RELATIVISTIC RADIAL EQUATIONS FOLLOWING FROM THE SALPETER EQUATION*

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The relativistic radial equations for two spin-1/2 particles, consistent with the hole theory, are derived from the one-time Salpeter equation. They may be of much help in relativistic calculations for leptonium and quarkonium.

The Salpeter equation is a relativistic one-time equation for two spin-1/2 particles with an instantaneous interaction consistent with the hole theory [1, 2][1],

$$[E-(\vec{\alpha}^{(1)}\cdot \vec{p}+\beta^{(1)}m^{(1)})-(-\vec{\alpha}^{(2)}\cdot \vec{p}+\beta^{(2)}m^{(2)})-P(\vec{p})V(\vec{r})]\psi(\vec{r}) = 0, \tag{1}$$

where

$$P(\vec{p}) = \Lambda^{(1)}_{+}(\vec{p})\Lambda^{(2)}_{+}(-\vec{p})-\Lambda^{(1)}_{-}(\vec{p})\Lambda^{(2)}_{-}(-\vec{p})$$
$$= \frac{1}{2}\left(\frac{\vec{\alpha}^{(1)}\cdot \vec{p}+\beta^{(1)}m^{(1)}}{\sqrt{\vec{p}^2+m^{(1)2}}}+\frac{-\vec{\alpha}^{(2)}\cdot \vec{p}+\beta^{(2)}m^{(2)}}{\sqrt{\vec{p}^2+m^{(2)2}}}\right) \tag{2}$$

is the hole theory projector which makes this equation different from the Breit equation valid in the single particle theory [3]. The potential $V(\vec{r})$ in Eq. (1) may have the Breit-like form

$$V(\vec{r}) = V(r)-BV'(r), \quad B = \frac{1}{2}\left[\vec{\alpha}^{(1)}\cdot \vec{\alpha}^{(2)}+\frac{(\vec{r}\cdot \vec{\alpha}^{(1)})(\vec{r}\cdot \vec{\alpha}^{(2)})}{r^2}\right], \tag{3}$$

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[1] The one-time Salpeter equation was obtained from the two-time Bethe–Salpeter equation in Ref. [1]. In general, the one-time approach to the relativistic two-body problem was derived from the two-time approach and also from the formal field theory in Refs. [2]. Full retardation was included. Also a general method of eliminating angular coordinates was described there. It was applied to the Breit equation in Refs. [6]. For another method of elimination cf. Ref. [8]. For the problem of reduction of the Bethe–Salpeter equation cf. Refs. [9].

which includes the lowest-order retardation. In the case of electromagnetic interactions $V(r) = V'(r) = \mp\alpha/r$, where $\alpha = e^2/4\pi$.

The projector (2) causes that effective calculations using the Salpeter equation (1) are much more difficult than those with the Breit equation, even in the electromagnetic case [4] or in the momentum space [5]. In particular, the radial equations following from the Salpeter equation seem to be not known in the literature, though they may be of much help in relativistic calculations for leptonium l^-l^+ and quarkonium $q\bar{q}$. In this note we derive the relativistic radial equations from Eq. (1) in a similar way as it was done previously in the case of the Breit equation [6]. Due to the projector (2), the new radial equations are integro-differential equations, even in the purely static case when $V(\vec{r}) = V(r)$.

First, we eliminate angular coordinates from Eq. (1) with the potential (3) by means of the unitary transformation

$$\bar{\psi}(\vec{r}) = U(\theta, \phi)\psi(\vec{r}), \qquad U(\theta, \phi) = e^{i\frac{\sigma_2{}^{(1)}+\sigma_2{}^{(2)}}{2}\theta}\, e^{i\frac{\sigma_3{}^{(1)}+\sigma_3{}^{(2)}}{2}\phi} \tag{4}$$

and the substitution (allowed by rotational invariance of Eq. (1))

$$\bar{\psi}(\vec{r}) = \frac{1}{\sqrt{2\pi}} Z_j^0(\cos\theta)\psi(r), \tag{5}$$

where

$$Z_j^0(\cos\theta) = U(\theta, \phi)\sqrt{2\pi}\,\langle\theta, \phi|j0\rangle = \frac{1-\sigma_3^{(1)}\sigma_3^{(2)}}{2} P_j^0(\cos\theta) + \frac{1+\sigma_3^{(1)}\sigma_3^{(2)}}{2} P_j^1(\cos\theta) \tag{6}$$

are the spinorial spherical harmonics corresponding to $m_j = 0$ [2, 6]. Here $P_j^{m_s{}^2}(\cos\theta)$ are the spherical harmonics normalized to 1.

After calculations, we get in the case of $m^{(1)} = m^{(2)}(=m)$ the following radial equation:

$$\begin{aligned}&\Bigg\{E + i(\alpha_3^{(1)} - \alpha_3^{(2)})\left[\frac{d}{dr} + \frac{1+\frac{1}{2}(\alpha_1^{(1)}\alpha_1^{(2)} + \alpha_2^{(1)}\alpha_2^{(2)})}{r}\right] \\ &- i(\alpha_1^{(1)} - \alpha_1^{(2)})\frac{\alpha_2^{(1)}\alpha_2^{(2)}\sqrt{j(j+1)}}{r} - (\beta^{(1)} + \beta^{(2)})m - V_{\text{eff}}\Bigg\}\psi(r) = 0,\end{aligned} \tag{7}$$

where

$$\begin{aligned}V_{\text{eff}}\psi(r) = &\Bigg\{-i(\alpha_3^{(1)} - \alpha_3^{(2)})\left[\frac{d}{dr} + \frac{1+\frac{1}{2}(\alpha_1^{(1)}\alpha_1^{(2)} + \alpha_2^{(1)}\alpha_2^{(2)})}{r}\right] \\ &+ i(\alpha_1^{(1)} - \alpha_2^{(2)})\frac{\alpha_2^{(1)}\alpha_2^{(2)}\sqrt{j(j+1)}}{r} + (\beta^{(1)} + \beta^{(2)})m\Bigg\} \\ &\times \tfrac{1}{2}\langle j0|\frac{1}{\sqrt{\vec{p}^2+m^2}}|j0\rangle\,[V(r) - \tfrac{1}{2}(\vec{\alpha}^{(1)}\cdot\vec{\alpha}^{(2)} + \alpha_3^{(1)}\alpha_3^{(2)})V'(r)]\psi(r).\end{aligned} \tag{8}$$

We can write here the formula

$$\langle j0|\frac{1}{\sqrt{\vec{p}^2+m^2}}|j0\rangle f(r) = \sum_l |\langle l|j0\rangle|^2 R_l f(r), \quad \sum_l |\langle l|j0\rangle|^2 = 1, \tag{9}$$

where

$$|\langle l|j0\rangle|^2 = \sum_{m_l} |\langle lm_l|j0\rangle|^2$$

$$= \delta_{lj}\frac{1-\sigma_2^{(1)}\sigma_2^{(2)}}{2}$$

$$+\left[\left(\delta_{lj-1}\frac{1-\sigma_3^{(1)}\sigma_3^{(2)}}{2}+\delta_{lj+1}\frac{1+\sigma_3^{(1)}\sigma_3^{(2)}}{2}\right)\frac{j-i\sigma_2^{(1)}\sqrt{j(j+1)}}{2j+1}\right.$$

$$\left.+\left(\delta_{lj+1}\frac{1+\sigma_3^{(1)}\sigma_3^{(2)}}{2}+\delta_{lj+1}\frac{1-\sigma_3^{(1)}\sigma_3^{(2)}}{2}\right)\frac{j+1+i\sigma_2^{(1)}\sqrt{j(j+1)}}{2j+1}\right]\frac{1+\sigma_2^{(1)}\sigma_2^{(2)}}{2} \tag{10}$$

and

$$R_l f(r) = \int_0^\infty r'^2 dr'\left[\frac{1}{\sqrt{rr'}}\int_m^\infty d\varepsilon J_{l+\frac{1}{2}}(\sqrt{\varepsilon^2-m^2}\, r)J_{l+\frac{1}{2}}(\sqrt{\varepsilon^2-m^2}\, r')\right]f(r'). \tag{11}$$

Here $J_{l+\frac{1}{2}}(kr)$ are the Bessel functions and $\varepsilon = \sqrt{k^2+m^2}$. In Eqs. (9–11) we make use of the orthogonality relation

$$\int_0^\infty k^2 dk j_l(kr) j_l(kr') = \frac{\pi}{2}\,\frac{\delta(r-r')}{r'^2}, \quad j_l(kr) = \sqrt{\frac{\pi}{2kr}}\,J_{l+\frac{1}{2}}(kr). \tag{12}$$

Next, we split the radial equation (7) into the system of 16 radial equations, taking into account the following representation of Dirac matrices:

$$\begin{aligned}
\alpha_1^{(1)} &= \sigma_3\times\sigma_1\times\mathbf{1}\times\mathbf{1}, & \alpha_1^{(2)} &= \mathbf{1}\times\sigma_1\times\sigma_3\times\mathbf{1},\\
\alpha_2^{(1)} &= \sigma_1\times\mathbf{1}\times\mathbf{1}\times\sigma_1, & \alpha_2^{(2)} &= \sigma_1\times\sigma_3\times\mathbf{1}\times\sigma_1,\\
\alpha_3^{(1)} &= \sigma_2\times\sigma_1\times\mathbf{1}\times\mathbf{1}, & \alpha_3^{(2)} &= \mathbf{1}\times\sigma_1\times\sigma_2\times\mathbf{1},\\
\beta^{(1)} &= \sigma_1\times\sigma_1\times\mathbf{1}\times\sigma_3, & \beta^{(2)} &= \mathbf{1}\times\sigma_1\times\sigma_1\times\mathbf{1},
\end{aligned} \tag{13}$$

where σ's and **1** are Pauli matrices.

We write Eq. (7) in this representation and combine the resulting radial components ψ's according to the scheme

$$\begin{aligned}
&9-14,\ 12-15,\ 9+14,\ 12+15 \to f_1^+, f_2^+, f_3^-, f_4^-,\\
&11-16,\ 10-13,\ 11+16,\ 10+13 \to g_1^+, g_2^+, g_3^-, g_4^-
\end{aligned} \tag{14}$$

and

$$1-6,\, 4-7,\, 1+6,\, 4+7 \rightarrow f_1^-, f_2^-, f_3^+, f_4^+,$$

$$3-8,\, 2-5,\, 3+8,\, 2+5 \rightarrow g_1^-,\, g_2^-,\, g_3^+,\, g_4^+ \qquad (15)$$

(where the normalization coefficient is $1/\sqrt{2}$). Then, the new radial components f_1 and f_2 correspond to $s = 0$, while the rest of f's and all g's — to $s = 1$. All f's have $m_s = 0$, while all g's mix $m_s = +1$ and $m_s = -1$. The components "+" and "−" refer to the intrinsic parity $\pi = +\eta$ and $\pi = -\eta$, respectively, where

$$\pi = \eta\beta^{(1)}\beta^{(2)} = \eta\sigma_1 \times \mathbf{1} \times \sigma_1 \times \sigma_3, \qquad \eta^2 = 1. \qquad (16)$$

The total parity $P = \pi(-1)^l$ is $P = +\eta$ and $P = -\eta$ for the components (14) and (15), respectively. So, the system of 16 radial equations splits into two independent subsystems of 8 equations containing the components (14) and (15), respectively. Here $l = j$ for $s = 0$ and $l = j-1, j, j+1$ for $s = 1$ if $j > 0$ ($l = s$ if $j = 0$ or $j = s$ if $l = 0$). The components "1" and "2" have $l = j$, while "3" and "4" mix $l = j-1$ and $l = j+1$ if $j > 0$ (they have $l = 1$ if $j = 0$).

After calculations, the resulting radial equations can be written in the form:

$$\begin{aligned}
\frac{d}{dr} F_2^{\pm} \pm \frac{m \mp m}{2} F_4^{\mp} + \tfrac{1}{2} E f_3^{\mp} \qquad\qquad\qquad &= 0,\\
\pm \frac{m \mp m}{2} F_3^{\mp} + \tfrac{1}{2} E f_4^{\mp} + \frac{i\sqrt{j(j+1)}}{r} G_2^{\pm} &= 0,\\
-\left(\frac{d}{dr} + \frac{2}{r}\right) F_3^{\mp} \pm \frac{m \pm m}{2} F_1^{\pm} + \tfrac{1}{2} E f_2^{\pm} + \frac{i\sqrt{j(j+1)}}{r} G_4^{\mp} &= 0,\\
\pm \frac{m \pm m}{2} F_2^{\pm} + \tfrac{1}{2} E f_1^{\pm} \qquad\qquad\qquad &= 0,\\
\left(\frac{d}{dr} + \frac{1}{r}\right) G_2^{\pm} \pm \frac{m \mp m}{2} G_4^{\mp} + \tfrac{1}{2} E g_3^{\mp} \qquad\qquad\qquad &= 0,\\
\pm \frac{m \mp m}{2} G_3^{\mp} + \tfrac{1}{2} E g_4^{\mp} - \frac{i\sqrt{j(j+1)}}{r} F_2^{\pm} &= 0,\\
-\left(\frac{d}{dr} + \frac{1}{r}\right) G_3^{\mp} \pm \frac{m \pm m}{2} G_1^{\pm} + \tfrac{1}{2} E g_2^{\pm} - \frac{i\sqrt{j(j+1)}}{r} F_4^{\mp} &= 0,\\
\pm \frac{m \pm m}{2} G_2^{\pm} + \tfrac{1}{2} E g_1^{\pm} \qquad\qquad\qquad &= 0, \qquad (17)
\end{aligned}$$

where we use the abbreviations

$$F_1^{\pm} = [1+\tfrac{1}{2}R_j(V-2V')]f_1^{\pm},$$

$$F_2^{\pm} = [1+\tfrac{1}{2}R_j(V+2V')]f_2^{\pm},$$

$$F_3^{\mp} = [1+\tfrac{1}{2}P_jV]f_3^{\mp} - \frac{i}{2}S_j(V-V')g_4^{\mp},$$

$$F_4^{\mp} = [1+\tfrac{1}{2}P_jV]f_4^{\mp} - \frac{i}{2}S_j(V+V')g_3^{\mp},$$

$$G_1^{\pm} = [1+\tfrac{1}{2}R_j(V-V')]g_1^{\pm},$$

$$G_2^{\pm} = [1+\tfrac{1}{2}R_j(V+V')]g_2^{\pm},$$

$$G_3^{\mp} = [1+\tfrac{1}{2}Q_j(V+V')]g_3^{\mp} + \frac{i}{2}S_jVf_4^{\mp},$$

$$G_4^{\mp} = [1+\tfrac{1}{2}Q_j(V-V')]g_4^{\mp} + \frac{i}{2}S_jVf_3^{\mp}. \tag{18}$$

Here

$$P_j = \frac{jR_{j-1}+(j+1)R_{j+1}}{2j+1}, \qquad Q_j = \frac{(j+1)R_{j-1}+jR_{j+1}}{2j+1},$$

$$S_j = \frac{\sqrt{j(j+1)}\,(R_{j-1}-R_{j+1})}{2j+1}, \tag{19}$$

while the integral operator R_l is defined in Eq. (11). We can see that the radial equations (17) are integro-differential equations.

Let us notice that from Eqs. (11) and (12) we get

$$R_lf(r) \to \frac{1}{m}f(r) \tag{20}$$

if $m \to \infty$. So, in this limit, the radial equations (17) become pure differential equations. Then

$$F_i^{\pm} \to f_i^{\pm}, \quad G_i^{\pm} \to g_i^{\pm} \quad (i = 1, 2, 3, 4), \tag{21}$$

unless they are multiplied by $\dfrac{m+m}{2} = m$ when mF_i^+ and $m\,G_i^+$contain potential terms additive to m. These terms cause that the Klein paradox, appearing in the Breit equation with an infinitely rising static potential $V(\vec{r}) = V(r)$ [7, 6], does not occur in the Salpeter equation with such a potential (at least in the limit of $m \to \infty$). The difference from the Breit equation [6] stems from the fact that for Salpeter equation

$$V_{\text{eff}} \to \tfrac{1}{2}(\beta^{(1)}+\beta^{(2)})\left[V(r)-\tfrac{1}{2}(\vec{\alpha}^{(1)}\cdot\vec{\alpha}^{(2)}+\alpha_3^{(1)}\alpha_3^{(2)})V'(r)\right] \tag{22}$$

if $m \to \infty$ (cf. Eq. (8)). The projector $\frac{1}{2}(\beta^{(1)}+\beta^{(2)})$ introduces here the difference.

We hope that the relativistic radial equations (17), which are consistent with the hole theory, will prove usefull in effective calculations for two spin-1/2 particles. In spite of their complicated form (which unfortunately cannot be avoided) they are perfectly tractable by numerical methods. In particular the kernel

$$K_l(r, r') = \frac{1}{\sqrt{rr'}}\int_m^\infty d\varepsilon J_{l+\frac{1}{2}}(\sqrt{\varepsilon^2-m^2}\, r)J_{l+\frac{1}{2}}(\sqrt{\varepsilon^2-m^2}\, r') \tag{23}$$

of the integral operator R_l given in Eq. (11) can be readily tabulated. In the limit of $m \to 0$ it takes the form

$$K_l(r, r') \to \frac{1}{rr'}\left[\frac{4rr'}{(r+r')^2}\right]^{l+1}\frac{l!}{2^{l+1}(2l+1)!!\pi}\, F\left(l+1, l+1, 2l+2; \frac{4rr'}{(r+r')^2}\right) \tag{24}$$

(involving the hypergeometrical function) which is evidently regular for $r \neq r'$. On the other hand, in the limit of $m \to \infty$ it becomes

$$K_l(r, r') \to \frac{1}{rr'}\,\frac{\delta(r-r')}{m} \tag{25}$$

in consistency with Eq. (20). So, only in this unattainable limit two Dirac particles interact as strictly point-like objects. In fact, they behave as extended objects due to the hole theory.

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