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METRIC-AFFINE UNIFICATION OF GRAVITY AND GAUGE THEORIES*

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A unified description of the gravitational and gauge fields in terms of a metric tensor and a linear connection on the bundle manifold is proposed. It is pointed out that the Lagrangian should contain, except of the Ricci scalar term, an additional term quadratic in torsion.

1. *Introduction*

The metric unification of gravity and gauge theories is constructed analogously to the Kaluza–Klein unification of gravitation and electromagnetism. The gravitational field described by a metric tensor on the space-time and the gauge field interpreted as a connection on a principal fibre bundle over the space-time are described jointly by a metric tensor on the bundle manifold. The metric tensor has n linearly independent Killing vector fields — the fundamental fields of the bundle. The Ricci scalar of this metric tensor, used as the Lagrangian density, leads to the equations of the gauge field and to the Einstein equations with the cosmological constant in the case of non-Abelian structure groups [1–3].

If we consider theories, such as the Einstein–Cartan theory, in which two quantities: the metric tensor and the affine connection describe the gravitational field, then the metric tensor on the bundle manifold is insufficient to describe completely the structure of the gravitational field. In the present approach we construct a linear connection and a metric tensor on the bundle manifold. They describe together the gravitational and the gauge fields.

2. *The metric tensor on the bundle manifold*

Let (P, M, π, G) be the principal fibre bundle with the structure group G. P is the bundle manifold, M is the base manifold, $\pi: P \to M$ is the projection. We assume that the

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base manifold M coincides with the space-time. If ξ is an element of the Lie algebra G', then ξ^* denotes fundamental vector field on P associated to ξ. Fundamental vector fields taken at the point p span the vertical subspace of the tangent space at p, i.e. the subspace consisting of vectors tangent to the fibre through p.

Let ω be the 1-form of connection on P. It means that ω is a G'-valued 1-form on P satisfying the following conditions: 1) $\omega(\xi^*) = \xi$ for each element ξ of the Lie algebra G', 2) $\omega(Xa) = a^{-1}\omega(X)a$ for each vector X on P and each element a of the group G. Given ω, we can define the horizontal subspace of the tangent space as consisting of all vectors X such that $\omega(X) = 0$. If x is a local vector field on M, $\hat{x}$ denotes its horizontal lift, i.e. a local horizontal vector field on P such that $\pi\hat{x} = x$.

We assume that the group G possesses an invariant metric h and that g is the metric tensor on the space-time M. Then we can construct a metric tensor $\tilde{g}$ on the bundle manifold characterized as follows: $\tilde{g}(X, Y) = g(\pi X, \pi Y) + h(\omega X, \omega Y)$. The metric tensor $\tilde{g}$ is invariant with respect to the action of the group G on P, $\pounds_{\xi^*}\tilde{g} = 0$. Inversely, if we have given an invariant metric tensor $\tilde{g}$ on the principal fibre bundle P, satisfying $\tilde{g}(\xi^*, \eta^*) = h(\xi, \eta)$, then we can determine the connection ω and the metric tensor g on M.

3. *The linear connection on the principal fibre bundle with a connection*

We construct now a linear connection $\tilde{\Gamma}$ on the principal fibre bundle P with the connection ω assuming that the space-time M is provided with the linear connection Γ. We wish to construct $\tilde{\Gamma}$ entirely determined by the structure of the principal fibre bundle with a connection, not using any additional structure (even the metric structure). It is our purpose to define the covariant derivatives of the fundamental vector fields and of the vector fields horizontally lifted from M, because each vector field on P is a linear combination of such fields. The derivatives of the fields ξ^* and $\hat{x}$ are defined as follows:

$$\tilde{\nabla}_{X_p}\xi^* = \alpha[\omega(X_p), \xi]^*_p, \qquad (1)$$
$$\tilde{\nabla}_{X_p}\hat{x} = (\nabla_{\pi X_p}x)^{\uparrow}_p + \beta(F(X_p, \hat{x}_p))^*_p.$$

In the above formulas $F = d\omega + \frac{1}{2}[\omega, \omega]$ is the curvature of the connection ω, α and β are arbitrary parameters. The connection $\tilde{\Gamma}$ is invariant with respect to the action of the group G on P. Inversely, assume that we have given an invariant linear connection $\tilde{\Gamma}$ on the principal fibre bundle P with the connection ω, which satisfies (1) and $\omega(\tilde{\nabla}_{X_p}\hat{x}) = \beta F(X_p, \hat{x}_p)$. Then we can define the linear connection Γ on M as follows $\nabla_{x_m}y = \pi\tilde{\nabla}_{\hat{x}_m}y$. The value of the right-hand side of this equality does not depend on the point to which we make horizontal lift of the vector x_m.

Let us introduce a basis (ω^α) of 1-forms on $P(\alpha, \beta, \gamma = 1, 2, \ldots, 4+n)$. We choose $\omega^\mu = \pi^*dx^\mu$, $\mu, \nu, \varrho = 1, 2, 3, 4$, where (x^μ) is a coordinate system on M. Furthermore, as ω^i, $i, j, k = 5, \ldots, 4+n$, we take components of the connection form ω, $\omega = (\omega^i)$.

In the above basis nonvanishing components of the metric tensor $\tilde{g}$ are following: $\tilde{g}_{\mu\nu} = g_{\mu\nu}$, $\tilde{g}_{ij} = h_{ij}$. The non-vanishing components of the linear connection Γ are following:

$$\tilde{\Gamma}^{\mu}{}_{\nu\varrho} = \Gamma^{\mu}{}_{\nu\varrho}, \quad \tilde{\Gamma}^{i}{}_{jk} = \alpha c^{i}{}_{kj}, \quad \tilde{\Gamma}^{i}{}_{\mu\nu} = \beta F^{i}{}_{\mu\nu},$$

where $c^{i}{}_{jk}$ are structure constants of the group G.

We can calculate the torsion tensor of the connection $\tilde{\Gamma}$:

$$\tilde{Q}^{\mu}{}_{\nu\varrho} = Q^{\mu}{}_{\nu\varrho}, \quad \tilde{Q}^{i}{}_{jk} = (2\alpha-1)c^{i}{}_{kj}, \quad \tilde{Q}^{i}{}_{\mu\nu} = (1-2\beta)F^{i}{}_{\mu\nu}$$

and the covariant derivatives of the metric tensor $\tilde{g}$:

$$\tilde{\nabla}_{\varrho}\tilde{g}_{\mu\nu} = \nabla_{\varrho}g_{\mu\nu}, \quad \tilde{\nabla}_{\nu}\tilde{g}_{i\mu} = \beta h_{ij}F^{j}{}_{\mu\nu}.$$

The curvature scalar appears to be

$$\tilde{R} = R+\alpha(\alpha-1)h^{ij}k_{ij},$$

where $k_{ij} = c^{k}{}_{il}c^{l}{}_{jk}$ is the Killing tensor on the group G.

4. The Einstein–Cartan theory in 4+n dimensions

In the Einstein–Cartan theory the connection Γ is metric, $\nabla g = 0$. We require that the connection $\tilde{\Gamma}$ be metric too: $\tilde{\nabla}\tilde{g} = 0$. Thus, we put $\beta = 0$.

What Lagrangian on the bundle manifold should we choose to obtain correct dynamical equations for the gravitational and gauge fields? The simplest choice, $L = \tilde{R}$, does not give the equations of the gauge field. It is better to choose

$$L = \tilde{R}+\frac{\mu}{2}\tilde{Q}^{\alpha}{}_{\beta\gamma}\tilde{Q}_{\alpha}{}^{\beta\gamma} = R+\frac{\mu}{2}Q^{\mu}{}_{\nu\varrho}Q_{\mu}{}^{\nu\varrho}-2\Lambda+\frac{\mu}{2}h_{ij}F^{i}{}_{\mu\nu}F^{j\mu\nu},$$

where

$$\Lambda = \left(\frac{\alpha}{2}(1-\alpha)+\mu(\alpha-\tfrac{1}{2})^{2}\right)h^{ij}k_{ij}.$$

Analogous Lagrangian with terms quadratic in torsion has been considered in the 4-dimensional case [4, 5]. In our approach we have two arbitrary parameters: α and μ. An additional degree of freedom is contained in h_{ij}. For a simple Lie group each invariant metric is proportional to the Killing tensor, $h_{ij} = \lambda k_{ij}$.

The cosmological constant Λ in our approach is arbitrary. After convenient choice of the parameters α and μ it may be put equal to zero. Arbitrariness of the cosmological constant differs essentially our metric-affine approach from the metric approach, in which the value of cosmological constant is fixed.

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