}\right) - mr_0\left(\frac{1}{r_1}+\frac{1}{r_2}\right) - \frac{4m^2}{r_0}\left(1+2q-\frac{15\pi}{8}\right). \tag{33}$$

Thus, the time-delays in TS1 and TS2 metrics are identical up to the order $(m/r_0)^3$. Of course, these expressions may be distinguished in higher order terms.

6. The gravitational redshift

As it has recently been shown in [20, 21], the higher order gravitational redshift may be measured in the near future owing to the new experimental possibilities (high-stability clocks). The first-order redshift is connected, to a great extent, with the equivalence principle and is not produced by field equations. But the last statement is not true for the higher order terms. Let us calculate this effect in the TS fields and reveal the influence of the rotation and the oblateness of the source of gravitation.

According to [22] the Doppler effect in the gravitational field with metric $g_{\lambda\mu}$ is determined by the following relation

$$\frac{\lambda + d\lambda}{\lambda} = \left(\frac{dx_2^0}{dx_1^0}\right)\left(g_{\mu\sigma}\frac{dx^\mu}{dx^0}\frac{dx^\sigma}{dx^0}\right)^{1/2}_{|x_2{}^\mu} \cdot \left(g_{\mu\sigma}\frac{dx^\mu}{dx^0}\frac{dx^\sigma}{dx^0}\right)^{-1/2}_{|x_1{}^\mu}, \tag{34}$$

where x_1^μ denotes the coordinates of the emitter with proper wavelenght λ, x_2^μ means coordinates of the observer, and the derivative (dx_2^0/dx_1^0) describes the relation between the moments of emitting and receiving a pulse of radiation. When the emitter and observer move in the equatorial plane of the TS space-time (5) we obtain from Eq. (26)

$$x_2^0 = x_1^0 + \int_{r_1}^{r_2} \frac{e^\gamma\sqrt{E}}{f\sqrt{\Delta}}\,\frac{\Delta + \omega f^2(b-\omega)}{\sqrt{\Delta - f^2(b-\omega)^2}}\,dr. \tag{35}$$

After differentiation we may find the derivative (dx_2^0/dx_1^0) and calculate the Doppler effect as a consequence of Eq. (34). For the sake of simplicity we shall consider the static emitter and observer at infinity (that is, $r_2 \gg r_1$). In this case the general formula (34) reduces to the expression for the gravitational redshift

$$\frac{\lambda + d\lambda}{\lambda} = f^{-1/2}.$$

In particular cases of TS1 and TS2 fields with an accuracy up to $(m/r)^3$ it follows

$$\left(\frac{d\lambda}{\lambda}\right)_{TS1} = \frac{m}{r} + \frac{3m^2}{2r^2} + \frac{5m^3}{2r^3}, \tag{36a}$$

$$\left(\frac{d\lambda}{\lambda}\right)_{TS2} = \frac{m}{r} + \frac{3m^2}{2r^2} + \frac{m^3(21-q^2)}{8r^3}, \tag{36b}$$

that is,

$$\left(\frac{d\lambda}{\lambda}\right)_{TS2} = \left(\frac{d\lambda}{\lambda}\right)_{TS1} + \frac{Q}{2r^3}. \tag{37}$$

7. *The physical discussion*

The expressions for the crucial effects of General Relativity in the TS1 and TS2 metrics are calculated with an accuracy up to the terms of the order of m^3 and q^2. The second-order terms in Eqs. (17a), (17b) are more exact than the expressions for the perihelion precession calculated in [23, 24]. The light deflections (24a), (24b) in the low orders are the same as calculated in [19]. The results from [23, 25] for light deflections are apparently wrong.

On the basis of Eqs. (18), (25), (33) and (37) it may be concluded that the perihelion precession in TS fields is more sensitive to the change of the quadrupole properties of the source. On the other hand, the time-delay effect is unchanged to the order of $(m/r_0)^3$.

The values of the additional terms in Eqs. (18), (25) and (37) depend on the parameter q, which is related to the parameters of the gravitational source as $q = a/m$. For $q < 1$ all terms stipulated by an additional quadrupole momentum Q are positive and the effects for $\delta = 2$ are greater than for $\delta = 1$. In the case of the TS naked singularity with $q > 1$ all these terms are negative, that is, the values of the effects in TS2 are smaller than in TS1. It is interesting to note that this situation must be realized in Sun's ($a_\odot = 1.26\, m_\odot$ [24]) and Earth's ($a_\oplus = 763.88\, m_\oplus$ [26]) gravitational fields. Therefore, after the experimental investigation of the higher order perihelion precession, light deflection and redshift, it would be possible to choose the appropriate metric for the description of these fields. Of course, the TS-corrections to the known and measured effects for the Schwarzschild field ($q = 0$) are extremely small and undetected by means of contemporary experimental tools.

As it is already known [7, 8], in the limit $q \to 1$ all metrics of the TS series coincide with the extreme Kerr space-time. It is also true for calculated effects. In fact, all the expressions for crucial effects in the TS1 are the same as those for TS2 when $q = 1$.

It should be noted that the first three calculated effects depend on the direction of the gravitational mass rotation or, strictly speeking, on the direction of the test particle and light ray motion with respect to the direction of the source rotation. In this paper the counter-clockwise rotation of mass m as seen from the positive direction of z-axis was supposed. For the clockwise rotation in the terms of the odd orders of a the sign must be replaced by its opposite. All effects in this paper for corotating test particles and null signals are given. As it follows from Eqs. (17a), (17b), (24a) and (24b) the perihelion precession and light deflection for corotating particles and rays are smaller than for counterrotating (such peculiarity was already known).

In the case of the time-delay (33) the value of retardation for corotating radar signal is smaller than for counter-rotating. This peculiarity ought to be discussed, in our opinion, as an independent new test of General Relativity in the Sun's system. The proposed gravitational experiment would consist in the comparison of the time-delays for co- and counter-rotating signals. The signal propagated from the Earth to two spacecrafts (or else from the spacecrafts to the Earth) at the opposite sides of Sun's rotation axis must be used without the retransmission back to the Earth (or back to the spacecrafts) since for the retransmitted signals the influence of the terms of odd orders of a on the total time-delay are compensated.

In the limit of Lense–Thirring metric the time retardation of corotating signal with respect to counterrotating must be equal to

$$\Delta x^0_{\text{co-rot}} - \Delta x^0_{\text{counter-rot}} = -\frac{16ma}{r_0}$$

that approximately gives $2.1 \cdot 10^{-10}$ seconds. It is interesting to point out, that a situation of two spacecrafts forming the positive and negative angles with respect to the Sun–Earth direction has already been realized in the "Mariner" relativistic experiment [27, 28] but the time-delay was measured with accuracy to $1.2 \cdot 10^{-6}$ s [29]. Thus, the feasibility of the proposed new experiment depends only on future improvements in the stabilities of clocks and in radar astronomy techniques.

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