ON THE MASS-GROUP ASSOCIATED TO HADRONIC MASS--SPECTRUM. THE ROLE AND THE PROBLEMS OF SL(2, *R*) AND SL(2,*Z*)*

By P. Fré

Istituto di Fisica Teorica, Università di Torino**

(Received July 1, 1978)

This paper is concerned with the study of the SL(2, *R*) group as a hadronic mass-group (= group, one of whose generators is the mass squared operator M^2). It is shown that, besides dual models, also conformal invariant quantum field models admit SL(2, *R*) as a mass-group. The mass-algebra is explicitly constructed and its representation on the field derived. The separation between kinematical and dynamical degrees of freedom results from the construction and the dynamical content of the theory is shown to be specified by the choice of unitary, irreducible representations of the mass-group appearing in the spectrum. The problem of interpreting the modular group invariance of dual mass-spectra is considered from the point of view of breakdown of SL(2, *R*) mass-group and a no-go theorem is proved. Therefore it is shown that the physical intepretation of the modular group in hadron physics must be indirect, in the sense that it is the shadow, on the 2-point function, of another, so far, hidden symmetry.

1. Introduction

The present investigation has been stimulated by the results of two recent rather different approaches to the study of the hadronic mass-spectrum, which, nevertheless, in my opinion, imply a common point of view. The first stimulus was the successful classification of mass-spectra of dual string models carried on by Nahm through the use of modular functions [1]. This approach shows that for all known dual models there is a discrete, for infinite, non-abelian subgroup of the projective group, invariance under which fully determines the mass-spectrum.

The second stimulus has been the algebraic construction devised by Stern and Leutwyler to study composite models of hadrons giving rise to Poincaré invariant theories [2].

* This work has been supported by Deutsche Forschung Gemeinschaft, when the author was at the University of Bielefeld, Federal Republic of Germany.

** Address: Istituto di Fisica Teorica, Università di Torino, C. SO M. d'Azeglio 42, 10125 Torino, Italy.

This latter work is based on a non-linear decomposition of Poincaré algebra decoupling an external from an internal algebra ***D***, which contains the mass-squared operator together with three spin operators spanning the SU(2) algebra. If nothing else than Poincaré invariance is required, the mass-squared operator M^2, which commutes with the spin operators J_1, J_2, J_3, stays alone in an abelian algebra and no information on the spectrum is obtainable algebraically. However it is tempting to assume that M^2 has a few partners and that together they span a non-abelian algebra. If an infinity of masses is to be accomodated in a unitary irreducible representation (UIR) the algebra must be non-compact. In this case the algebra determines the properties of the spectrum. We call mass-algebra the Lie algebra which M^2 belongs to and mass-group the group generated by it.

The hypothesis that the mass-group is non-compact and non-abelian matches the situation of dual models (SU(1,1) $\sim$ SL(2, R)) and, I am going to show, also of irreducible conformal field models. From this point of view the results of paper [1] are both encouraging and intriguing; in fact:

(*i*) the spectrum is associated to a group, yet

(*ii*) the spectrum determining group is non-Lie and does not coincide with the mass--group. However it is at least isomorphic to a subgroup of the mass-group itself. In fact it is a discrete subgroup of SL(2, R).

This suggests that the classifying groups discovered by Nahm could be the outcome of breaking an SL(2, R) symmetry group and the question is how this latter is implemented on the states, related to the mass-group and finally broken. I have not yet found the answer to all these questions, however I have the following two clusters of results and problems to present.

First cluster (Mass-group). Both dual models and conformal invariant field models admit SL(2, R) as mass-group. The two theories differ in the use they make of the group generators: the first one associates the mass-squared operator with the compact subgroup O(2) generator and therefore obtains a discrete spectrum. The second one associates M^2 with a non-compact subgroup and therefore generates a continuous spectrum. The continuity of the spectrum for the conformal model is, on the other hand, a necessary consequence of the fact that conformal symmetry is not the direct product of Poincaré group with an internal group [14]. Therefore what distinguishes the two theories is the connection between the mass-algebra and the space-time algebra which, while is clear by construction for conformal models, is still obscure for dual models.

If the field (dual or conformal) belongs to the discrete series of UIR of SL(2, R) then the complete Fock-space of the theory necessarily contains an infinite number of UIR or the theory is free. In any case the two and three point invariant functions are calculated and exhibited.

Second cluster (Modular subgroups). We have two possibilities of interpreting them. Either they are subgrups of the full mass-group surviving its breakdown as a symmetry of the system, or they act at a different level and are related to the mass group in an indirect way. I show that the second is the only acceptable interpretation because the first is ruled out by a no-go theorem. The proof of the theorem also shows that modular invariance of the two-point function is similar to the requirement of an infinity of periodic boundary conditions, which, however, the theorem forbids to implement on the states in a simple way. The implementation of the modular conditions in Fock-space and the consequent physical interpretation of the modular function weight (critical dimension of dual models) seems a rather difficult unsolved problem which needs a breakthrough in understanding.

The paper is organized as follows. In Section 2 I introduce the non-abelian mass--group and confront the situation of dual and conformal models. In Section 3 I connect the UIR of the mass-group with the fields transformation laws. Two- and three-point functions are derived in Section 4. The discussion of the modular two-point fuctions in the context emerging from the previous sections is effected in Section 5. In the same section I give the proof of the no-go theorem. Section 6 is devoted to conclusions.

2. *Non-abelian mass-group in dual and conformal models*

In Ref. [2] the concept of dynamical algebra arises from the following construction. Let P_μ, $M_{\mu\nu}$ be the generators of Poincaré group satisfying

$$[P_\mu, P_\nu] = 0, \quad [M_{\lambda\mu}, P_\nu] = i(g_{\mu\nu}P_\lambda - g_{\lambda\nu}P_\mu),$$
$$[M_{\mu\nu}, M_{\varrho\sigma}] = -i(g_{\mu\varrho}M_{\nu\sigma} + g_{\nu\sigma}M_{\mu\varrho} - g_{\nu\varrho}M_{\mu\sigma} - g_{\mu\sigma}M_{\nu\varrho}), \tag{2.1}$$

and let the light-plane components of Lorentz vectors and tensors be defined in the usual way; for any v_μ and in space-time dimensions d

$$v_+ = \frac{1}{\sqrt{2}}(v_0 + v_{d-1}), \quad v_- = \frac{1}{\sqrt{2}}(v_0 - v_{d-1}), \quad v_\perp = (v_1, \ldots, v_r, \ldots, v_{d-2}). \tag{2.2}$$

We can rewrite the Lie algebra (2.1) in a new form if we introduce the notations

$$S_r = M_{+,r} = \frac{M_{0r} + M_{d-1,r}}{\sqrt{2}},$$
$$J_{rs} = M_{rs}, \quad B_r = M_{-,r}, \quad N_3 = -M_{+,-} = -M_{0,d-1}, \quad P_\mu = (P_+, P_-, P_\perp). \tag{2.3}$$

The generators $P_+, P_\perp, S_\perp, J_{rs}, N_3$ span a subalgebra $\boldsymbol{K}$, (the kinematical algebra) of dimension $(4(d-1)+(d-2)(d-3))/2$ which generates the stability subgroup of the light-like plane

$$x_+ = 0 \tag{2.4}$$

and has the following Lie structure

$$[P_+, J_{rs}] = [P_+, S_r] = [P_+, P_\perp] = [P_\perp, N_3] = 0,$$
$$\left[\begin{Bmatrix} P_t \\ S_t \end{Bmatrix}, J_{rs}\right] = i\left(\delta_{ts}\begin{Bmatrix} P_r \\ S_r \end{Bmatrix} - \delta_{tr}\begin{Bmatrix} P_s \\ S_s \end{Bmatrix}\right),$$
$$[P_r, S_s] = i\delta_{rs}P_+, \quad [N_3, P_+] = iP_+, \quad [N_3, S_r] = iS_r. \tag{2.5}$$

The remaining three generators lying outside $K(P_-, B_\perp)$ satisfy the following commutation relations among themselves and with the elements of K

$$[B_r, P_-] = [B_r, B_s] = 0, \quad [S_r, P_-] = [B_r, P_+] = -iP_r, \quad [P_r, B_s] = i\delta_{rs}P_-,$$
$$[N_3, B_s] = -iB_s, \quad [S_r, B_s] = -iJ_{rs} + i\delta_{rs}N_3. \tag{2.6}$$

Now let D be a Lie algebra with the following properties

$$D = S \oplus M^2, \quad [M^2, S] = 0, \tag{2.7}$$

where M^2 is a one-dimensional abelian algebra generated by an operator M^2 and S is an SO$(d-1)$ algebra generated by $(d-1)(d-2)/2$ spin operators J_i. In the case $d = 4$, for instance the Lie algebra D is

$$[M^2, J_i] = 0, \quad [J_i, J_j] = i\varepsilon_{ijk}J_k. \tag{2.8}$$

If the algebra D commutes with the kinematical algebra

$$[D, K] = 0 \tag{2.9}$$

the authors of [2] prove the following:
Poincaré invariance reconstruction theorem: If H is the carrying space of a unitary (in general reducible) representation of the stability subgroup generated by K and $\bar{H}$ is the carrying space of a unitary (in general reducible) representation of the group generated by D, then the space $\mathscr{H} = H \otimes \bar{H}$ carries a unitary (in general reducible) representation of the Poincaré group. The three operators $(P_-, B_\perp)$ which are required, for instance in the case $d = 4$, to close Poincaré algebra are represented in the following way in $\mathscr{H}$

$$P_- = (P_\perp^2 + M^2)/2P_+, \tag{2.10a}$$

$$B_r = [P_-S_r - P_rN_3 - \varepsilon_{rs}(P_sJ_3 + \sqrt{M^2}\, J_s)]/P_+. \tag{2.10b}$$

Moreover the following identification is made

$$J_3 = Q_3 \equiv J_{rs} + (S_rP_s - S_sP_r)/P_+, \tag{2.11}$$

where Q_3 is the Casimir operator of the Lie algebra K. From the point of view of this theorem all the information about the spectrum of a relativistic system comes from the study of the dynamical algebra D. On the other hand, at this stage, any spectrum can be accomodated because the only requirement is that M^2 commutes with S and therefore it is a perfectly arbitrary operator. However M^2, in a hadronic theory, must be an infinite rank operator with an infinite spectrum of eigenvalues. Moreover, as the hadrons interact with each other, the theory must contain additional operators T_i (i belonging to an appropriate set of indices) which connect different mass-states. It follows that the T_i do not commute with M^2,

$$[T_i, M^2] \neq 0, \tag{2.12}$$

and are not Poincaré invariant operators. The first statement is self-evident; to prove the second assume, for instance, that the T_i commute at least with the kinematical algebra

$$[T_i, \boldsymbol{K}] = 0, \tag{2.13}$$

then it follows:

$$[T_i, P_-] = [T_i, M^2]/2P_+, \tag{2.14a}$$

$$[T_i, B_r] = P_+^{-1}[T_i, P_-]S_r - \varepsilon_{rs}P_+^{-1}[T_i, \sqrt{M^2}\, J_s]. \tag{2.14b}$$

So Poincaré invariance of T_i is consistent only with $[T_i, M^2] = 0$, which contradicts (2.12).

We can conclude that, in a hadronic theory the mass-squared operator M^2 belongs to a non-abelian algebraic structure containing operators T_i with non-trivial Poincaré transformations. It is very much tempting to assume that the set $\{M^2, T_i\}$ closes a Lie algebra. In this case we would have a new dynamical algebra D' such that

$$\boldsymbol{D}' = \boldsymbol{S} \oplus \boldsymbol{M}^{2'}, \tag{2.15}$$

where $\boldsymbol{M}^{2'}$ is now a non-abelian algebra containing M^2. I call it the non-abelian mass--algebra. In this more general case it is not necessary that $\boldsymbol{S}$ commutes with the whole $\boldsymbol{M}^{2'}$ but we only require that it commutes with M^2. Therefore in general we have

$$[\boldsymbol{S}, \boldsymbol{M}^{2'}] \subseteq \boldsymbol{D}, \qquad [\boldsymbol{S}, M^2] = 0. \tag{2.16}$$

However, as I am mainly interested in $\boldsymbol{M}^{2'}$ I forget, for the moment, about $\boldsymbol{S}$ and I behave as if all the hadrons were spinless (this means that I select the trivial identity representation of SU(2)).

As I already pointed out, in order to accomodate an infinite spectrum $\boldsymbol{M}^{2'}$ must be non-compact. The smallest non-abelian, non-compact algebra is that of traceless 2-dimensional matrices, which is isomorph to the Lie algebra of the groups SU(1, 1), SL(2, R) and O(2, 1). It is rather remarkable that we have indeed two classes of models with such a mass-group, namely dual models and conformal invariant field models.

For dual models (Veneziano model for instance) we write [3]

$$M_{\mathrm{D}}^2 = \frac{1}{\alpha}\sum_{n=1}^{\infty} n \sum_{i=1}^{d-2} a_n^{i\dagger} a_n^i - \alpha_0/\alpha, \tag{2.17}$$

where α is a universal scale-fixing parameter (Regge-slope), d is the number of space-time dimensions and the transverse mode operators satisfy the following Weyl algebra

$$[a_n^i, a_m^{j\dagger}] = \delta_{ij}\delta_{nm}. \tag{2.18}$$

In the theory one also introduces two other operators

$$T_+ = \sum_{n=1}^{\infty} \sqrt{n(n+1)} \sum_{i=1}^{d-2} a_{n+1}^{i\dagger} a_n^i, \tag{2.19a}$$

$$T_- = \sum_{n=2}^{\infty} \sqrt{n(n-1)} \sum_{i=1}^{d-2} a_{n-1}^{i\dagger} a_n^i, \tag{2.19b}$$

which, together with the operator

$$T_0 = \alpha M_D^2 + \alpha_0, \tag{2.20}$$

as a consequence of (2.18) satisfy the algebra

$$[T_+, T_-] = -2T_0, \quad [T_0, T_\pm] = \pm T_\pm, \tag{2.21}$$

and the relations

$$T_+^\dagger = T_-, \quad T_0^\dagger = T_0. \tag{2.22}$$

The set of equations (2.21) and (2.22) implies that the Hilbert space on which the T_i act carriers a unitary, in general reducible, representation of the SL(2, R) group and that T_0 is the generator of the maximal compact subgroup O(2) [4, 5]. Hence the spectrum of T_0 is discrete and equally spaced. The relation of the operators T_i with the usual Virasoro operators of dual models and space-time generators is the following:

$$T_0 = L_0^{\mathrm{tr}} - p_\perp^2 \alpha, \tag{2.23a}$$

$$T_+ = L_{-1}^{\mathrm{tr}} - a_1^{(\perp)\dagger} \cdot p_\perp \sqrt{\alpha}\,, \tag{2.23b}$$

$$T_- = L_{+1}^{\mathrm{tr}} - p_\perp \cdot a_1^{(\perp)} \sqrt{\alpha}\,, \tag{2.23c}$$

where the symbol "tr" means that we have taken only the contribution from the transverse modes.

The Poincaré transformation properties of the operators $T_\pm$ are intricate as it is evident from Eqs (2.23) because of the simultaneous gauge invariance of the model. The generators of the physical mass-group, as previously defined, are however the T_i and not the Poincaré invariant L_n so that the merging of Poincaré group and gauge group seems to be the non-trivial mechanism which defines correct transformation laws of $T_\pm$ consistent with $M^2 = T_0$ and hence with the discrete mass-spectrum. The subtleties which must be involved become apparent if we confront Eqs (2.21), (2.22) with their analogues emerging in a conformal field model.

Let me enlarge the space-time algebra (2.1) by adding the generators of special conformal transformations K_μ and the dilatation generator D. The full conformal algebra is given by the following commutation relations together with those reported in (2.1) (I follow the notations of Ref. [6])

$$[K_\mu, K_\nu] = [D, M_{\mu\nu}] = 0, \quad [D, K_\mu] = iK_\mu, \quad [D, P_\mu] = -iP_\mu,$$
$$[K_\varrho, M_{\mu\nu}] = i(g_{\varrho\mu}K_\nu - g_{\varrho\nu}K_\mu), \quad [K_\mu, P_\nu] = -2i(g_{\mu\nu}D + M_{\mu\nu}). \tag{2.24}$$

Now considering the light-plane components of the vector operator K_μ, using (2.5), (2.6) and (2.24), it is lenghty, but straightforward to prove that the following operators

$$\mathcal{T}_+ = M^2 = P_\mu P^\mu, \tag{2.25a}$$

$$\mathcal{T}_0 = \frac{i}{2}(D + N_3 - S_\perp \cdot P_\perp / P_+) + \frac{\nu}{2}\,, \tag{2.25b}$$

$$\mathcal{T}_- = -\tfrac{1}{4}(K_+/P_+ - S_\perp^2/P_+^2), \tag{2.25c}$$

(where $\nu = (d-2)/2$) satisfy SL(2, R) algebra in the form

$$[\mathcal{T}_+, \mathcal{T}_-] = -2\mathcal{T}_0, \quad [\mathcal{T}_0, \mathcal{T}_\pm] = \pm\mathcal{T}_\pm, \tag{2.26}$$

and commute with the generators of the kinematical algebra

$$[\mathcal{T}_n, \boldsymbol{K}] = 0. \tag{2.27}$$

We see therefore that for a system having conformal symmetry the mass-operators M^2 is embedded in an SL(2, R) algebra. However the difference with the dual case is also apparent. Here we have in fact

$$\mathcal{T}_0^\dagger = -\mathcal{T}_0, \quad \mathcal{T}_\pm^\dagger = \mathcal{T}_\pm, \quad M_C^2 = \mathcal{T}_+, \tag{2.28}$$

which is to be compared with (2.20), (2.21), (2.22).

To understand the implications of (2.28) it is covenient to consider the relation of the operators (2.21) and (2.28) with the three independent generators of O(2, 1) $\sim$ SU(1,1) in its standard form. This latter is given by three pseudo-spin operators satisfying [7]

$$[X_0, X_1] = iX_2, [X_0, X_2] = -iX_1, [X_1, X_2] = -iX_0. \tag{2.29}$$

X_0 generates the compact O(2) subgroup of rotations of the xy-plane, while X_1 and X_2 generate the O(1, 1) subgroups of Lorentz transformations in the x and y direction respectively. The lowest dimensional representation of the Lie-algebra (2.29) is given by the following 2×2 matrices:

$$X_0 = -\tfrac{1}{2}\sigma_3 = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \tag{2.30a}$$

$$X_1 = \frac{i}{2}\sigma_2 = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}, \tag{2.30b}$$

$$X_2 = \frac{i}{2}\sigma_1 = \begin{pmatrix} 0 & -i/2 \\ -i/2 & 0 \end{pmatrix}, \tag{2.30c}$$

where $(\sigma_1, \sigma_2, \sigma_3)$ are the usual Pauli matrices. The representation (2.30) of the algebra generates the defining representation of the group SU(1, 1) [7] and we have the three standard one-parameter subgroups

$$\exp(i\mu X_0) = \begin{pmatrix} e^{i\mu/2} & 0 \\ 0 & e^{-i\mu/2} \end{pmatrix}, \tag{2.31a}$$

$$\exp(i\nu X_1) = \begin{pmatrix} \cosh \nu/2 & i\sinh \nu/2 \\ -i\sinh \nu/2 & \cosh \nu/2 \end{pmatrix}, \tag{2.31b}$$

$$\exp(i\xi X_2) = \begin{pmatrix} \cosh \xi/2 & \sinh \xi/2 \\ \sinh \xi/2 & \cosh \xi/2 \end{pmatrix}. \tag{2.31c}$$

In a unitary representation of SU(1, 1) the spin-operators X are represented by hermitean operators and it is an easy task to verify that defining

$$T_0 \equiv X_0, \; T_\pm = X_1 \pm iX_2, \tag{2.32}$$

we satisfy the equations (2.21), (2.22), while defining

$$\mathscr{T}_0 = -iX_1, \quad \mathscr{T}_+ = X_0 - X_2, \quad \mathscr{T}_- = -X_0 - X_2, \tag{2.33}$$

we satisfy the equations (2.26) and (2.28). So from the algebraic point of view we have: In dual models

$$M_D^2 = X_0 - \alpha_0/\alpha \quad \text{(compact)}. \tag{2.34}$$

In conformal models

$$M_C^2 = X_0 - X_2 \quad \text{(non-compact)}. \tag{2.35}$$

The mathematical structure of the two models is therefore similar although the physical interpretation of the mathematical entities is different. In the next two sections I discuss how the mass group is implemented on the fields and the physical states.

3. *Unitary irreducible representations of the mass-group and the fields*

The UIR of the group SL(2, R) are very well known [4, 5, 7]. They are classified according to the value of the Casimir operator

$$Q = X_1^2 + X_2^2 - X_0^2 \tag{3.1}$$

and the spectrum of the eigenvalues x_0 of X_0. Parametrizing the eigenvalue q of Q in the following way:

$$q = (1-k/2)k/2 \tag{3.2}$$

all possible cases are listed below:

A) Continuous class, non exceptional interval, integral case C_q^0

$$1/4 \leqslant q < \infty \; (k = 1 + i\sigma, \sigma \in R), \; x_0 = \pm 0, \pm 1, \pm 2, \ldots$$

B) Continuous class, non exceptional interval, half-integral case $C_q^{\frac{1}{2}}$

$$1/4 \leqslant q < \infty \; (k = 1 + i\sigma, \sigma \in R), \; x_0 = \pm \tfrac{1}{2}, \pm(\tfrac{1}{2}+1), \ldots$$

C) Continuous class exceptional interval E_k

$$0 < q < 1/4 \; (k \in R, 0 \leqslant k < 2), \; x_0 = \pm 0, \pm 1, +2, \ldots$$

D) Discrete class D_k^+

$$k = 1, 2, 3, \ldots, \; q = k/2(1-k/2), \; x_0 = k/2, \; k/2+1, \; k/2+2, \ldots$$

E) Discrete class, D_k^-

$$k = 1, 2, 3, \ldots, \; q = k/2(1-k/2), \; x_0 = -k/2, \; -k/2-1, \ldots$$

For the discussion of mass-group only the discrete class and the continuous class in the exceptional interval are relevant because (a) in dual models the operator X_0 is M^2 and the positivity of the mass-spectrum requires x_0 to be bounded from below. This selects the class D_k^+ of representations. (b) In conformal models k is the anomaly of the scale dimension which must be real positive. So only E_k and $D_k^\pm$ are admitted. The statement (b) must be proved. Actually I shall show that for a conformal field there are two different regimes. If $k < 2$ the mass-states fall off in a UIR of type E_k and a field of scale dimension

$$l = v+k \qquad \left(v = \frac{d-2}{2}\right) \tag{3.3}$$

can be coupled to itself in a non trivial three-point function. On the other hand if $k > 2$ then the mass-states necessarily fall off in a UIR of type D_k^+ and the expansion of the two fields product contains only fields of higher scale dimension. The field is not coupled to itself.

Let me then consider a quantum field $\phi_k(x)$ which transforms irreducibly under conformal algebra. For simplicity let me take $\phi_k(x)$ scalar and hermitean. I can write [6]

$$[P_\mu, \phi_k(x)] = i\partial_\mu \phi_k(x), \tag{3.4a}$$

$$[M_{\mu\nu}, \phi_k(x)] = i(x_\mu\partial_\nu - x_\nu\partial_\mu)\phi_k(x), \tag{3.4b}$$

$$[K_\mu, \phi_k(x)] = i(2x_\mu x\cdot\partial - x^2\partial_\mu + 2lx_\mu)\phi_k(x), \tag{3.4c}$$

$$[D, \phi_k(x)] = i(x\cdot\partial + l)\phi_k(x), \tag{3.4d}$$

where the scale dimension l is given in (3.3). Hermiticity of $\phi_k(x)$ yields

$$\phi_k(x) = \phi_k^{(+)}(x) + \phi_k^{(-)}(x), \tag{3.5}$$

where

$$\phi_k^{(\pm)}(x) = \int \frac{d^d p}{(2\pi)^{d/2}} \exp(\mp ip\cdot x)\varphi_k^{(\pm)}(p), \tag{3.6a}$$

$$\phi_k^{(-)}(x)^\dagger = \phi_k^{(+)}(x), \tag{3.6b}$$

and the negative frequency part $\phi_k^{(-)}(x)$ annihilates the vacuum state

$$\phi_k^{(-)}(x)\,|0\rangle = 0. \tag{3.7}$$

Making use of the light-plane variables (2.2) I rewrite

$$\phi_k^{(\pm)}(x) = \int d\underset{\sim}{p}\,\psi^{(\pm)}(\underset{\sim}{p}, x) \int_0^\infty d\mu \exp\left(\mp i\mu \frac{x_+}{2p_+}\right) \varphi_k^{(\pm)}(\underset{\sim}{p}, \mu), \tag{3.8}$$

where $\underset{\sim}{p} = (p_+, p_\perp)$ and the integration measure is

$$d\underset{\sim}{p} = \frac{dp_+}{2p_+}\, d^{2v}p_\perp.$$

The kinematical wave-functions $\psi^{(\pm)}(\underset{\sim}{p}, x)$ have the form

$$\psi^{(\pm)}(\underset{\sim}{p}, x) = (2\pi)^{-d/2} \exp\left[\mp i\left(\underset{\sim}{p}\tilde{x} + p_\perp^2 \frac{x_+}{2p_+}\right)\right] \tag{3.10}$$

and

$$\underset{\sim}{p}\tilde{x} = p_+x_- - p_\perp \cdot x_\perp. \tag{3.11}$$

The notation (3.8) inspires the definition of a new operator valued distribution

$$A_k^{(\pm)}(u, \underset{\sim}{p}) = \int_0^\infty d\mu \exp(-i\mu u)\varphi_k^{(\pm)}(\mu, \underset{\sim}{p}), \tag{3.12}$$

which I can use to rewrite the field in the amusing form

$$\phi_k^{(\pm)}(x) = \int d\underset{\sim}{p}\, \psi^{(\pm)}(\underset{\sim}{p}, x) A^{(\pm)}\left(\frac{x_+}{2p_+}, \underset{\sim}{p}\right). \tag{3.13}$$

Now let $f \in \mathscr{F}$ where $\mathscr{F}$ is the test-function space for the field $\phi_k(x)$. The states

$$|f\rangle = \int d^dx \overline{f(x)} \phi_k^{(+)}(x) \,|0\rangle \tag{3.14}$$

form a pre-Hilbert space which, after completion, becomes the one-particle sector. The scalar product is

$$\langle f_2|f_1\rangle_k = \int d^dx_1 \int d^dx_2 \overline{f_1(x_1)} \Delta_k(x_1, x_2) f_2(x_2), \tag{3.15}$$

where

$$\Delta_k(x_1, x_2) = \langle 0|\phi^{(-)}(x_2)\phi^{(+)}(x_1)\,|0\rangle \tag{3.16}$$

is the 2-point function.

If we introduce the transformation

$$f(\underset{\sim}{p}, u) = \int_0^\infty d\mu \exp(-i\mu u) \int \frac{d^dx}{(2\pi)^{d/2}} \exp\left(i\underset{\sim}{p}\tilde{x} + i\frac{p_\perp^2+\mu}{2p_+} x_+\right) f(x), \tag{3.17}$$

the states (3.14) admit the representations

$$|f\rangle = \int d\underset{\sim}{p} \int_{-\infty}^{+\infty} du \overline{f(\underset{\sim}{p}, u)} A^{(+)}(u, \underset{\sim}{p})\,|0\rangle \tag{3.18}$$

and the scalar product becomes

$$\langle f_2|f_1\rangle_k = \int d\underset{\sim}{p} \int d\underset{\sim}{q} \int_{-\infty}^{+\infty} du_1 \int_{-\infty}^{+\infty} du_2 \overline{f_1(\underset{\sim}{p}, u_1)} \Delta_k(\underset{\sim}{p}, \underset{\sim}{q}, u_1, u_2) f_2(\underset{\sim}{q}, u_2), \tag{3.19}$$

where

$$\Delta_k(\underset{\sim}{p}, \underset{\sim}{q}, u_1, u_2) = \langle 0|A_k^{(-)}(\underset{\sim}{q}, u_2) A_k^{(+)}(\underset{\sim}{p}, u_1)\,|0\rangle. \tag{3.20}$$

Poincaré invariance of the vacuum gives

$$\langle 0|\varphi_k^{(-)}(p)\varphi_k^{(+)}(p')\,|0\rangle = \delta^d(p-p')\varrho_k(p^2), \tag{3.21}$$

where $\varrho_k(p^2)$ is a distribution in squared mass only. From Eq. (3.21) it follows that

$$\langle 0|A_k^{(-)}(\underset{\sim}{q}, u_2)A_k^{(+)}(\underset{\sim}{p}, u_1)\,|0\rangle = 2p_+\delta(\underset{\sim}{p}-\underset{\sim}{q})\overline{G_k(u_2-u_1)}, \tag{3.22}$$

where

$$G_k(u_1-u_2) = \int_0^\infty d\mu \exp\left[i\mu(u_1-u_2)\right]\varrho_k(\mu). \tag{3.23}$$

Now let $b_N(\underset{\sim}{p})$ be an orthonormal basis in the space $L^2(R^{2\nu+1})$ with the integration measure (3.9). We can write

$$f(\underset{\sim}{p}, u) = \sum_{N=0}^\infty b_N(\underset{\sim}{p})\chi_N(u), \tag{3.24}$$

and the scalar product decomposes in the following way:

$$\langle f_2|f_1\rangle_k = \sum_{N=0}^\infty (\chi_N^{(1)}, \chi_N^{(2)})_k, \tag{3.25}$$

where

$$(\chi_N^{(1)}, \chi_N^{(2)})_k = \int_{-\infty}^{+\infty} du_1 \int_{-\infty}^{+\infty} du_2\overline{\chi_N^{(1)}(u_1)}G_k(u_1-u_2)\chi_N^{(2)}(u_2). \tag{3.26}$$

This suggests to define

$$^{(k)}O_N^{(+)}(u) = \int d\underset{\sim}{p}\overline{b_N(\underset{\sim}{p})}A_k^{(+)}(u, \underset{\sim}{p}), \tag{3.27a}$$

$$^{(k)}O_N^{(-)}(u) = {}^{(k)}O_N^{(+)}(u)^\dagger, \tag{3.27b}$$

and write

$$A_k^{(+)}(u, \underset{\sim}{p}) = \sum_{N=0}^\infty {}^{(k)}O_N^{(+)}(u)b_N(\underset{\sim}{p}). \tag{3.28}$$

The field operators $O_N(u)$ have the following vacuum expectation value

$$\langle 0|{}^{(k)}O_N^{(-)}(u_2){}^{(k)}O_M^{(+)}(u_1)\,|0\rangle = \delta_{N,M}G_k(u_1-u_2). \tag{3.29}$$

Eq. (3.29) follows from (3.27) and (3.22).

The field is therefore decomposed into the sum of an infinite number of orthogonal components which, however, as we shall presently see, behave in an isomorphic way under the mass-group. To prove the last statement let me consider the generalized states $A_k^{(+)}(u, \underset{\sim}{p})|0\rangle$, and look for the representation of the mass-group generators (2.25) on them. I can write

$$\mathcal{T}_n A_k^{(+)}(u, \underset{\sim}{p})\,|0\rangle = [\mathcal{T}_n, A_k^{(+)}(u, \underset{\sim}{p})]\,|0\rangle + A_k^{(+)}(u, \underset{\sim}{p})\mathcal{T}_n|0\rangle. \tag{3.30}$$

If the vacuum is conformal invariant, the second term in the r.h.s. of Eq. (3.30) is known and the transformation laws (3.4) of the field induce the behaviour of $A^{(+)}(u, \underset{\sim}{p})$ under $\mathscr{T}_n$. In the appendix A I show that

$$\mathscr{T}_+ A_k^{(+)}(u, \underset{\sim}{p})\,|0\rangle = i\frac{d}{du}A_k^{(+)}(u, \underset{\sim}{p})\,|0\rangle, \tag{3.31a}$$

$$\mathscr{T}_0 A_k^{(+)}(u, \underset{\sim}{p})\,|0\rangle = -\left(u\frac{d}{du}+\frac{k}{2}\right)A_k^{(+)}(u, \underset{\sim}{p})\,|0\rangle, \tag{3.31b}$$

$$\mathscr{T}_- A_k^{(+)}(u, \underset{\sim}{p})\,|0\rangle = i\left(u^2\frac{d}{du}+ku\right)A_k^{(+)}(u, \underset{\sim}{p})\,|0\rangle, \tag{3.31c}$$

where k is the anomaly of the scale dimension introduced in Eq. (3.3). From Eq. (3.31) I can conclude that for any N the generalized states ${}^{(k)}O_N^{(+)}(u)|0\rangle$ transform according to an irreducible unitary representation of the group SL(2, R). In fact using the relations (2.34) we obtain the representation of the standard generators X_0, X_1, X_2

$$X_0\,{}^{(k)}O_N^{(+)}(u)\,|0\rangle = \frac{i}{2}\left[(1+u^2)\frac{d}{du}+ku\right]{}^{(k)}O_N^{(+)}(u)\,|0\rangle, \tag{3.32a}$$

$$X_1\,{}^{(k)}O_N^{(+)}(u)\,|0\rangle = -i\left[u\frac{d}{du}+\frac{k}{2}\right]{}^{(k)}O_N^{(+)}(u)\,|0\rangle, \tag{3.32b}$$

$$X_2\,{}^{(k)}O_N^{(+)}(u)\,|0\rangle = -\frac{i}{2}\left[(1-u^2)\frac{d}{du}-ku\right]{}^{(k)}O_N^{(+)}(u)\,|0\rangle, \tag{3.32c}$$

and by confrontation with the results collected in appendix B you see that those on the r.h.s. of Eq. (3.32 a–c) are the infinitesimal generators of the representation D_k^+ for k integer and larger than zero and of the representation E_k for k real in the interval from zero to two.

Once this has been checked it opens the possibility of con structing the whole Fock space of the theory by purely group theoretical tools. In fact the one particle states created by the field

$${}^{(k)}O_N(u) = {}^{(k)}O_N^{(-)}(u)+{}^{(k)}O_N^{(+)}(u) \tag{3.33}$$

due to the transformation laws (3.32), must fill up the carrying space of the UIR selected by the value of k. The one-particle sector is therefore fully determined. The many-particle states, on the other hand, are created from the vacuum by the polynomials of the field and their limit points. They are therefore Kronecker products of irreducible states and can be decomposed into irreducible components once the Clebsch–Gordan coefficients (CGc) are known. The N-point functions are all determined as appropriate combinations of CGc. The three-point function in particular is just proportional to such a coefficient. The details of this programme are developed in the next section.

4. *Two- and three-point functions and the Fock-space*

I have already introduced the two-point function in Eq. (3.29). It is the metric of the one particle sector. The index N is unaffected by the action of the mass-group and therefore I shall drop it. In the following ${}^{(k)}O^{(\pm)}(u)$ will denote any N-th component of the field $A_k^{(\pm)}(u)$ and $\chi(u)$ any N-th component of a test-function (3.24). Following a method by Rühl [9] I want to show that $G_k(u_1, u_2)$ is fully determined by group-theory. For this purpose let $R_0^{(k)}, R_1^{(k)}, R_2^{(k)}$ denote the differential operators representing X_0, X_1, X_2 in Eq. (3.32) and let me introduce the states

$$|\chi\rangle = \int_{-\infty}^{+\infty} du \overline{\chi(u)}{}^{(k)}O^{(+)}(u)\,|0\rangle. \tag{4.1}$$

Using Eq. (3.32) and integrating by parts I obtain

$$X_n|\chi\rangle = \int_{-\infty}^{+\infty} du \overline{\chi(u)} R_n^{(k)} O^{(+)}(u)\,|0\rangle = |{}^*R^{(k)}\chi\rangle \tag{4.2}$$

where the operators ${}^*R_n^{(k)}$ are the so called shadows of $R_n^{(k)}$. A very elementary algebra shows that

$${}^*R_n^{(k)} = R_n^{k^*}, \qquad k^* = 2-k. \tag{4.3}$$

So if the field transforms according to the representation k the test functions transform according to the representation k^*; the difference between the two regimes $k < 2$ and $k > 2$ is evident at this level. In the first case the representations k and k^* are unitarily equivalent because they correspond to the same value of the Casimir and have the same spectrum. In the second case if k is positive k^* must be negative and, although the Casimir has the same value, the representations cannot be unitarily equivalent; the representation k has in fact the spectrum bounded from below, while k^* has the spectrum bounded from above (discrete series of type D^+ and D^-). The second case is the less usual but more interesting for the present considerations because the representation involved is the same as in dual models.

As I have recalled in appendix B, the representation D_k^+ is realized in a Hilbert space of holomorphic (or antiholomorphic) functions $f(\omega)$ defined over the upper complex plane. The scalar product is local and is given in Eq. (B11). However, as shown by Rühl [9], the elements of this space, called $\overline{\mathscr{H}}_k$, are completely specified by their boundary values on the real axis, which are interpreted as distributions. The connection with field theory is provided by an intertwining operator which maps the space of these boundary values into the space of test functions $\chi(u)$. It turns out that such an intertwining operator is just the inverse of the two-point function.

I proceed as follows. I consider two arbitrary elements $f_1, f_2 \in \overline{\mathscr{H}}_k$. Their scalar product is

$$(f_1, f_2)_k = \frac{2^{k-2}(k-1)}{\pi} \int_{-\infty}^{+\infty} du \int_0^{\infty} dv v^{k-2} \overline{f_1(u+iv)} f_2(u+iv). \tag{4.4}$$

Introducing a Fourier–Laplace representation for f_1 and f_2

$$f_{1,2}(\omega) = \int_0^\infty d\mu \exp(i\mu\omega) g_{1,2}(\mu), \tag{4.5}$$

where

$$\omega = u + iv, \tag{4.6}$$

and inserting (4.5) into (4.4) one obtains a new representation of the scalar product

$$(f_1, f_2)_k = \int_0^\infty \overline{g_1(\mu)} g_2(\mu) \frac{\Gamma(k)}{\mu^{k-1}} d\mu. \tag{4.7}$$

The boundary values of f_1 and f_2 can be defined from (4.5) by posing $v = 0$

$$\hat{f}_{1,2}(u) = \int_0^\infty d\mu \exp(i\mu u) g_{1,2}(\mu). \tag{4.8}$$

In terms of the boundary values we have a third reprentation of the scalar product

$$(f_1, f_2)_k = \int_{-\infty}^{+\infty} du_1 \int_{-\infty}^{+\infty} du_2 \overline{\hat{f}_1(u_1)} S_k(u_1 - u_2) \hat{f}_2(u_2), \tag{4.9}$$

where

$$S_k(u_1 - u_2) = \int_0^\infty d\mu \exp[i\mu(u_1 - u_2)] \frac{\Gamma(k)}{(2\pi)^2 \mu^{k-1}} = \frac{(2\pi)^{-2}\Gamma(k)\Gamma(2-k)(i)^{k-2}}{[u_1 - u_2 + i\varepsilon]^{2-k}}. \tag{4.10}$$

I can consider $S_k(u_1 - u_2)$ as the kernel of an integral operator

$$[S_k f](u_1) = \int_{-\infty}^{+\infty} S_k(u_1 - u_2) f(u_2) du_2 \tag{4.11}$$

and denoting by ((,)) the usual scalar product of $L^2(R)$:

$$((f, g)) = \int_{-\infty}^{+\infty} du \bar{f}(u) g(u) \tag{4.12}$$

I can write:

$$(f_1, f_2)_k = ((\hat{f}_1, S_k \hat{f}_2)). \tag{4.13}$$

The differential operators $R_n^{(k)}$ representing the generators X_n are self-adjoint with respect to the scalar product of $\mathscr{H}_k$. This means

$$(R_n^{(k)} f_1, f_2)_k = (f_1, R_n^{(k)} f_2)_k. \tag{4.14}$$

Inserting (4.13) into (4.14) and integrating by parts I obtain

$$((\hat{f}_1, {}^*R_n^{(k)} S_k f_2)) = ((f_1, S_k R_n^{(k)} f_2)), \tag{4.15}$$

which shows that S_k intertwines between the representation generated by $R_n^{(k)}$ and that generated by $*R_n^{(k)}$. Because of this property of S_k we can identify the boundary values $\hat{f}(u)$ with the images of the test-functions under S_k^{-1}

$$\hat{f}_\chi(u) = S_k^{-1}\chi(u). \tag{4.16}$$

The scalar product of two test-functions is determined by the two-point function $G_k(u_1-u_2)$ and has been given in Eq. (3.26). On the other hand, as the mass-group is unitarily and irreducibly reproduced in the one-particle sector, we must have

$$\langle\chi_2|\chi_1\rangle = (\hat{f}_{\chi_1}, \hat{f}_{\chi_2})_k \equiv ((\hat{f}_{\chi_1} S_k \hat{f}_{\chi_2})) = ((\chi_1, G_k\chi_2)), \tag{4.17}$$

which finally gives the two-point function

$$G_k(u_1-u_2) = S_k^{-1}(u_1-u_2) = i^k(u_1-u_2+i\varepsilon)^{-k}. \tag{4.18}$$

Comparing (4.18) with (3.23) I also obtain the spectral function $\varrho_k(\mu)$

$$\varrho_k(\mu) = \frac{\mu^{k-1}}{\Gamma(k)}. \tag{4.19}$$

Eq. (4.19) is a standard result for conformal invariant field theories [10, 11]. Anyhow, the performed exercise shows that Eqs (4.18), (4.19) are a consequence of the SL(2, R) group structure and do not depend on the interpretation of the parameter μ as mass squared which is typical of conformal models where we have $X_0-X_2 = M_C^2$ (see Eqs (2.34), (2.35)). The structural analogy between the two models (dual and conformal) becomes fully evident when the field is expanded in eigenfunctions forming an orthonormal basis of the representation space.

As explained in appendix B, the Hilbert space $\mathscr{H}_k$ is separable and an orthonormal discrete basis is provided by the eigenfunctions of the compact generator X_0 which are given in Eq. (B23). If I call $\hat{h}_n(\mu)$ the boundary values of these eigenfunctions on the real axis and $\xi_n(u)$ their representatives in the space of test-functions

$$\xi_n(u) = [S_k h_n](u) \tag{4.20}$$

I can introduce the normal mode operators

$${}^{(k)}\alpha_n^\dagger = \int_{-\infty}^{+\infty} du\overline{\xi_n(u)}{}^{(k)}O^\dagger(u), \tag{4.21a}$$

$${}^{(k)}\alpha_n = ({}^{(k)}\alpha_n^\dagger)^\dagger, \tag{4.21b}$$

which, because of the orthonormality of the $\xi_n(u)$ satisfy

$$\langle 0|{}^{(k)}\alpha_m{}^{(k)}\alpha_n^\dagger|0\rangle = \delta_{n,m}. \tag{4.22}$$

Rewriting (4.21a) in the form

$${}^{(k)}\alpha_n^\dagger = ((h_n^{(k)}, S_k^{(k)}O^\dagger)), \tag{4.23}$$

it becomes evident that $\alpha_n^\dagger$ is the coefficient of h_n in the expansion of the field $O(u)$ in $R_0^{(k)}$ eigenfunctions

$$ {}^{(k)}O^\dagger(u) = \sum_{n=0}^{\infty} {}^{(k)}\alpha_n^\dagger h_n^{(k)}(u). \tag{4.24}$$

Hence, recalling that ${}^{(k)}O^\dagger(u)$ is any N-th component of the field $A^{(+)}(u, \underset{\sim}{p})$ I can write

$$A^{(+)}(u, \underset{\sim}{p}) = \sum_{N=0}^{\infty}\sum_{n=0}^{\infty} {}^{(k)}\alpha_{N,n}^\dagger b_N(\underset{\sim}{p}) h_n^{(k)}(\underset{\sim}{p}), \tag{4.25}$$

where

$$\langle 0| {}^{(k)}\alpha_{N,n} {}^{(k)}\alpha_{M,m}^\dagger |0\rangle = \delta_{N,M}\delta_{n,m}; \tag{4.26}$$

the operators ${}^{(k)}\alpha_{N,n}$ are completely analogous to the normal mode operators of dual models recalled in Eq. (2.18). Indeed they have an external index N which transforms under the kinematical group as the transverse index of the a_n^i transforms under SO($d-2$), and an internal index n which, as the n of a_n^i transforms irreducibly under the mass-group SL(2, R). The difference is that in dual models n is a quantum of mass, while in conformal models it is a quantum of the following physical quantity

$$X_0 = \tfrac{1}{2}(\mathscr{T}_+ - \mathscr{T}_-), \tag{4.27}$$

that is

$$X_0 = \tfrac{1}{2}(M^2 + \tfrac{1}{4}K_+P_+^{-1} - \tfrac{1}{4}S_\perp^2 P_+^{-2}). \tag{4.28}$$

A distinction between the two cases still comes from the fact that Eq. (4.26) holds true, for the moment, only as vacuum expectation value, while Eq. (2.15) is an operator identity. This depends on having used only invariance arguments which enforce only weak-topology relations. The strong-topology version of Eq. (4.26) is a dynamical assumption equivalent to considering $O_N(u)$ as a generalized free field (c-numeric field commutator) [8]

$$[{}^{(k)}\alpha_{M,m}, {}^{(k)}\alpha_{N,n}^\dagger] = \delta_{N,M}\delta_{n,m} \Leftrightarrow [{}^{(k)}O_N^{(-)}(u_2), {}^{(k)}O_M^{(+)}(u_1)] = \delta_{N,M}G_k(u_1-u_2). \tag{4.29}$$

However if we assume (4.29) then the structure of the conformal model matches completely that of dual models, because in that case the Fock space of the quantum field $\phi_k(x)$ is just the space spanned by the harmonic oscillators ${}^{(k)}\alpha_{N,n}^\dagger$. Indeed, recalling (3.18), an arbitrary multiparticle state can be rewritten

$$|f_1, \ldots, f_l\rangle = \iint_{-\infty}^{+\infty} \prod_{i=1}^{l} d\underset{\sim}{p}^i du^i \overline{f_i(\underset{\sim}{p}^i, u^i)} A^{(+)}(u_i, \underset{\sim}{p}_i)\,|0\rangle. \tag{4.30}$$

On the other hand a test function f_i admits the decomposition

$$f_i(\underset{\sim}{p}^i, u^i) = \sum_{N=0}^{\infty}\sum_{n=0}^{\infty} C_i^{(N,n)} b_N(\underset{\sim}{p}^i)\xi_n(u), \tag{4.31}$$

so Eq. (4.30) becomes

$$|f_1, \ldots, f_l\rangle = \prod_{i=1}^{l} \{\sum_{N,n} C_i^{(N,n)}\, {}^{(k)}\alpha_{N,n}^\dagger\}\,|0\rangle, \tag{4.32}$$

which is a linear combination of states of the form

$$|\{N_i, n_i\}\rangle = {}^{(k)}\alpha^\dagger_{N_1,n_1}\,{}^{(k)}\alpha^\dagger_{N_2,n_2} \cdots {}^{(k)}\alpha^\dagger_{N_l,n_l}|0\rangle. \tag{4.33}$$

Hence the states (4.33), analogous to dual states, span the entire Fock space of the generalized free conformal field $\phi_k(x)$. From the group-theoretical point of view the states (4.33) are just Kronecker products of states of the UIR D_k^+ which can be decomposed into irreducible components via CGc.

The CGc of the discrete series of UIR of SL(2, R) have been derived in Ref. [5] with a spinorial method. The result is the following. Let $|k, n\rangle$ be the eigenstates of the compact generator X_0 in the the representation D_k^+

$$X_0|k, n\rangle = \left(\frac{k}{2}+n\right)|k, n\rangle. \tag{4.34}$$

They are an orthonormal basis for the representation Hilbert space. If we introduce the basis of $D_{k_1}^+ \otimes D_{k_2}^+$

$$|k_1, n_1; k_2, n_2\rangle = |k_1, n_1\rangle \otimes |k_2, n_2\rangle, \tag{4.35}$$

then the CGc are

$$C(k_3 n_3|k_1 n_1; k_2 n_2) = \langle k_3 n_3|k_1, n_1; k_2, n_2\rangle \tag{4.36}$$

and have the following properties. They are different from zero only if

(*i*) $$k_3 = k_1+k_2+2l$$

(*ii*) $$n_3 = n_1+n_2-l\ , \tag{4.37}$$

where l is any positive integer. Moreover we have

$$\langle k_1 n_1; k_2, n_2|k_1+k_2+2l, n_1+n_2-l\rangle$$
$$= B_l \sum_{\lambda=0}^{l} (-1)^l \binom{l}{\lambda} \frac{N^2(k_1+l, n_1-l+\lambda)N^2(k_2+l, n_2-l)}{N(k_1, n_1)N(k_2, n_2)N(k_1+k_2+2l, n_1+n_2-l)}, \tag{4.38}$$

where B_l is a normalization factor depending only on l and $N(k, l)$ is the normalizer of the irreducible states introduced in appendix B, Eq. (B22).

From Eq. (4.37) it follows that the reduction of the two-particle sector spanned by ${}^{(k)}\alpha^\dagger_{N,n}\,{}^{(k)}\alpha^\dagger_{M,m}|0\rangle$ takes contributions only from the representations $D_{k'}^+$, where

$$2(l-k) = k_l', \qquad l = \text{integral}. \tag{4.39}$$

Introducing new creation-annihilation operators for the basis states of these representations:

$$[{}^{(k_l)}\alpha_{N,n}, {}^{(k_l)}\alpha^\dagger_{N',n'}] = \delta_{N,N'}\delta_{n,n'} \tag{4.40}$$

we can write the operator product expansion

$${}^{(k_i)}\alpha^\dagger_{N,n}\,{}^{(k_j)}\alpha^\dagger_{M,m} = \sum_L \mathscr{C}(L|MN) \sum_{l=0}^{n+m} C(k_l^{ij}, n+m-l|k_i n; k_j m)^{(k_l^{ij})}\alpha^\dagger_{L,n+m-l}, \tag{4.41}$$

where:

$$k_l^{ij} = k_i + k_j + 2l, \tag{4.42}$$

and $\mathscr{C}(L|MN)$ is an appropriate CGc for the kinematical group generated by $\boldsymbol{K}$. The explicit structure of $\mathscr{C}(L|MN)$ is irrelevant for the concern of this paper and has not been studied here. A new field can be associated to each representation k_l by defining, in analogy to Eq. (4.25)

$$A_{k_l}^{(+)}(u, \underset{\sim}{p}) = \sum_{N=0}^{\infty} \sum_{n=0}^{\infty} b_N(\underset{\sim}{p}) \xi_n^{(k_l)}(u)^{(k_l)} \alpha_{N,n}^{\dagger}, \tag{4.43}$$

$$A_{k_l}^{(-)}(u, \underset{\sim}{p}) = [A_{k_l}^{(+)}(u, \underset{\sim}{p})]^{\dagger}, \tag{4.44}$$

and then, in analogy to (3.13)

$$\phi_{k_l}^{(\pm)}(x) = \int d\underset{\sim}{p}\, \psi^{(\pm)}(\underset{\sim}{p}, x) A^{(\pm)}\left(\frac{x_+}{2p_+}, \underset{\sim}{p}\right). \tag{4.45}$$

Now let H be the full Fock-space of the quantum field-theory

$$H = \bigoplus_{r=0}^{\infty} H_r, \tag{4.46}$$

where H_r is the r particle sector

$$H_r = \underbrace{H_1 \otimes H_1 \otimes \ldots \otimes H_1 \otimes H_1}_{(r \text{ times (symmetrized)})}, \tag{4.47}$$

and $\overline{\mathscr{H}}_k$ the carrying spaces of the UIR of the series $D_{k_l}^+$ $(k_0 = k)$ we have

$$H_1 \equiv \overline{\mathscr{H}}_{k_0}, \tag{4.48}$$

and by the previous results

$$H_2 = \bigoplus_{l=0}^{\infty} \overline{\mathscr{H}}_{k_l}. \tag{4.49}$$

Considering the three-particle sector and using twice the Clebsch–Gordan decomposition (4.41) we find that the representations contributing to its reduction have the form

$$k' = 2k_0 + 2l + k_0 + 2p, \tag{4.50}$$

where l and p are arbitrary positive integers. At this point we have two possibilities. If k_0 is even k' is still of the form (4.39):

$$k' = 2k_0 + 2l' \tag{4.51}$$

for any l and p. So all the spaces appearing in the reduction of the three, and hence r-particle sector are already contained in the reduction of the two particle sector. For k_0 even we obtain:

$$H = 1 \oplus \overline{\mathscr{H}}_{k_0} \bigoplus_{l=0}^{\infty} \overline{\mathscr{H}}_{k_l}. \tag{4.52}$$

If k_0 is odd k' is either of the form (4.51) or of the form:

$$k' = 2k_0+2l'+1, \tag{4.53}$$

in this case calling:

$$\bar{k}_l = 2k_0+2l+1 \tag{4.53a}$$

we have for k_0 odd

$$H = 1\oplus\overline{\mathscr{H}}_{k_0} \overset{\infty}{\underset{l=0}{\oplus}} \overline{\mathscr{H}}_{k_l} \overset{\infty}{\underset{l=0}{\oplus}} \overline{\mathscr{H}}_{\bar{k}_l}. \tag{4.54}$$

The complete description of the theory is obtained once the vertex function connecting the particles belonging to different representations is explicitly given. From (4.41) it follows:

$$\langle 0|^{(k_l{}^{ij})}\alpha^{(-)}_{L,n+m-l}{}^{(k_i)}\alpha^{\dagger}_{N,n}{}^{(k_j)}\alpha^{\dagger}_{M,m}|0\rangle = \mathscr{C}(L|MN)\,\langle k_l^{ij}n+m-l|k_in;\,k_jm\rangle, \tag{4.55}$$

hence recalling (4.24)

$$\begin{aligned}&\langle 0|^{(k_l{}^{ij})}O_L^{(-)}(u_3)^{(k_i)}O_N^{(+)}(u_1)^{(k_j)}O_M^{(+)}(u_2)\,|0\rangle \\ &= \mathscr{C}(L|MN)\sum_{n+m\geqslant l}\langle k_l^{ij},\,n+m-l|k_in;\,k_jm\rangle\bar{h}^{(k_l{}^{ij})}_{n+m-l}(u_3)h_n^{(k_i)}(u_2)h_m^{(k_j)}(u_1)\end{aligned} \tag{4.56}$$

and inserting the explicit form of the CGc given in Eq. (4.38) I obtain the explicit form of the vertex function

$$\begin{aligned}&\langle 0|^{(k_l{}^{ij})}O_L^{(-)}(u_3)^{(k_i)}O_N^{(+)}(u_2)^{(k_j)}O_M^{(+)}(u_1)\,|0\rangle \\ &= \mathscr{C}(L|NM)C(l)\,(u_3-u_1)^{-k_i-l}(u_3-u_2)^{-k_j-l}(u_2-u_1)^l.\end{aligned} \tag{4.57}$$

5. *SL*(2, *R*) *mass-group and modular functions*

In the previous sections I have tried to enlighten the mathematical analogy between the dual and conformal models due to the isomorphism of their mass-groups. The purpose of the present section is to discuss the results of Ref. [1] in the context emerging from the considerations of Sections 2, 3, 4.

Ref. [1] deals with dual models and the author concentrates on the following generating function

$$G(\omega) = \sum_n d_n \exp(i2\pi\alpha m_n^2\omega), \tag{5.1}$$

where d_N is the degeneracy of a mass-level m_N^2 and ω is a parameter taking values in the upper complex plane. The function $G(\omega)$ has the following physical interpretation

$$G(\omega) = \mathrm{Tr}_{(\underset{\sim}{p})}\exp(i2\pi\alpha\omega M_{\mathrm{D}}^2), \tag{5.2}$$

where M_{D}^2 is the dual mass-squared operator and the trace $Tr_{(\underset{\sim}{p})}$ is performed over all states with a given, fixed, kinematical momentum $\underset{\sim}{p}$ (see Eq. (3.9)). The main result of [1] is that $G(\omega)$ is, for all known dual models a modular function with respect to a convenient subgroup Γ of the modular group $\mathscr{G} = \mathrm{SL}(2, Z)$ (Γ depends on the specific model). This means the following.

Let $\mathscr{M}$ be the vector space of meromorphic functions on the upper complex plane and let D_k^+ be a UIR of SL(2, R) belonging to the discrete class discussed in appendix B. The carrying space $\mathscr{H}_k$ of D_k^+ is a subspace of $\mathscr{M}$ and the representation of SL(2, R) can be extended to all $\mathscr{M}$ allowing the operators $U_k^{(+)}(\tilde{g})$ defined in Eq. (B 12a), to act on any element $f \in \mathscr{M}$. So we have

$$\forall f \in \mathscr{M} : [U_k^{(+)}(\tilde{g})f](\omega) = (a + b\tilde{g}^{-1}\omega)^k f(\tilde{g}^{-1}\omega). \tag{5.3}$$

Eq. (5.3) is now meaningful also if k is negative integral, because the new poles introduced by the multiplier when $k < 0$ spoil holomorphicity but not meromorphicity. Then let $\Gamma \subset \mathscr{G} \subset \mathrm{SL}(2, Z)$ be a discrete subgroup of SL(2, R) which is contained in the modular group $\mathscr{G}$ and is of finite index in it ($|\mathscr{G}/\Gamma| < \infty$). We say that an element $f_0 \in \mathscr{M}$ is a Γ-modular function of weight k if

$$\forall \gamma \in \Gamma : U_k^{(+)}(\gamma) f_0 = f_0. \tag{5.4}$$

In [1] $G(\omega)$ is seen to be Γ-modular for Γ = the invariance group of the lattice of periods of the string and

$$k = -\frac{D_c - 2}{2}, \tag{5.5}$$

where D_c is the so called critical number of space-time dimensions of the model [3].

Recalling the results of Section 2 and in particular Eqs (2.20) and (2.33) I can write

$$G(\omega) = [\mathrm{Tr}_{(\underset{\sim}{p})} \exp(i2\pi\omega X_0)] \exp(-i2\pi\alpha_0\omega), \tag{5.6}$$

where X_0 is the generator of the compact one-parameter subgroup of SU(1,1) given explicitly in Eq. (2.31a). The SU(1, 1) group of which X_0 is a generator acts canonically as a transformation group on the Koba–Nielsen variable [3]

$$z = \exp(2\pi i\omega). \tag{5.7}$$

The transformations are those given in appendix B, Eq. (B3)

$$gz = \frac{\bar{\alpha}z + \bar{\beta}}{\beta z + \alpha}. \tag{5.8}$$

The subgroup (2.31a) generated by X_0 is therefore the subgroup of translations of $2\pi\omega$

$$2\pi\omega' = 2\pi\omega - \mu. \tag{5.9}$$

For each state of momentum $\underset{\sim}{p}$, mass

$$m_n^2 = n/\alpha, \tag{5.10}$$

and additional quantum numbers $\{\lambda\}$, I can define a pair of creation-annihilation operators

$$[\Delta^{(-)}_{\{\lambda\}(\underset{\sim}{p},n)}, \Delta^{(+)}_{\{\lambda'\}(\underset{\sim}{p},n')}] = \delta_{n,n'}\delta_{\{\lambda\},\{\lambda'\}}, \tag{5.11}$$

which will be convenient combinations of the operators (2.18), and with them I define a field

$$Q^{(\pm)}(\underset{\sim}{p}, \omega) = \sum_{n,\lambda} \Delta^{(\pm)}_{n,\{\lambda\}}(\underset{\sim}{p})e^{\pm in2\pi\omega}, \tag{5.12}$$

which will transform in a covariant, reducible way under SU(1, 1). If we now let ω approach the real axis ($v \to 0$ in $\omega = u+iv$) we can reinterpret the trace (5.6) as the two point function of the field (5.12). In fact

$$e^{2\pi i\alpha_0(u_1-u_2)}G(u_1-u_2) = \langle 0|Q^{(-)}(\underset{\sim}{p}, u_2)Q^{(+)}(\underset{\sim}{p}, u_1)\,|0\rangle$$

$$= \langle 0|Q^{(-)}(\underset{\sim}{p}, 0)\exp(2\pi i[u_1-u_2]X_0)Q^{(+)}(\underset{\sim}{p}, 0)\,|0\rangle = \mathrm{Tr}_{(\underset{\sim}{p})}\exp[2\pi i(u_1-u_2)X_0]. \tag{5.13}$$

From the point of view of Eq. (5.13) it is natural to ask what is the relation between the SU(1, 1) transformations of the field $Q(\underset{\sim}{p}, u)$ and the modular transformations (5.4) leaving the trace (5.13) invariant. Γ is in fact a subgroup of SL(2, R) which is globally isomorphic to SU(1, 1) under the Cayley transformation (B4). However it is essential to note that the modular transformations are implemented on the variable

$$u = u_1-u_2 = \frac{1}{2\pi i}(\lg z_1 - \lg z_2), \tag{5.14}$$

and are of the type

$$\forall\gamma\in\Gamma\subset \mathrm{SL}(2, R): \gamma u = \frac{du+c}{bu+a}. \tag{5.15}$$

Therefore due to the logarithm difference in Eq. (5.14) there is no hope that the rational transformations (5.8) on the Koba–Nielsen variable induce rational transformations (5.15) on the variable (5.14).I can conclude that the modular transformations on the two point function are the descendants of another type of transformation on the fields which is not an element of the SU(1, 1) group of dual models, that is of the mass-group. In particular these new transformations could connect different UIR of SU(1, 1) contained in the theory. A very natural and attractive interpretation would be the following: (I) "*The modular transformations canonically implemented on the 2-point function are the descendants of modular transformations canonically implemented on the fields*". (By canonically implemented I mean that it is of the type (5.3) with $\tilde{g} = \gamma\in\Gamma$).

I want first to discuss why (I) is natural and attractive and then *to show that it is ruled out by a no-go theorem.* If (I) were true we could always consider Γ as a subgroup of SL(2, R) and start with a theory characterized by an SL(2, R) mass-group canonically implemented on the variable u conjugate to mass-squared. This theory is the conformal model with integral anomalous dimension k, described in Sections 2, 3, 4, so that the physical interpretation of the weight k of the modular function would be fixed from the beginning. As we know, the mass-spectrum of such a model is necessarily continuous because the mass-operator is associated to $\mathscr{T}_+ = X_0 - X_2$ (see Eqs (2.28), (2.36)) which, in the fundamental representation (2.30) of the Lie algebra, is given by the following matrix

$$\mathscr{T}_+ = \begin{pmatrix} 1/2 & i/2 \\ i/2 & -1/2 \end{pmatrix}, \tag{5.16}$$

and which generates the non-compact one-parameter subgroup of SU(1, 1) isomorph to the translation subgroup N_0 of SL(2, R)

$$\exp(iu\mathscr{T}_+) = C^\dagger \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} C = n_0(u) \in N_0 \subset \mathrm{SL}(2, R), \tag{5.17}$$

where C is the Cayley matrix (B1) realizing the SU(1, 1) $\sim$ SL(2, R) isomorphism.

If by an additional physical principle we enforce the quantization of the spectrum of $\mathscr{T}_+ \equiv M^2$, we can do it only with an overall breakdown of the SL(2, R) symmetry. In fact, for instance, the dilatation subgroup $\mathscr{D}$

$$d(s) \in \mathscr{D} \Leftrightarrow d(s) = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \tag{5.18}$$

must be broken in the presence of mass quantization. It is conceivable that the surviving symmetry could be the discrete subgroup $\Gamma \subset \mathrm{SL}(2, R)$ plus the translation subgroup N_0 which must always be a symmetry of the system if the squared mass is to be a conserved quantum number. Therefore the attitude underlying interpretation (I) is that the modular mass-spectrum should result from the breakdown of conformal symmetry, performed in a proper way. In this interpretation the parameter (5.5) would be related to the anomalous dimension and the modular transformations (5.4) would be canonically implemented in the Hilbert space of states.

Unfortunately interpretation (I) *is impossible because of the following theorem*:

No-go theorem: "Let H_0 be a Hilbert space on which the translation group N_0 is unitarily represented. If a subgroup $\Gamma \subset \mathscr{G} \subset \mathrm{SL}(2, R)$ of finite index in $\mathscr{G}$ is also unitarily represented in H_0, then the whole group SL(2, R) is unitarily represented in it."

Before proving the theorem I point out the consequences. The essence of the statement is that either we have a modular invariant theory in which the mass is not a good quantum number, which is physically unacceptable, or a fully SL(2, R) invariant theory with continuous mass-spectrum, or a theory in which N_0 is a symmetry and the spectrum of its generator, continuous or discrete, violates invariance of the theory under any discrete subgroup of finite index in $\mathscr{G}$. This latter statement does not forbid modularity of the function (5.13) in the variable $u_1 - u_2$ but forbids to implement modular transformations on the fields in the simple natural way. The modularity of (5.13) implies therefore that there are more complicated and so far unknown representations of the modular group on the fields which produce the simpler modularity of the two-point function. Although I have no recipe to build them explicitly I believe that finding them would be a consistent breakthrough in understanding.

Proof of the no-go theorem. The techniques which I shall use for the proof are based on the very powerful concept of Γ-cuspidal subgroups of SL(2, R) and have been developed in Ref. [13], for instance. First I need some definitions and lemmas.

Definition 1: A one-parameter subgroup $N \subset \mathrm{SL}(2, R)$ is called unipotent if its generator Θ is nilpotent

$$\Theta^2 = 0. \tag{5.19}$$

Definition 2: Let Γ be a discrete subgroup of SL(2, R). A Γ-cuspidal subgroup of SL(2, R) is a unipotent subgroup N such that

$$\Gamma_N = \Gamma \cap N \neq \{e\}.$$

In other words Γ contains at least a non identity element of N and hence all the cyclic group generated by it.

Lemma 1: The standard translation group (5.17) is unipotent and all the other possible unipotent subgroups are the following conjugates of N_0

$$N_x = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} N_0 \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} \quad 0 \leqslant x < \infty, \tag{5.20a}$$

and

$$\overline{N}_0 = N_\infty = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} N_0 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{5.20b}$$

(For the proof see [13].)

Lemma 2: Let $\mathscr{G}$ be the modular group. A unipotent subgroup N_x is $\mathscr{G}$-cuspidal if and only if $x = q$ where q is a rational number (proof see [13]).

Lemma 3: Let $\mathscr{N}_\Gamma$ be the set of Γ-cuspidal subgroups. They are all conjugate under some element of Γ

$$\forall N, N' \in \mathscr{N}_\Gamma \exists \gamma \in \Gamma / N' = \gamma N \gamma^{-1} \tag{5.21}$$

(proof see [13]).

Lemma 4: Let $N_{q'} = \gamma N_q \gamma^{-1}$ be the conjugate of N_q under the element γ. Then we have

$$q' = \gamma q \tag{5.22}$$

(proof by direct check).

Lemma 5: If Γ is of finite index in $\mathscr{G}$ then a unipotent sugroup $N \subset \mathrm{SL}(2, R)$ is Γ-cuspidal if and only if it is $\mathscr{G}$-cuspidal (proof see [13]).

Lemma 6: The generator of the cuspidal subgroup N_q is

$$\Theta_q = (1+q^2)X_0 - 2qX_1 - (1-q^2)X_2, \tag{5.23}$$

where X_n are the standard generators of SL(2, R) recalled in Eq. (2.29). (Proof: from lemma 1

$$\exp(iu\Theta_q) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + iu\Theta_q = \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix} \tag{5.24}$$

so we have

$$\Theta_q = \begin{pmatrix} -iq & i \\ -iq^2 & iq \end{pmatrix}. \tag{5.25}$$

The decomposition of the matrix (5.25) in the basis provided by the Cayley transforms of the generators (2.30) is just (5.23).)

Proof of the main theorem: Let H_0 be the Hilbert space which we are interested in and let the scalar product of any two $\psi, \phi \in H_0$ be denoted by (ψ, ϕ). Let $U(n_0)$ be the unitary operators representing the translation subgroup N_0 and $U(\gamma)$ be the unitary operators representing the discrete subgroup Γ of finite index in $\mathscr{G}$. Because of unitarity we can write

$$(U(\gamma)\,U(n_0)\,U(\gamma^{-1})\,\psi,\ U(\gamma)\,U(n_0)\,U(\gamma^{-1})\phi) = (U(\gamma n_0 \gamma^{-1})\,\psi,\ U(\gamma n_0 \gamma^{-1})\phi) = (\psi, \phi) \tag{5.26}$$

for any $n_0 \in N_0$ and any $\gamma \in \Gamma$. From Eq. (5.26) we deduce that the whole Γ-conjugacy class of the group N_0 is unitarily represented in H_0. It follows that the generators Θ_q of the cuspidal subgroups of this conjugacy class are hermitean operators in H_0. So is any linear combination of them which therefore generates a unitary symmetry of the system. On the other hand because of lemma 3 and 4 there are infinitely many elements in the conjugacy class of N_0 (all N_q, where q belongs to the orbit of zero under Γ). So we have infinitely many generators of the form (5.23), provided by lemma 6, which are symmetry generators for the system. It follows by linear combination that any element of the Lie algebra of SL(2, R) is self-adjoint and well-defined. The whole group is unitarily represented and this proves the no-go theorem.

From the technique used in the above proof we learn something very interesting. Because of lemma 5 every $\mathscr{G}$-cuspidal subgroup N_q contains a subgroup Γ_{N_q} of any Γ of finite index in $\mathscr{G}$. Therefore if $G(\omega)$ is the 2-point function of type (5.1) which is modular under a finite index Γ, it follows that $G(\omega)$ is a periodic function with an infinite number of periods (one for each cuspidal N_q); these periods are not however independent, because all the cuspidals belonging to the same conjugacy class determine the same period. Periodicity of $G(u)$) in u means that in the Hilbert space for which $G(u_1 - u_2)$ is the metric, the whole group Γ_{N_0} is the identity. In fact let $\gamma_0 \in \Gamma_{N_0}$ we have

$$\begin{aligned}(\psi_1 \gamma_0 \psi_2) &= \int_{-\infty}^{+\infty} du_1 \int_{-\infty}^{+\infty} du_2 \bar{\psi}_1(u_1) G(u_1 - u_2) \psi_2(u_2 + np_0) \\ &= \int_{-\infty}^{+\infty} du_1 \int_{-\infty}^{+\infty} du_2 \bar{\psi}_1(u_1) G(u_1 - u_2 - np_0) \psi_2(u_2) = (\psi_1, \psi_2),\end{aligned} \tag{5.27}$$

where p_0 is the period associated to Γ_{N_0}. So the condition (5.27) streaming from modularity is a sort of boundary condition on the allowed physical states. However this interpretation cannot be extended to the periods associated to the other subgroups. In the general case in fact the transformation on u_1 or u_2 cannot be transferred to the difference $u_1 - u_2$ because of the non-linearity. This is the essence of the no-go theorem. So we have learned that modularity results from a sort of boundary conditions but so far we do not know how to impose them on the states and the fields in a meaningful way.

6. Conclusions

In this paper it has been emphasized the possibility of approaching the hadronic mass--spectrum from a mass-group point of view. The case of SL(2, R) has been studied in detail because it encludes two relevant cases (dual and conformal models). The very attractive possibility of considering theories with dual type of mass-spectrum as broken versions of conformal theories is however excluded and this, unfortunately, excludes also the possibility of interpreting the weight of the modular functions introduced in Ref. [1] as anomalous dimension of the field.

The mass-group approach shows that it is very critical to understand the Poincaré transformation properties of the other members of the mass-algebra because it is on this properties that the possibility of having a discrete mass-spectrum relies. I feel that the solution of the problem of how the modular transformations are implemented on the fields requires first an understanding of the previous problem.

The author is particularly grateful to Professor H. Satz for the hospitality at the Department of Physics, University of Bielefeld where the major part of this work was performed and to the Deutsche Forschung Gemeinschaft for financial support. Essential and enlightening discussions with Dr. E. Etim during the whole development of the work are also acknowledged.

APPENDIX A

The purpose of the present appendix is to show that Eqs (3.31) hold true. The ingredients of the proof are Eqs (3.30), (3.4), (3.12), (2.25) and the assumption of conformal invariance of the vacuum

$$M_{\mu\nu}|0\rangle = P_\mu|0\rangle = K_\mu|0\rangle = D|0\rangle = 0. \tag{A1}$$

In order to verify Eq. (3.30) it is convenient to start by checking the following identities, which are a consequence of the definition (3.12) of $A^{(+)}(u, \underset{\sim}{p})$ and of the conformal transformation laws (3.4)

$$[S_\perp, A^{(+)}(u, \underset{\sim}{p})] = -ip_+ \frac{\partial}{\partial p_\perp} A^{(+)}(u, \underset{\sim}{p}), \tag{A2}$$

$$[D+N_3, A^{(+)}(u, \underset{\sim}{p})] = i\left(2x_+ \frac{\partial}{\partial x_+} - p_\perp \frac{\partial}{\partial p_\perp} + l\right) A^{(+)}(u, \underset{\sim}{p}), \tag{A3}$$

$$[K_+, A^{(+)}(u, \underset{\sim}{p})] = i\left(2x_+^2 \frac{\partial}{\partial x_+} + p_+ \frac{\partial}{\partial p_\perp} \cdot \frac{\partial}{\partial p_\perp} + 2(l-\nu)x_+\right) A^{(+)}(u, \underset{\sim}{p}). \tag{A4}$$

Using also the invariance of the vacuum (A1) one obtains

$$S_\perp^2 P_+^{-2} A^{(+)}(u, \underset{\sim}{p})\,|0\rangle = -\frac{\partial}{\partial p_\perp} \cdot \frac{\partial}{\partial p_\perp} A^{(+)}(u, \underset{\sim}{p})\,|0\rangle, \tag{A5}$$

$$S_\perp P_\perp P_\perp^{-1} A^{(+)}(u, \underset{\sim}{p}) |0\rangle = -i p_\perp \frac{\partial}{\partial p_\perp} A^{(+)}(u, \underset{\sim}{p}) |0\rangle, \tag{A6}$$

$$P_\mu P^\mu A^{(+)}(u, \underset{\sim}{p}) |0\rangle = i \frac{d}{du} A^{(+)}(u, \underset{\sim}{p}) |0\rangle. \tag{A7}$$

(A7) is already the first of Eqs (3.31). The other ones are obtained inserting the results (A2 – A6) into (3.30) and using the conformal invariance of the vacuum. Note in particular the role of the c-number part of $\mathscr{T}_0 : v/2$. It contributes a term

$$\frac{v}{2} A^{(+)}(u, \underset{\sim}{p}) |0\rangle, \tag{A8}$$

which cancels with the analogous term due to canonical part of the scale dimension l.

APPENDIX B

The UIR of the group SU(1, 1) SL(2, R) have been constructed long time ago in [4] and are very well-known. In this appendix I have collected all the formulae which I need in the main text and also the derivation of the form of representation basis functions in mass-space which, although not used in this paper, could be useful elsewhere.

The isomorphism between SU(1, 1) and SL(2, R) is realized by the unitary Cayley matrix

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \tag{B1}$$

An element of SU(1, 1) is a complex 2×2 matrix of the form

$$g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad |\alpha|^2 - |\beta|^2 = 1, \tag{B2}$$

and can be implemented as a projective transformation in the complex plane

$$gz = \frac{\bar{\alpha} z + \bar{\beta}}{\beta z + \alpha}. \tag{B3}$$

Under such a group of transformations the unit circle $|z|^2 < 1$ is left invariant. If $g \in$ SU(1,1) and we define

$$\tilde{g} = C g C^\dagger = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{B4}$$

then $\tilde{g}$ is an element of SL(2, R) that is a real 2×2 matrix of determinant one. The Cayley matrix transforms the unit-circle into the upper complex plane

$$Cz = -i \frac{z+i}{z-i} = \omega \qquad \text{Im}\, \omega > 0 \text{ if } |z|^2 < 1, \tag{B5}$$

and the group SL(2, R) becomes the group of projective transformations of this half plane

$$\tilde{g}\omega = \frac{d\omega+c}{b\omega+a}. \tag{B6}$$

The UIR of SU(1, 1) belonging to the discrete classes D_k^+ and D_k^- are realized on a Hilbert space $\mathscr{H}_k$ of functions holomorphic inside the unit-circle and of finite norm with respect to the following scalar product

$$(F_1, F_2)_k = \frac{k-1}{\pi} \int\limits_{|z|<1} (1-|z|^2)^{k-2} F_1(z) F_2(z) d^2 z. \tag{B7}$$

The representation of the group elements is given by

$$[U_k^{(+)}(g)F](z) = (\alpha+\beta g^{-1}z)^k F(g^{-1}z) \tag{B8}$$

for the representation D_k^+, and

$$[U_k^{(-)}(g)F](z) = (\bar{\alpha}+\bar{\beta} g^{*-1}z)^k F(g^{*-1}z) \tag{B9}$$

for the representation D_k^-. These representations are unitary with respect to the scalar product (B7). If we define the Cayley transformation

$$f(\omega) = (\omega+i)^{-k} F(z), \tag{B10}$$

where the relation between z and ω is given by (B5), we map the space $\mathscr{H}_k$ into the space $\overline{\mathscr{H}}_k$ of analytic functions in the upper complex plane. The scalar product (B7) is transformed into

$$(f_1, f_2)_k = \frac{2^{k-2}(k-1)}{\pi} \int\limits_{-\infty}^{+\infty} d\,\mathrm{Re}\,\omega \int\limits_0^{\infty} d\,\mathrm{Im}\,\omega (\mathrm{Im}\,\omega)^{k-2} \overline{f_1(\omega)} f_2(\omega), \tag{B11}$$

and the representation of the group elements becomes

$$[U_k^{(+)}(\tilde{g})f](\omega) = (a+b\tilde{g}^{-1}\omega)^k f(\tilde{g}^{-1}\omega), \tag{B12a}$$

$$[U_k^{(-)}(\tilde{g})f](\omega) = (d+c\underline{\tilde{g}}^{-1}\omega)^k f(\underline{\tilde{g}}^{-1}\omega), \tag{B12b}$$

where

$$\tilde{g}\omega = \frac{a\omega+b}{c\omega+d}. \tag{B13}$$

Let X_0, X_1, X_2 be the generators of the standard one-parameter subgroups introduced in the main text in Eqs (2.26), (2.27), (2.28). Their representation in the spaces $\mathscr{H}_k$ and $\overline{\mathscr{H}}_k$

and for the classes D_k^+ and D_k^- can be easily obtained by differentiation of Eqs (B8), (B9) and (B12a), (B12b). The result is

$\mathscr{H}_k$:

D_k^+	D_k^-
$X_0F(z) = \left(z\dfrac{d}{dz} + \dfrac{k}{2}\right)F(z)$	$X_0F(z) = -\left(z\dfrac{d}{dz} + \dfrac{k}{2}\right)F(z)$
$X_1F(z) = -\frac{1}{2}\left[(1+z^2)\dfrac{d}{dz} + kz\right]F(z)$ (B14)	$X_1F(z) = \frac{1}{2}\left[(1+z^2)\dfrac{d}{dz} + kz\right]F(z)$ (B16)
$X_2F(z) = -\dfrac{i}{2}\left[(1-z^2)\dfrac{d}{dz} - kz\right]F(z)$	$X_2F(z) = -\dfrac{i}{2}\left[(1-z^2)\dfrac{d}{dz} - kz\right]F(z)$

$\overline{\mathscr{H}}_k$:

D_k^+	D_k^-
$X_0f(\omega) = \dfrac{i}{2}\left[(1+\omega^2)\dfrac{d}{d\omega} + k\omega\right]f(\omega)$	$X_0f(\omega) = -\dfrac{i}{2}\left[(1+\omega^2)\dfrac{d}{d\omega} + k\omega\right]f(\omega)$
$X_1f(\omega) = -i\left(\omega\dfrac{d}{d\omega} + \dfrac{k}{2}\right)f(\omega)$ (B15)	$X_1f(\omega) = i\left(\omega\dfrac{d}{d\omega} + \dfrac{k}{2}\right)f(\omega)$ (B17)
$X_2f(\omega) = -\dfrac{i}{2}\left((1-\omega^2)\dfrac{d}{d\omega} - k\omega\right)f(\omega)$	$X_2f(\omega) = -\dfrac{i}{2}\left((1-\omega^2)\dfrac{d}{d\omega} - k\omega\right)f(\omega)$

The spaces $\mathscr{H}_k$ and $\overline{\mathscr{H}}_k$ admit an orthonormal basis composed of eigenfunctions of the compact generator X_0

$$X_0|n\rangle = \left(n + \frac{k}{2}\right)|n\rangle \tag{B18}$$

for D_k^+ and

$$X_0|n\rangle = \left(-n - \frac{k}{2}\right)|n\rangle \tag{B19}$$

for D_k^-. We have

$$\langle n|n\rangle = 1 \qquad n = 0, 1, 2, 3, \ldots \tag{B20}$$

in $\mathscr{H}_k$ we have

$$|n\rangle = z^n N(k, n)(-1)^n, \tag{B21}$$

where

$$N^2(k, n) = \frac{\Gamma(k+n)}{\Gamma(k)\Gamma(n+1)}. \tag{B22}$$

The corresponding states in $\mathscr{H}_k$ are obtained by Cayley transformation (B10)

$$|\bar{n}\rangle = (-i)^n N(k, n)\,(\omega+i)^{-k-n}(\omega-i)^n = h_n(\omega). \tag{B23}$$

The inverse Cayley transformation gives

$$(-1)^n N(k, n) z^n = (z-i)^{-k} h_n(\omega(z)). \tag{B24}$$

Their correspondents in μ-space are defined through

$$h_n(\omega) = \int_0^\infty d\mu \exp(i\mu\omega) k_n(\mu). \tag{B25}$$

From (B24) we have

$$z^n = (z-i)^{-k} \int_0^\infty d\mu \exp\left(\mu\,\frac{z+i}{z-i}\right) \varrho_n(\mu), \tag{B26}$$

where

$$\varrho_n(\mu) = \frac{k_n(\mu)}{(-1)^n N(k, n)}. \tag{B27}$$

The expansion ([12] page 1038)

$$(z-i)\exp\left(\mu\,\frac{z+i}{z-i}\right) = \sum_{m=0}^{\infty} z^m \Delta_m(\mu), \tag{B28}$$

$$\Delta_m(\mu) = (i)^{k-m} L_m^{(k-1)}(2\mu) e^{-\mu}, \tag{B29}$$

implies that $\varrho_n(\mu)$ must be such that

$$\int_0^\infty \Delta_m(\mu)\varrho_n(\mu) d\mu = \delta_{n,m}, \tag{B30}$$

which is solved by

$$\varrho_n(\mu) = 2(i)^{m-k} e^{-\mu}(2\mu)^{k-1} L_n^{(k-1)}(2\mu)\,\frac{\Gamma(n+1)}{\Gamma(k+n)}. \tag{B31}$$

We obtain therefore

$$k_n(\mu) = 2(-i)^{n+k}\left(\frac{\Gamma(n+1)}{\Gamma(k+n)\Gamma(k)}\right)^{1/2} e^{-\mu}(2\mu)^{k-1} L_n^{(k-1)}(2\mu), \tag{B32}$$

where $L_n^{(k-1)}(2\mu)$ are Laguerre's polynomials and (B32) is the orthonormal basis for the space with the scalar product (4.7) of the main text.

The representations of the continuous class in the exceptional interval E_k have also two realizations: one in a Hilbert space of functions defined on the boundary of the unit circle and one, obtained by Cayley transformation, in a Hilbert space of functions on the real axis. In the two spaces the scalar products are of the form, respectively

$$(\mathscr{I}, \mathscr{G})^{E_k} = \oint dx \oint dy \overline{\mathscr{I}}(x) \mathscr{L}(x, y) \mathscr{G}(y), \tag{B33}$$

$$(f, g)^{E_k} = \int_{-\infty}^{+\infty} du_1 \int_{-\infty}^{+\infty} du_2 \overline{f(u_1)} \mathscr{L}(u_1, u_2) g(u_2), \tag{B34}$$

where $\mathscr{L}(x, y)$ and $(\mathscr{L} u_1, u_2)$ are the appropriate invariant densities. The infinitesimal generators of the representation have the same form as in the representations of the discrete class but with k restricted to the appropriate interval $0 < k < 2$. I do not need further details on this class of representations. The relevant point is that given the infinitesimal generators in the form (B16) or (B17) they can be integrated to a representation of the class E_k or $D_k^\pm$ depending on the value of K.

REFERENCES

[1] W. Nahm, *Nucl. Phys.* **B114**, 174 (1976).
[2] H. Leutwyler, J. Stern, Orsay preprint IPNO/TH 77–11 (1977).
[3] M. Jacob, *Dual Theory*, North-Holland 1974.
[4] V. Bargmann, *Ann. Math.* **48**, 568 (1947).
[5] A. O. Barut, C. Fronsdal, *Proc. R. Soc. (London)* **A287**, 532 (1965).
[6] G. Mack, A. Salam, *Ann. Phys.* **53**, 174 (1969).
[7] N. Mukunda, *J. Math. Phys.* **8**, 2210 (1967).
[8] N. N. Bogolubov, A. A. Logunov, I. T. Todorov, *Introduction to Axiomatic Quantum Field Theory*, W. A. Benjamin 1975.
[9] W. Rühl, Proc. International Advanced Study Institute on Math. Phys., Istanbul 1972, p. 185.
[10] G. Mack, I. T. Todorov, *Phys. Rev.* **D8**, 1764 (1973).
[11] W. Rühl, *Commun. Math. Phys.* **34**, 149 (1973).
[12] I. S. Gradshteyn, I. V. Ryzhik, *Tables of Integrals, Series and Products*, Academic Press 1975.
[13] S. Lang, *SL*(2, *R*), Addison-Wesley 1974.
[14] S. Coleman, J. Mandula, *Phys. Rev.* **159**, 1251 (1967).