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GEOMETRY AT INFINITY

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The geometry of the space-time at coordinate infinity is considered. It is shown that time becomes absolute in the limit and that the spatial section of the manifold there is an euclidean two sphere.

1. Introduction

In a recent work (Ref. [1]) it was shown that as a local observer's radial distance r tends to infinity (in his equally local view), the metric of the space-time becomes

$$ds^2 = dt^2 - r_0^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1}$$

where r_0 is a constant. In other words, the spatial part of the space-time manifold of any observer is, "at infinity", an Euclidean sphere of constant and finite radius. It is the observer's limit or boundary of the universe. This seems to be an almost inevitable consequence of a new interpretation of the non-symmetric unified field theory (Ref. [2]) in which the metric of a "background", Riemannian space is determined by a physically meaningful law. We say "almost inevitable" because the conclusion is based on a restricted (spherically symmetric, static) solution of the field equations.

It is the purpose of this note to discuss in some detail the geometry of the manifold with the metric given by (1). In particular, we shall determine the groups of motion allowed by the latter. Let us denote the above manifold by I. We can evidently choose units so that we also have $r_0 = 1$, and we shall adopt these units from now on.

It is immediately obvious from (1) (with $r_0 = 1$) that I admits minimum varieties (Ref. [3]) given respectively by

$$\phi = \beta, \text{ a constant,} \quad \text{and} \quad \theta = \alpha, \text{ a constant.} \tag{2}$$

On these sections we have usual (special) Lorentz transformations relating t and (the arc measure) θ with t' and θ' in the first, and t and ϕ with t' and ϕ', in the second case. The "velocity of light" is 1 for the former, but cosec α for the latter type of transformation.

We shall show that in the general case, I requires an absolute time.

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2. *Killing vectors of I*

In the sequel we shall use a co-ordinate notation in which

$$(x^1, x^2, x^3) \equiv (t, \theta, \phi), \tag{3}$$

so that, for example $\xi_{1,2} = \partial\xi_1/\partial\theta$ for the derivative of the time like component of a vector ξ_λ with respect to θ. Our first step is to determine the Killing vectors ${}^a\xi^\lambda$ and the infinitesimal generators of the group of motions allowed by *I*. Greek indices go from 1 to 3 and will be used as tensor indices. Latin indices will be used to distinguish different Killing vectors we may obtain. The equations we have to solve are

$$\xi_{\lambda;\mu} + \xi_{\mu;\lambda} = 0. \tag{4}$$

Now in the manifold *I* only two Christoffel brackets are not identically zero, namely

$$\Gamma^2_{33} = -\sin\theta\cos\theta, \quad \Gamma^3_{23} = \cot\theta. \tag{5}$$

In spite of its extreme simplicity *I* is not a space of constant curvature (though its space-like part, of course, is). This may be immediately verified since the only significant, non-zero component of the Riemannian–Christoffel tensor is $R_{2332} = \sin^2\theta$, and so, the condition $R_{\lambda\mu\nu x} = K(g_{\lambda\nu}g_{\mu x} - g_{\mu\nu}g_{\lambda x})$ is not satisfied when, for example $\lambda = \nu = 1$, $\mu = x = 2$. ($g_{\mu\nu}$ is the metric tensor of *I*: diag $(1, -1, -\sin^2\theta)$.)

After this aside, the Killing equations (4) become

$$\xi_{1,1} = 0, \quad \xi_{1,2} + \xi_{2,1} = 0, \quad \xi_{1,3} + \xi_{3,1} = 0, \quad \xi_{2,2} = 0,$$
$$\xi_{2,3} + \xi_{3,2} - 2\cos\theta\xi_3 = 0, \quad \xi_{3,3} + \sin\theta\cos\theta\xi_2 = 0. \tag{6}$$

From the first form of these equations it follows easily that

$$\xi_1 = -f\theta + g, \quad \xi_2 = ft + h, \quad \xi_3 = (f'\theta - g')t + w(\phi, \theta), \tag{7}$$

where f, g and h are at most functions of ϕ only; dashes denote differentiation with respect to ϕ while w is, as indicated, a function of θ and ϕ. Substituting (7) into the last two of the equations (6), we obtain:

$$f't + h' + f't + \frac{\partial w}{\partial\theta} - 2\cot\theta[(f'\theta - g')t + w] = 0,$$

$$(f''\theta - g'')t + \frac{\partial w}{\partial\phi} + \sin\theta\cos\theta(ft + h) = 0. \tag{8}$$

Equations (8) can be satisfied only if $f = 0$ and g is constant $= g_0$ say, when they become respectively,

$$\frac{\partial w}{\partial\theta} - 2\cot\theta w + h' = 0, \quad \frac{\partial w}{\partial\phi} + h\sin\theta\cos\theta = 0. \tag{9}$$

The condition of integrability, $\partial^2\omega/\partial\theta\partial\phi = \partial^2\omega/\partial\phi\partial\theta$, implies immediately that $h'' + h = 0$, or

$$h = h_0\cos(\phi + \varepsilon), \tag{10}$$

where h_0 and ε are constants. It is then easy to integrate equations (9) for w, the result being

$$w = -h_0 \sin(\phi+\varepsilon) \sin\theta \cos\theta + k_0 \sin^2\theta, \tag{11}$$

where k_0 is yet another constant. Thus the general solution of the Killing equation is

$$\xi = (g_0, h_0 \cos(\phi+\varepsilon), -h_0 \sin(\phi+\varepsilon) \sin\theta \cos\theta + k_0 \sin^2\theta), \tag{12}$$

ξ is clearly a linear combination of the four basic, covariant, Killing vectors

$$^1\xi_\lambda = (1, 0, 0), \quad ^2\xi_\lambda = (0, \cos\phi, -\sin\phi \sin\theta \cos\theta),$$

$$^3\xi_\lambda = (0, -\sin\phi, -\cos\phi \sin\theta \cos\theta), \quad ^4\xi_\lambda = (0, 0, -\sin^2\theta). \tag{13}$$

Raising the index with the help of the metric tensor of I, the corresponding contravariant Killing vectors are

$$^1\xi^\lambda = (1, 0, 0), \quad ^2\xi^\lambda = (0, -\cos\phi, \sin\phi \cot\theta),$$

$$^3\xi^\lambda = (0, \sin\phi, \cos\phi \cot\theta), \quad ^4\xi^\lambda = (0, 0, 1). \tag{14}$$

Using vectors (14) we can construct (Ref. [3]) the infinitesimal generators of the complete group G_4 of motions in I:

$$X_a = {}^a\xi^\lambda \frac{\partial}{\partial x^\lambda}. \tag{15}$$

In fact

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = -\cos\phi \frac{\partial}{\partial\theta} + \sin\phi \cot\theta \frac{\partial}{\partial\phi},$$

$$X_3 = \sin\phi \frac{\partial}{\partial\theta} + \cos\phi \cot\theta \frac{\partial}{\partial\phi}, \quad X_4 = \frac{\partial}{\partial\phi}. \tag{16}$$

Hence

$$(X_1, X_a) = 0, \tag{17}$$

while

$$(X_2, X_3) = X_4, \quad (X_3, X_4) = X_2, \quad (X_4, X_2) = X_3. \tag{18}$$

Thus the G_4 splits (as was perhaps intuitively obvious) into a translation G_1 in t and the euclidean rotation G_3 or O_3. This proves also the last statement of the Introduction that time in I is absolute in the sense that only simple translations (choice of origin) in it are permitted.

In the next section we shall record for the sake of completeness the finite equations of transformation of I into itself.

3. *Finite automorphisms*

To determine finite equations of the automorphisms of I it turns out to be simplest to write (Ref. [3])

$$p^{\mu}{}_{\nu} = \frac{\partial x^{\mu}}{\partial x'^{\nu}} \tag{19}$$

and to solve the equations of transformation of the affine connection

$$p^{\sigma}{}_{\mu,\nu} = \Gamma'^{\lambda}_{\mu\nu} p^{\sigma}{}_{\lambda} - \Gamma^{\sigma}_{\beta\gamma} p^{\beta}{}_{\mu} p^{\gamma}{}_{\nu}, \tag{20}$$

where $\Gamma'^{2}_{33} = -\sin\theta' \cos\theta'$, $\Gamma'^{3}_{23} = \cot\theta'$. Indeed, it follows immediately from equations (20) (as well as from the result of the last section) that $t = t' + \text{const}$. Consequently we have

$$p^{1}{}_{2} = p^{1}{}_{3} = p^{2}{}_{1} = p^{3}{}_{1} = 0, \tag{21}$$

and the equations to be solved reduce to

$$\frac{\partial w}{\partial \theta'} = \sin\theta \cos\theta y^2, \qquad \frac{\partial w}{\partial \phi'} = \cot\theta' x + \sin\theta \cos\theta yz,$$

$$\frac{\partial x}{\partial \theta'} = \cot\theta' x + \sin\theta \cos\theta yz, \qquad \frac{\partial x}{\partial \phi'} = -\sin\theta' \cos\theta' w + \sin\theta \cos\theta z^2,$$

$$\frac{\partial y}{\partial \theta'} = -2\cot\theta wy, \qquad \frac{\partial y}{\partial \phi'} = \cot\theta' z - \cot\theta(wz + xy),$$

$$\frac{\partial z}{\partial \theta'} = \cot\theta' z - \cot\theta(wz + xy), \qquad \frac{\partial z}{\partial \phi'} = -\sin\theta' \cos\theta' y - 2\cot\theta xz, \tag{22}$$

where we have written

$$w = p^{2}{}_{2}, \quad x = p^{2}{}_{3}, \quad y = p^{3}{}_{2} \quad \text{and} \quad z = p^{3}{}_{3}. \tag{23}$$

Integrability conditions

$$\frac{\partial^2 w}{\partial \theta' \partial \phi'} = \frac{\partial^2 w}{\partial \phi' \partial \theta'}, \text{ etc.,}$$

of equations (22) give now the following relations

$$xy - wz = \frac{y}{x} \sin^2\theta' = -\frac{w}{z} \frac{\sin^2\theta'}{\sin^2\theta} = \frac{x}{y} \operatorname{cosec}^2\theta = -\frac{z}{w}, \tag{24}$$

whence

$$z \sin\theta = w \sin\theta', \qquad x = -y \sin\theta \sin\theta',$$

or

$$z \sin \theta = -w \sin \theta', \quad x = +y \sin \theta \sin \theta', \tag{25}$$

and

$$y^2 \sin^2 \theta + w^2 = 1 = \operatorname{cosec}^2 \theta'(x^2+z^2 \sin^3 \theta). \tag{26}$$

Let us consider the first pair of the equations (25). From the fifth of equations (22), $\partial y/\partial \theta' = -2 \cot \theta wy$, we get

$$y = f'(\phi') \operatorname{cosec}^2 \theta, \tag{27}$$

where $f' = df/d\phi'$ is a function of ϕ' only. The second of the equations (25) ($x = -y \sin \theta \sin \theta'$) now becomes

$$\frac{\partial \theta}{\partial \phi'} = -f' \frac{\sin \theta'}{\sin \theta}, \quad \text{or} \quad \frac{\partial \cos \theta}{\partial f} = \sin \theta',$$

whence

$$\cos \theta = f \sin \theta' + g(\theta').$$

Substituting this result into the first of the equations (26) we find readily that

$$g = 0, \quad f = \cos(\phi' + \phi'_0), \tag{28}$$

where ϕ'_0 is a constant. Thus

$$\cos \theta = \cos(\phi' + \phi_0') \sin \theta' \tag{29}$$

becomes the first of our finite automorphism equations as we might have expected.

From the first equation (25)

$$\frac{\partial \phi}{\partial \phi'} = \frac{\partial \theta}{\partial \theta'} \frac{\sin \theta'}{\sin \theta} = -\frac{\cos(\phi' + \phi_0') \sin \theta' \cos \theta'}{\sin^2 \theta},$$

and equation (27)

$$\frac{\partial \phi}{\partial \theta'} = -\frac{\sin(\phi' + \phi_0')}{\sin^2 \theta}.$$

Hence ϕ is given by the partial differential equation

$$\tan(\phi' + \phi_0') \frac{\partial \phi}{\partial \phi'} = \sin \theta' \cos \theta' \frac{\partial \phi'}{\partial \theta'}, \tag{30}$$

or

$$\frac{\partial \phi}{\partial \xi} = \frac{\partial \phi}{\partial \eta}, \tag{31}$$

where

$$\xi = \ln \tan \theta', \quad \eta = \ln \sin(\phi' + \phi_0'). \tag{32}$$

The general solution

$$\phi = \phi(\xi+\eta) = \phi(\ln \sin(\phi'+\phi_0')\tan\theta') = \phi(\sin(\phi'+\phi_0')\tan\theta').$$

But, if we let $z = \xi+\eta$, then

$$\frac{\partial\phi}{\partial\theta'} = -\frac{\sin(\phi'+\phi_0')}{\sin^2\theta} = -\frac{\sin(\phi'+\phi_0')}{1-\cos^2(\phi'+\phi_0')\sin^2\theta'} = \frac{d\phi}{dz}\frac{1}{\sin\theta'\cos\theta'},$$

$$\frac{d\phi}{dz} = -\frac{e^{\xi+\eta}}{1+e^{2(\xi+\eta)}} = -\frac{e^z}{1+e^{2z}},$$

and

$$\tan(\phi \neq \phi_0) = -\sin(\phi'+\phi_0')\tan\theta'. \tag{33}$$

Equations (29) and (33) give now the required general transformations for which

$$d\theta^2+\sin^2\theta d\phi^2 = d\theta'^2+\sin^2\theta' d\phi'^2.$$

The solution to the second set of the equations (25) is obtained by letting

$$\theta \to -\theta \quad \text{and} \quad \phi \to -\phi. \tag{34}$$

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