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ALGEBRAIC PROPERTIES OF S_2 SEPARABLE SPACE-TIMES*

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We consider a four dimensional Lorentzian space-tmie (V_4, g) which admits a S_2 separability structure and we investigate the algebraic classification of its Weyl tensor. We show that S_2 separability does not impose algebraic restrictions on the Weyl tensor of (V_4, g) or in other words empty space-time of any Petrov type could admit the S_2 separability structure.

1. *Introduction*

Integrability of the geodesic equation in a Lorentzian space-time (V_4, g) plays an important role in investigations of the global and physical properties of the gravitational field. Recently and less recently several papers appeared on this subject (see the review paper [1]). In connection with the property of separability of the Hamilton–Jacobi equation, Benenti introduced a concept of local separability structure [2]. A separability structure in a (V_n, g) (where $n = \dim V_n$ and g is any pseudo-Riemannian metric) is an equivalence class of coordinate charts in which the geodesic equation separates.

Separability structures are classified and characterized by the maximal number $r\,(0 \leqslant r \leqslant n)$ of ignorable coordinates which they allow. We say that a separability structure is of class S_r if it allows exactly r ignorable coordinates. A geometrical characterization of separability structures has been given by Benenti [2] and it is summarized in the following theorem:

Theorem 1. A pseudo-Riemannian manifold (V_n, g) admits a separability structure of class S_r if and only if there exist (locally) r Killing vectors $\underset{\alpha}{X}\,(\alpha = 1, \ldots, r)$ and $n-r$ Killing tensors $\underset{a}{K}$ of order 2 $(a = r+1, \ldots, n)$ altogether independent, such that:

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1) the following commutation relations hold

$$[\underset{\alpha}{X}, \underset{\beta}{X}] = 0 \quad \forall\, \alpha, \beta, \tag{1}$$

$$[\underset{\alpha}{X}, \underset{a}{K}] = 0 \quad \forall\, \alpha, a, \tag{2}$$

$$[\underset{a}{K}, \underset{b}{K}] = 0 \quad \forall\, a, b, \tag{3}$$

in the Lie algebra of Killing tensors with the Schouten–Nijenhuis brackets [3];

2) the $n-r$ Killing tensors have in common $n-r$ independent eigenvectors $\underset{a}{X}$ such that:

$$[\underset{a}{X}, \underset{b}{X}] = 0 \quad \forall\, a, b, \tag{4}$$

$$[\underset{a}{X}, \underset{\alpha}{X}] = 0 \quad \forall\, a, \alpha, \tag{5}$$

$$g(\underset{a}{X}, \underset{\alpha}{X}) = 0 \quad \forall\, a, \alpha. \tag{6}$$

One of the Killing tensors $\underset{\alpha}{K}$ coincides with the metric tensor g.

From the physical point of view, when we consider separability structures in space-time (V_4, g) it turns out that the most interesting ones are those of class S_2. Those space-times admit two commuting Killing vectors ξ and η and a second order Killing tensor K independent of g. In particular this class includes all separable, stationary axisymmetric space-times. Motivated by the fact that those space-times all belong to the Petrov type D family, we investigate the relations between S_2 separability and the algebraic properties of the Weyl tensor, to find out whether S_2 separability restricts the allowed Petrov type of (V_4, g). In this note we show that this is not the case.

2. *Consequences of S_2 separability conditions*

In this Section we investigate some useful differential consequences of the necessary and sufficient separability conditions (1)–(6) when space-time (V_4, g) admits a separability structure of class S_2. First of all we rewrite (1)–(6) in a general coordinate form.

$$\xi_{(i;j)} = 0, \qquad \eta_{(i;j)} = 0, \tag{7}$$

$$\xi_{i;j}\eta^j - \eta_{i;j}\xi^j = 0, \tag{8}$$

$$K_{ij;l} + K_{li;j} + K_{jl;i} = 0, \tag{9}$$

$$K^{ij}{}_{;l}\xi^l - 2K^{l(i}\xi^{j)}{}_{;l} = 0, \tag{10}$$

$$K^{ij}{}_{;l}\eta^l - 2K^{l(i}\eta^{j)}{}_{;l} = 0, \tag{11}$$

$$\xi_{i;j}X^j - X_{i;j}\xi^j = 0, \qquad \eta_{i;j}X^j - X_{i;j}\eta^j = 0, \tag{12}$$

$$\xi_{i;j}Y^j - Y_{i;j}\xi^j = 0, \quad \eta_{i;j}Y^j - Y_{i;j}\eta^j = 0, \tag{13}$$

$$X_i\xi^i = X_i\eta^i = Y_i\xi^i = Y_i\eta^i = 0, \tag{14}$$

$$X_{i;j}Y^j - Y_{i;j}X^j = 0. \tag{15}$$

Here X and Y are the eigenvectors of K which belong to the separability structure.[1]

We will also use the well known integrability conditions for Killing's equations:

$$\xi_{i;j;k} = \xi_l R^l{}_{kji}. \tag{16}$$

Taking into account (7), (12), and (14) one can easily prove that:

$$(X_{i;j} - X_{j;i})\xi^i = 0, \quad (Y_{i;j} - Y_{j;i})\xi^i = 0, \tag{17}$$

as well as similar relations obtained by subsituting η for ξ. Using (14) and (15) one gets

$$\xi_{i;j}(X^iY^j - X^jY^i) = 0 \tag{18}$$

which, by (7), implies that

$$\xi_{i;j}X^iY^j = 0. \tag{19}$$

Similar relations hold with η replacing ξ.

According to conditions (15) the vectors X and Y span a distribution. This distribution is known to be integrable (see [1], Theorem (3.8)). Integrability conditions are the well known relations:

$$X^i{}_{;j}X^j = \alpha X^i + \beta Y^i, \quad X^i{}_{;j}Y^j = aX^i + bY^i, \tag{20}$$

together with similar ones obtained by substituting X by Y and Y by X.

From (12) and (19) one can show that[2]:

$$X^i{}_{;j}\eta^j = A\xi^i + B\eta^i, \quad X^i{}_{;j}\xi^j = C\xi^i + D\eta^i. \tag{21}$$

Using (16) together with (19) and (20) one can easily show that:

$$R_{ijkl}\xi^iX^jY^kX^l = 0, \quad R_{ijkl}\xi^iY^jY^kX^l = 0, \quad R_{ijkl}\eta^iX^jY^kX^l = 0,$$
$$R_{ijkl}\eta^iY^jY^kX^l = 0. \tag{22}$$

Again from (16) and (21) and the fact that the norms of ξ and η are Lie-dragged along ξ and η, after some manipulations one gets:

$$R_{ijkl}\xi^i\eta^j\xi^kX^l = 0, \quad R_{ijkl}\xi^i\eta^j\xi^kY^l = 0, \quad R_{ijkl}\eta^i\xi^j\eta^kX^l = 0,$$
$$R_{ijkl}\eta^i\xi^j\eta^kY^l = 0. \tag{23}$$

It seems that S_2 separability conditions do not imply any further simple algebraic relations involving the Riemann tensor.

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[1] Throughout the rest of this paper Latin indices, $i, j, \ldots$ will run from 0 to 3.

[2] In order to prove (21) one should expand $X^i{}_{;j}\eta^j$ as a linear combination of ξ, η, X and Y and take into account the fact that non-degeneracy of g implies $[g(\xi,\xi)g(\eta,\eta) - g(\xi,\eta)^2][g(X,X)g(Y,Y) - g(X\,Y)^2] \neq 0$. A similar method could be used to prove (20).

3. Petrov types and S_2 separable space-times

In this Section we show that in an empty space-time (V_4, g) admitting a S_2 separability structure all the Petrov types are a priori allowed.

We first introduce an orthonormal tetrad $\{\underset{i}{e}\}$,

$$\underset{0}{e} = \xi, \quad \underset{1}{e} = \eta - \frac{g(\xi, \eta)}{g(\xi, \xi)}\xi, \quad \underset{2}{e} = X, \quad \underset{3}{e} = Y - \frac{g(X, Y)}{g(X, X)}X. \tag{24}$$

Defining this tetrad we assumed that ξ and X are non-isotropic vectors. This does not restrict our discussion, since a theorem due to Benenti [5] assures that null vectors cannot belong to the separability structure. According to the prescription given in [4] (page 340) we construct a matrix $D_{\alpha\beta} = A_{\alpha\beta} + iB_{\alpha\beta}$ $(\alpha, \beta = 1, 2, 3)$ where

$$A_{\alpha\beta} = \hat{R}_{0\alpha 0\beta}, \qquad B_{\alpha\beta} = \tfrac{1}{2}\, \varepsilon_{\alpha\gamma\delta} \hat{R}_{0\beta}{}^{\gamma\delta}. \tag{25}$$

Here $\hat{R}_{ijkl}$ denotes the tetrad components $R(\underset{i}{e}, \underset{j}{e}, \underset{k}{e}, \underset{l}{e})$ of the Riemann tensor. Taking into account (22) and (23) we get

$$||D_{\alpha\beta}|| = \begin{Vmatrix} A_{11}+iB_{11}, & 0, & 0 \\ 0, & A_{22}+iB_{22}, & A_{23}+iB_{23} \\ 0, & A_{23}+iB_{23}, & -(A_{11}+A_{22})-iB(_{11}+B_{22}) \end{Vmatrix} = \left\Vert \begin{array}{c|c} A_{11}+iB_{11} & 0 \\ \hline 0 & \Gamma \end{array} \right\Vert . \tag{26}$$

The eigenvalues of $||D_{\alpha\beta}||$ are given by:

$$\lambda_1 = A_{11} + iB_{11}, \tag{27}$$

$$\alpha\lambda_{2,3} = -\lambda_1 \pm \sqrt{(\lambda_1)^2 - 4 \operatorname{Det} ||\Gamma||}\,. \tag{28}$$

Investigating (27) and (28) and noticing that

$$\lambda_1 + \lambda_2 + \lambda_3 = 0 \tag{29}$$

we come to the conclusion that the matrix (26) can be a priori reduced to all possible canonical forms. This means that empty space-time (V_4, g) can admit all the possible Petrov types when it also admits a S_2 separability structure.

It would be interesting to find explicit examples of S_2 separable solutions of the Einstein equations which do not belong to the already known classes of space-times of type D or N (see [1], part II).

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