SEMI-INCLUSIVE SCATTERING IN QED

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We argue that the conventional approach to the infrared catastrophe with a very small resolution energy with respect to the rest electron mass is not convenient for large energies. We conjecture that for the high energy QED it is better to consider the semi-inclusive (or inclusive) scattering. This scattering has the positive cross section and is relativistically invariant. It is shown by the explicit calculations that to the fourth and sixth orders of the Coulomb scattering in the Born approximation, leading logarithms can be obtained treating formally the real photons as the soft photons. Conjecturing that this is true for higher orders of perturbation theory and summing an infinite set of diagrams, we obtain the anomalous dimension, which is a function of the coupling constant only. We calculate the cross sections for the electron-electron and electron-proton scattering showing that the anomalous dimension does not depend upon the statistical correlation of the final particles.

1. *Introduction and motivation*

It is a well known fact that there does not exist the S-matrix in the Quantum Electrodynamics (QED). The vanishing mass of the photon and therefore, the infrared catastrophe are the cause of this nonexistance. If we take into consideration an arbitrary process with some ingoing and/or outgoing fermion lines and compute all virtual corrections we obtain the renormalized amplitude equal to zero. Only pure photon processes have nonvanishing amplitudes. This feature can be removed if we notice that in experiments we cannot exactly measure the total energy of all outgoing particles, but we measure this quantity with the error ΔE, which is called the resolution energy. We must say here that it is not the statistical error of the measured quantity but the error which is attributed to the hardware of the apparatus used in the experiment. Physically this error means that we cannot distinguish processes which contain different amounts of soft outgoing photons, i.e., photons whose total energy is less than, or equal to ΔE. It was shown in [1, 2, 3] that if we sum up all the probabilities of processes for which the final states differ solely as to the number of soft, photons, then we obtain the nonvanishing transition probabilities. Since "soft" is not an invariant concept, the discussion of soft photons must be restricted

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to a specific Lorentz-frame $\mathscr{L}$. Usually one assumes that in $\mathscr{L}$ the energy loss due to the soft photon emission does not exceed ΔE, with $\Delta F \ll m$, where m is the electron mass. Therefore, the transition probability with soft inelastic corrections is not relativistically invariant. The other trouble is that this restriction can be satisfied only in experiments performed at very low energy, and we know that radiative corrections are unimportant at low energies. If one uses this restriction for high energy processes, one obtains negative cross sections, i.e., nonphysical results [5]. We argue that for the high energy QED it is more convenient to take a different approach. In this paper we will take into consideration semi-inclusive cross sections. "Semi-inclusive", since we will consider processes which differ solely as to the number of photons in the final state and the total energy of these photons is restricted only by the energy-momentum conservation law. Such processes have nonvanishing transition probabilities which are relativistically invariant and the cross sections are positive.

It is well known that the infrared catastrophe corresponds to the long-range character of the electromagnetic force. Recently, using this fact, many physicists obtained, by changing asymptotic conditions, nonvanishing expressions for renormalized amplitudes [1, 4, 13]. But these expressions contain an arbitrary function $\varphi_\mu(k, p)$, which, for soft k, satisfies the condition:

$$\varphi_\mu(k, p) \to \frac{p_\mu}{k \cdot p}.$$

It means that these amplitudes are not, in contradistinction to transition probabilities, uniquely determined and in opposite to the amplitudes, the transition probabilities have a physical significance too. These are the reasons why, in our opinion, it is better, from the physical point of view, to consider nonvanishing transition probabilities.

The plan of this paper is the following: In Section 2 we sum up an infinite set of the leading diagrams and obtain the anomalous dimension for the Coulomb scattering. In Section 3 we do the same for other processes and show that the anomalous dimension does not depend on the statistical correlation.

2. *Semi-inclusive Coulomb scattering*

In the previous paper [5] we calculated the inclusive Coulomb scattering cross section in the fourth order of QED. We obtained there the following result for the high energy scattering

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_0^{\text{as}} \left[1 + \frac{2\alpha}{\pi}\left(F(\theta) + \tfrac{13}{6} \ln \frac{E}{m}\right)\right]. \tag{1}$$

We broke this expression into two terms; the first came from the elastic processes and the second from the inelastic ones. We noted there that after the summation the infrared poles from these two terms cancel each other. From this simple example we see that in the calculation of finite radiative and real corrections to the cross section of an arbitrary process, we can discard from the beginning the pole parts of all expressions involved and apply a finite subtraction to the remaining finite expressions for radiative corrections. But for the semi-inclusive, and also for the inclusive, scattering we can say a little more. We noticed that elastic and inelastic processes have terms proportional to $\ln E/\mu$, where μ is an arbitrary mass parameter, and these terms cancel each other if we sum up elastic and inelastic cross sections. Then we can generalize the Meuldermans kitchen recipe [6]: we can discard from the beginning, in inclusive or semi-inclusive processes, terms proportional to $\ln E/\mu$ too. It is so if we dimensionally regularize infrared divergences. When we use the ficticious photon mass λ to the regularization, we observe that from the Marciano–Sirlin substitution [7] we can discard terms proportional to $\ln E/\lambda$. These terms have

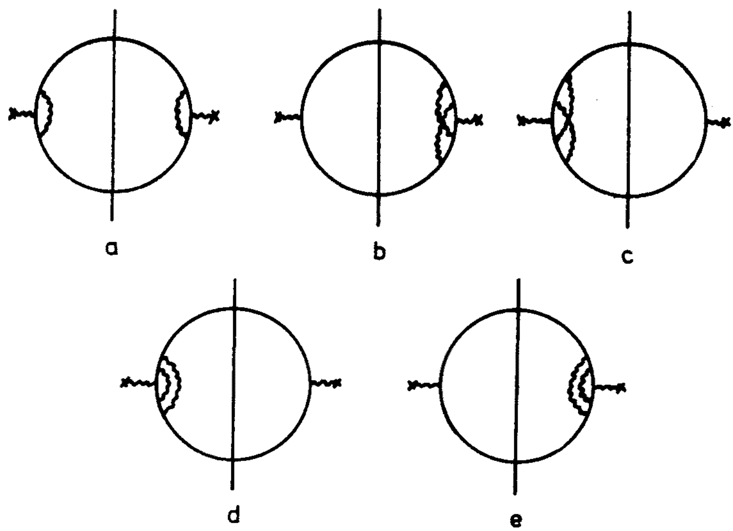

Fig. 1

direct connection with the Sudakov doubly logarithmic asymptotics of the Green functions. From this analysis we see that the doubly logarithmic terms are cancelled in semi-inclusive and inclusive cross sections. Of course, if we confine ourselves to the standard cross sections with a very small resolution energy ΔE, than these terms will appear. But, as we remarked, we cannot use the restriction $\Delta E \ll m$ for the high energy scattering and, therefore, the Sudakov doubly logarithmic asymptotics gives no contribution, i. e., the doubly logarithmic terms are cancelled. Since the dimensional and λ regularization schemes are equivalent, we will use these schemes exchangeably.

In [5] we calculated the inclusive Coulomb scattering cross section to the fourth order of perturbation theory. Now we calculate for the high energy region the semi-inclusive cross section to the sixth order taking leading terms only, i. e., we seek terms proportional to $\alpha^3(\ln E/m)^2$. Figs 1, 2, 3 and 4 contain diagrams which must be taken into consideration. Diagrammatic representations of the transition probabilities are drawn according to the rules given in [9, 12]. To these diagrams we must add sixteen purely inelastic diagrams

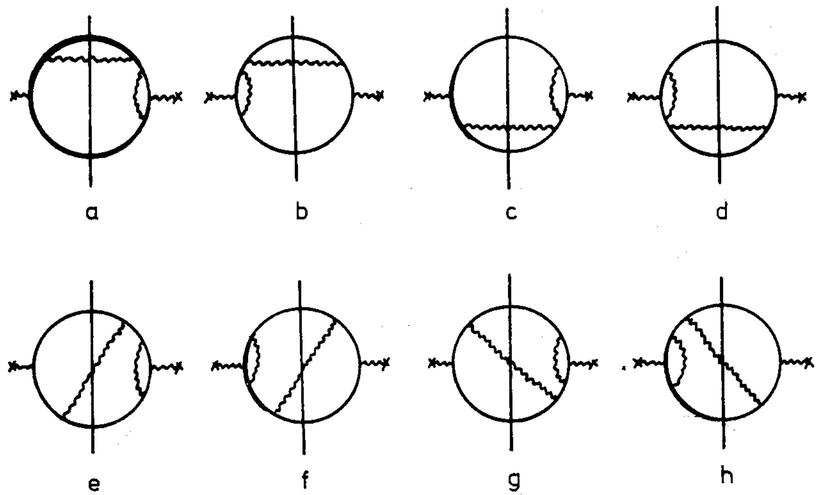

Fig. 2

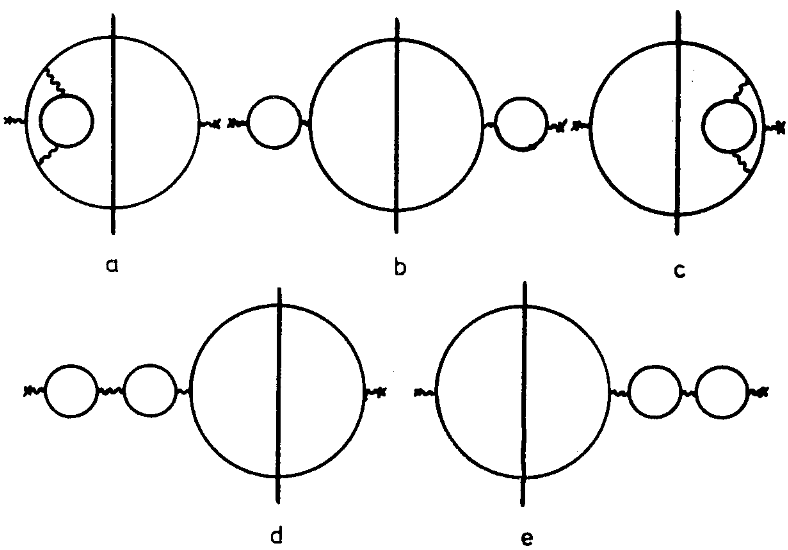

Fig. 3

which we will not draw here. The leading terms of these diagrams are the following:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{L}} (\text{Fig. 1a}) = \frac{\alpha Q^2 A(\theta)}{E^2}\,\frac{\alpha^2}{\pi^2}\left(\ln\frac{E}{m}\right)^2,$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{L}} (\text{Figs 1b+1e}) = \frac{\alpha Q^2 A(\theta)}{E^2}\,\frac{\alpha^2}{2\pi^2}\left(\ln\frac{E}{m}\right)^2,$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{L}} (\text{Figs 1c+1d}) = \frac{\alpha Q^2 A(\theta)}{E^2}\,\frac{\alpha^2}{2\pi^2}\left(\ln\frac{E}{m}\right)^2,$$

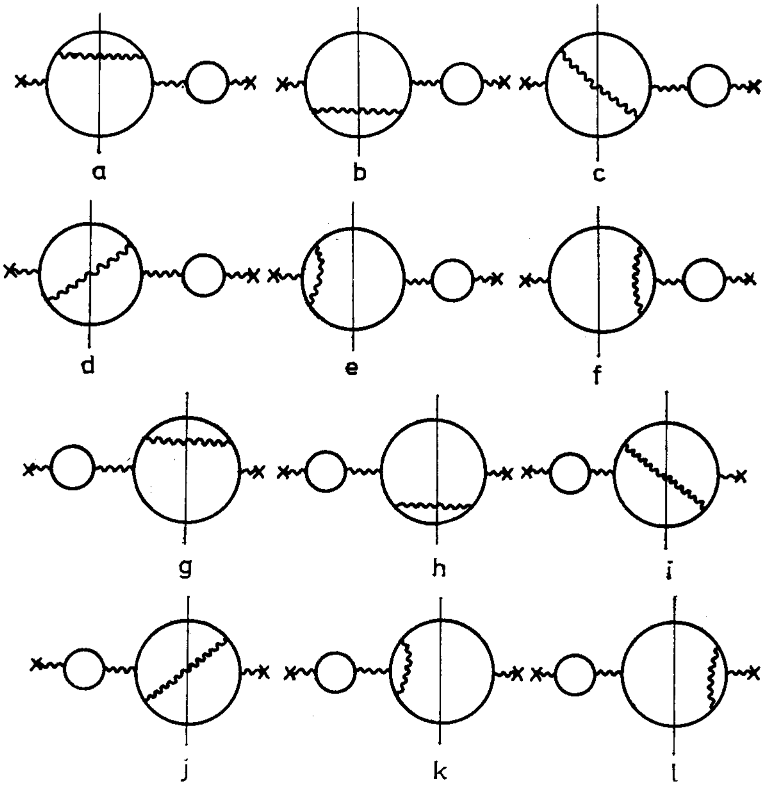

Fig. 4

$$\left(\frac{d\sigma}{d\Omega}\right)_L \text{(Fig. 2a)} = \left(\frac{d\sigma}{d\Omega}\right)_L \text{(Fig. 2b)} = \left(\frac{d\sigma}{d\Omega}\right)_L \text{(Fig. 2c)} = \left(\frac{d\sigma}{d\Omega}\right)_L \text{(Fig. 2d)}$$

$$= \frac{\alpha Q^2 A(\theta)}{E^2} \frac{\alpha^2}{2\pi^2} \left(\ln \frac{E}{m}\right)^2,$$

$$\left(\frac{d\sigma}{d\Omega}\right)_L \text{(Fig. 2e)} = \left(\frac{d\sigma}{d\Omega}\right)_L \text{(Fig. 2f)} = \left(\frac{d\sigma}{d\Omega}\right)_L \text{(Fig. 2g)} = \left(\frac{d\sigma}{d\Omega}\right)_L \text{(Fig. 2h)}$$

$$= \frac{\alpha Q^2 A(\theta)}{E^2} \frac{\alpha^2}{\pi^2} \left(\ln \frac{E}{m}\right)^2,$$

$$\left(\frac{d\sigma}{d\Omega}\right)_L \text{(Figs 3a+3c)} = -\frac{\alpha Q^2 A(\theta)}{E^2} \frac{\alpha^2}{6\pi^2} \left(\ln \frac{E}{m}\right)^2,$$

$$\left(\frac{d\sigma}{d\Omega}\right)_L \text{(Figs 3b+3d+3e)} = \frac{\alpha Q^2 A(\theta)}{E^2} \frac{\alpha^2}{3\pi^2} \left(\ln \frac{E}{m}\right)^2,$$

$$\left(\frac{d\sigma}{d\Omega}\right)_L \text{(Fig. 4)} = -\frac{\alpha Q^2 A(\theta)}{E^2} \frac{10\alpha^2}{3\pi^2} \left(\ln \frac{E}{m}\right)^2,$$

$$\left(\frac{d\sigma}{d\Omega}\right)_L \text{(purely inelastic diagrams)} = \frac{\alpha Q^2 A(\theta)}{E^2} \frac{9\alpha^2}{2\pi^2} \left(\ln \frac{E}{m}\right)^2.$$

Thus the final expression for the leading part of the semi-inclusive Coulomb scattering cross section in the sixth order is

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm L} = \frac{\alpha Q^2 A(\theta)}{E^2}\left[1+\frac{13\alpha}{3\pi}\ln\frac{E}{m}+\frac{28\alpha^2}{3\pi^2}\left(\ln\frac{E}{m}\right)^2\right]. \tag{2}$$

If we confine ourselves to the diagrams which do not contain vacuum polarization diagrams, then we obtain

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm L,NV} = \frac{\alpha Q^2 A(\theta)}{E^2}\left[1+\frac{5\alpha}{\pi}\ln\frac{E}{m}+\frac{1}{2!}\left(\frac{5\alpha}{\pi}\ln\frac{E}{m}\right)^2\right]. \tag{3}$$

We could also obtain the leading terms in the sixth order by formally treating outgoing photons as the soft photons of energies $E \ll m$. In the previous paper [5] we showed this in the fourth order. We conjecture that this prescription works in an arbitrary order. It will be proved in the future paper. Using this fact we can sum up an infinite set of diagrams obtaining

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm L,NV} = \frac{\alpha Q^2 A(\theta)}{E^2}\sum_{n=0}^{\infty}\sum_{\substack{i_1,i_2,j_1,j_2,j_3,j_4\geqslant 0\\ i_1+i_2+j_1+j_2+j_3+j_4=n}}\frac{A_1^{i_1}A_2^{i_2}B_1^{j_1}B_2^{j_2}B_3^{j_3}B_4^{j_4}}{i_1!\,i_2!\,j_1!\,j_2!\,j_3!\,j_4!}, \tag{4}$$

where (from [5]) we have

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm L}(\text{Fig. 2c}) = \frac{\alpha Q^2 A(\theta)}{E^2}A_1, \quad \left(\frac{d\sigma}{d\Omega}\right)_{\rm L}(\text{Fig. 2d}) = \frac{\alpha Q^2 A(\theta)}{E^2}A_2,$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm L}(\text{Fig. 3a}) = \frac{\alpha Q^2 A(\theta)}{E^2}B_1, \quad \left(\frac{d\sigma}{d\Omega}\right)_{\rm L}(\text{Fig. 3b}) = \frac{\alpha Q^2 A(\theta)}{E^2}B_2,$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm L}(\text{Fig. 3c}) = \frac{\alpha Q^2 A(\theta)}{E^2}B_3, \quad \left(\frac{d\sigma}{d\Omega}\right)_{\rm L}(\text{Fig. 3d}) = \frac{\alpha Q^2 A(\theta)}{E^2}B_4,$$

$$A_1+A_2+B_1+B_2+B_3+B_4 = \frac{5\alpha}{\pi}\ln\frac{E}{m}.$$

Therefore, we have

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm L,NV} = \frac{\alpha Q^2 A(\theta)}{E^2}\left(\frac{E}{m}\right)^{\frac{5\alpha}{\pi}}. \tag{5}$$

Here we must define the asymptotic expansion of the cross section.

$$\left(\frac{d\sigma}{d\Omega}\right)^{\rm as} = \frac{\alpha Q^2 A(\theta)}{m^d\left(\dfrac{E}{m}\right)^{d+\gamma(\alpha)}}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\alpha^n f_{nl}(\theta)\left(\ln\frac{E}{m}\right)^l, \tag{6}$$

where d and $\gamma(\alpha)$ are called the dimension and the anomalous dimension, respectively. We have in perturbation theory

$$\gamma(\alpha) = \sum_{l=0}^{\infty} g_l \alpha^{l+1}, \tag{7}$$

and from (5) we have

$$\gamma(\alpha) = -\frac{5\alpha}{\pi} + O(\alpha^2). \tag{8}$$

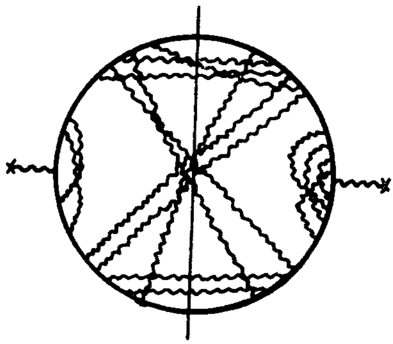

Fig. 5

This anomalous dimension is independent of θ. We find this reasonable since the anomalous dimension should be a global characteristic of processes, i. e., it cannot depend upon such conditions like experimentally determined initial energies of particles, or the installation of detectors which fix momentum transfers. Fig. 5 contains an example of the diagram which gives a contribution to the lowest order of $\gamma(\alpha)$. Other diagrams of the sixth order of perturbation theory (see Appendix) give the contribution to the higher order of $\gamma(\alpha)$ or to the functions $f_{nl}(\theta)$.

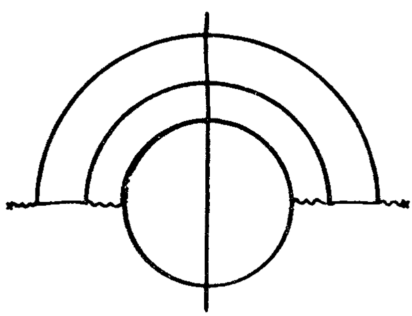

Fig. 6

In order to obtain the inclusive cross section we would have to include the pair production diagrams like that of Fig. 6. But these diagrams for the high energy limit are dominant and the inclusive cross section in the sixth order of perturbation theory is just the pair production cross section [8].

3. *Other processes*

In this chapter we will consider electron-proton and electron-electron scattering. We will study these processes in the high energy limit, i. e., when the initial total energy is considerably larger than all masses of particles taking part in the scattering. Of course, the first process is rather academic in this limit since in reality we would have to take into

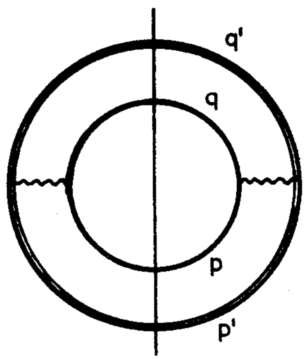

Fig. 7

consideration other interactions beside electromagnetic ones. But this process is manifestly relativistic, i. e., the expression for the cross section has an invariant structure. Fig. 7 contains the diagram which we must take into account. The external loop represents the proton. We introduce two Lorentz invariants,

$$s = (p+p')^2,\ t = (q-p)^2/s \tag{9}$$

tf we take all virtual and real corrections to this process, we obtain the following expression Ior the semi-inclusive (in this order inclusive too) electron-proton scattering cross section fo the sixth order of perturbation theory:

$$\frac{d\sigma}{dt} = \frac{\alpha^2 A\left(t, \frac{M}{m}\right)}{s}\left[1 + \frac{2\alpha}{\pi}\left(F\left(t, \frac{M}{m}\right) + \tfrac{11}{3}\ln\frac{s}{m^2}\right)\right]. \tag{10}$$

One can see from the previous considerations that if we take leading diagrams without vacuum polarization diagrams we obtain

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{L,NV}} = \frac{\alpha^2 A\left(t, \frac{M}{m}\right)}{s}\left(\frac{s}{m^2}\right)^{\frac{8\alpha}{\pi}}, \tag{11}$$

i. e., the anomalous dimension is equal to

$$\gamma(\alpha) = -\frac{16\alpha}{\pi} + O(\alpha^2). \tag{12}$$

The electron-electron scattering is not so academic as the previous one. Here we can consider only electromagnetic interactions of particles. Moreover this process is interesting for other reasons. We have two identical particles at the final state and one can hope

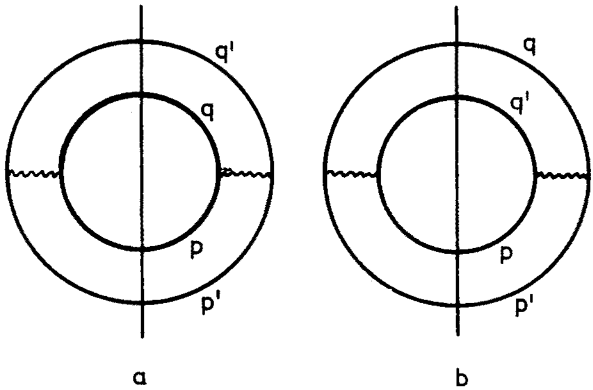

Fig. 8

for the statistical correlation effects. In the lowest order of perturbation theory we must calculate diagrams which are shown in Fig. 8. Considering all virtual and real corrections to the sixth order we have

$$\frac{d\sigma}{dt} = \frac{\alpha^2 A(t)}{s}\left[1+\frac{2\alpha}{\pi}\left(F(t)+\frac{23}{6}\ln\frac{s}{m^2}\right)\right], \tag{13}$$

and, as in the previous example, summing leading diagrams without vacuum polarization diagrams we obtain

$$\left(\frac{d\sigma}{dt}\right)_{\mathrm{L,NV}} = \frac{\alpha^2 A(t)}{s}\left(\frac{s}{m^2}\right)^{\frac{8\alpha}{\pi}}, \tag{14}$$

and the anomalous dimension is equal to

$$\gamma(\alpha) = -\frac{16\alpha}{\pi}+O(\alpha^2). \tag{15}$$

We see that it is the same as the previous one. The tree diagrams alone give no contribution to the leading terms of the cross section (see Appendix). The diagrams which contain self energy and vertex correction diagrams give the leading terms which is not true for the forward scattering [10].

I would like to thank Professor I. Białynicki-Birula for suggesting this problem and for discussions.

APPENDIX

In Section 2 we noticed that, apart from vacuum polarization diagrams, Figs 1 and 2 give all leading diagrams with virtual and real radiative corrections. In this Appendix we prove that the diagram of Fig. 8 is not leading. Let the function $F(\alpha, \theta, \kappa. m)$ have the following asymptotic expansion ($\kappa = E/m$)

$$F^{\mathrm{as}}(\alpha, \theta, \kappa, m) = \frac{1}{\kappa^n} f_0(\theta, m)\alpha^k(\ln \kappa)^l + \frac{1}{\kappa^n} \sum_{i=0}^{k-1} \sum_{j=0}^{l-1} f_{ij}(\theta, m)\alpha^i(\ln \kappa)^j. \tag{A.1}$$

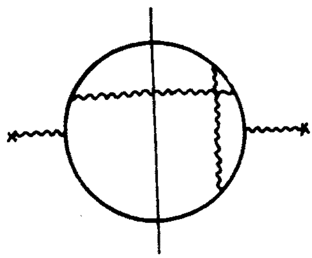

Fig. 9

We will show that for the diagram of Fig. 9

$$l = 1 \tag{A.2}$$

or

$$l = 0. \tag{A.3}$$

It means that we must calculate

$$\alpha^3 \int \frac{(1, k_\mu, k_\mu k_\nu)}{(r^2+2pr)(r^2+2r(p'+k)+2p'k+\lambda^2)(2p'k-\lambda^2)(r^2-2p'r)} \times \delta(E-E'-\omega)\frac{d^4r}{r^2-\lambda^2}\frac{d^3k}{\omega}|\vec{p}'|dE'. \tag{A.4}$$

As an example we will calculate the first integral noticing that others can be calculated in a similar manner. We must determine the asymptotical behavior of the integral

$$\int \frac{\sqrt{E'^2-m^2}}{\omega} d^3k \int_0^1 dxydyz^2dzM(\kappa, \theta, k, E'), \tag{A.5}$$

where

$$M(\kappa, \theta, k, E') = (z(y \times p + y(1-x)p' + (1-y)(p'+k))^2 + 2(1-y)p'k - \lambda^2)^{-2}(2p'k-\lambda^2)^{-1}. \tag{A.6}$$

M is the function of ω through the energy of the final electron $E' = E-\omega$ too. We make the expansion

$$M(\kappa, \theta, k, E') = M^{(0)}(\kappa, \theta, k, E) - \omega M^{(1)}(\kappa, \theta, k, E) + \frac{\omega^2}{2} M^{(2)}(\kappa, \theta, k, E) + \dots . \quad \text{(A.7)}$$

The integration over ω can be done and we have integrals over Feynman parameters only. To calculate these integrals asymptotically in the limit $\kappa \to \infty$ with θ held finite, it is convenient to make the Mellin transformation of F. We have

$$\bar{F}(\theta, \zeta) = \int_0^\infty \kappa^{-\zeta} F(\theta, \kappa) d\kappa. \quad \text{(A.8)}$$

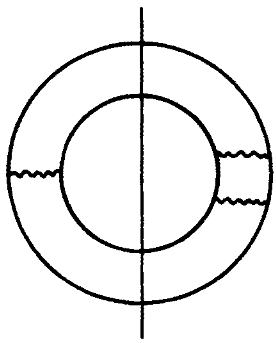

Fig. 10

The behavior of $\bar{F}(\theta, \zeta)$ at $\zeta \sim 0$ is related to the behaviour of $F(\theta, \kappa)$ for $\kappa \to \infty$. Specifically, a term ζ^{-n} in $\bar{F}(\theta, \zeta)$, $n > 0$, corresponds to a term $[(n-1)!]^{-1} \kappa^{-1}(\ln \kappa)^{n-1}$ in $F(\theta, \kappa)$. Therefore, obtaining the asymptotic form of $F(\theta, \kappa)$ for high energies is equivalent to obtaining all the pole terms in $\bar{F}(\theta, \zeta)$ at $\zeta = 0$ [10, 11]. One can prove that $\bar{F}(\theta, \zeta)$ has the pole of the first rank. Therefore, (A.3) is fulfilled.

In a similar manner one can prove that the diagram of Fig. 10 is not leading either.

At the end we will show, taking into account the diagram of Fig. 2c, how we calculate the leading term. The contribution to the cross section from this diagram is proportional to

$$\int_\lambda^{E-m} \omega d\omega G(p, p', \omega) \delta(E-E'-\omega) |\vec{p}'| dE', \quad \text{(A.9)}$$

where

$$G(p, p', \omega) = \int \frac{d^4 q}{q^2-\lambda^2} d\Omega_k \operatorname{Tr} \left(\frac{\not{p}-\not{k}+m}{(p-k)^2-m^2} \gamma_\mu (\not{p}+m)\gamma^\mu \right.$$
$$\left. \times \frac{\not{p}-\not{k}+m}{(p-k)^2-m^2} \gamma_\nu \frac{\not{p}-\not{q}-\not{k}+m}{(p-q-k)^2-m^2} \gamma^0 \frac{\not{p}'-\not{q}+m}{(p'-q)^2-m^2} \gamma^\nu (\not{p}'+m)\gamma^0 \right). \quad \text{(A.10)}$$

We expand this function in the following manner:

$$G(p, p', \omega) = \omega^{-2}G_{-2}(p, p')+\omega^{-1}G_{-1}(p, p')+G_0(p, p')+\omega G_1(p, p')+ \ldots . \quad \text{(A.11)}$$

The leading term comes from $G_{-2}(p, p')$ what means that this term one can obtain treating formally the outgoing photon as a soft one. Of course, the result contains expressions proportional to $\ln E/\lambda$ but, remembering our recipe from Section 2 we disregard these terms.

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In Fig. 4 the quantity s changes from 0 to 10, not to 1.0.

In Appendix there should be

$$b_0 = \frac{2(1-y)p^2(1-\cos\theta)}{\omega}, \quad c_0 = \frac{2xp^2(1-\cos\theta)}{\omega}.$$