CONSERVATION OF ENERGY IN UNIFIED FIELD THEORY

By A. H. Klotz

Department of Applied Mathematics, University of Sydney*

(Received August 1, 1978)

The problem of finding the energy-momentum tensor in the non-symmetric unified field theory is investigated. It is shown that the definition of such a quasi-tensor due to Einstein and Kaufman leads to the unlikely conclusion that the energy density of matter in the universe vanishes. A tentative solution of this difficulty which gives a non-zero distribution of matter is proposed. The article contains a brief discussion of some unsolved problems of the theory.

1. *Introduction*

There are now good reasons to believe (Ref. [1]) that the non-symmetric unified field theory (Refs. [2–4]) of Einstein and Straus contained in the weak system of the field equation

$$g_{\mu\nu,\lambda}-\tilde{\Gamma}^{\sigma}_{\mu\lambda}g_{\sigma\nu}-\tilde{\Gamma}^{\sigma}_{\lambda\nu}g_{\mu\sigma} = 0, \quad \tilde{R}_{\underline{\mu\nu}} = 0, \quad \tilde{R}_{\underset{\vee}{\mu\nu,\lambda}} = 0, \quad \tilde{\Gamma}_{\mu} = 0, \tag{1}$$

represent the correct generalisation of the theory of gravitation. I shall refer to it as UFT and distinguish the Einstein–Straus affine connection $\tilde{\Gamma}^{\lambda}_{\mu\nu}$ from any other by the twiddle, denoting by it also any tensor, such as Ricci or Riemann–Christoffel, formed from this connection.

By "correct" I mean logically consistent, free from most obvious contradictions as far as physical reality is concerned, and offering some hope of eventual empirical confirmation or otherwise. I cannot see at present any alternative unified field theory of equal merits, with the exception of Weyl's which seems to be related to quantum mechanical fields rather than to macroscopic gravitation and electromagnetism. The singular virtue of the UFT lies in foundation on a meaningful physical hypothesis that the principle of transposition invariance represents charge conjugation invariance and replaces the equivalence principle of General Relativity in providing a means for choice of the field equations.

* Address: Department of Applied Mathematics, University of Sydney, Sydney N. S. W., 2006, Australia.

Empirical testing of the new theory presents grave difficulties. General Relativity found its confirmation in a pure gravitational field or in the phenomena associated with neutral electromagnetic radiation in such fields. Gravitational fields of sufficient strength to yield observable deviations from classical predictions are readily available but comparable electromagnetic fields are not. Or, rather, they are, but investigation of their geometrical nature involves a self-defeating, two-fold problem. Paucity of known solutions of the field equations virtually necessitates operating with single particles, electrons if you like. But isolated elementary particles are subject to quantum laws whose incorporation is beyond the scope of the theory either conceptually or empirically. Experiments giving testable predictions can be devised (Ref. [5]) only if one ignores quantum effects which may of course, influence validity of any results or conclusions. Moreover, the proposed test employs exact equations of motion of a charged particle and these can only be guessed at the present time. In view of these remarks it seems clear that confirmation of the theory can be sought now only beyond a terrestrial laboratory and, because of its apparently electrically quasi-neutral state, beyond the solar system. Similarly, because of the interplay between concrete observations and their interpretation, it is unlikely that a suitably convincing test of the interaction between electromagnetic and gravitational fields indicated by UFT can be constructed immediately in the realm of current astrophysics. Hence, any hope of subjecting the theory to a relevant test of physical validity appears to be confined to cosmology.

I have shown recently (Ref. [6]) that UFT can be interpreted in such a way that it forecasts a unique model of the universe at least if one confines oneself to static case in which solutions of the field equations are known. As a consequence of this limitation it cannot be insisted that the model obtained represents the actual universe. It is more important that a unique model independent of ad hoc assumptions other than are vital to the completeness of the theory (determination of the metric in Ref. [1]) should result at all. In any case, whether we can already say anything about the validity of UFT (and of my metric assumption) depends on knowing what is the predicted distribution of matter in the predicted model. Some results of necessarily only preliminary calculations will be described in this article.

It must be borne in mind that the problem of matter in the UFT is very different than in General Relativity although it is solved by similar means of relying on the idea of conservation. We write down the general relativistic field equations

$$G_{\mu\nu} = -\kappa T_{\mu\nu}, \tag{2}$$

because the Einstein-tensor $G_{\mu\nu}$ is conserved in the invariant sense and the energy-momentum tensor $T_{\mu\nu}$ is assumed to be so. It is of absolute importance to the UFT that there is no known empirical confirmation of these equations. I assert that Einstein–Maxwell theory (when $T_{\mu\nu}$ is the electromagnetic energy-stress-momentum tensor) is simply wrong. Validity of this assertion depends on equations (2) remaining unconfirmed and on an eventual confirmation of the predictions of UFT.

We do not have in the latter an energy-momentum tensor, at least not as far as the gravitational and electromagnetic fields are concerned. If these are the only macroscopic fields (and some quantum mechanical considerations seem to suggest that they are, Ref. [7]) then the structure of the universe will be completely described by the equations (1). As far as UFT goes, physics, that is physical fields, are described by the fundamental tensor $g_{\mu\nu}$ (non-symmetric) and the contracted, skew-symmetric part Γ_μ of a generalised affine connection. (It must be observed that it is this limiting assumption that may have to be changed as knowledge accumulates. The assumption is good enough at present). Geometry of the space-time manifold in which the physical fields subsist is described by the affine connection $\tilde{\Gamma}^\lambda_{\mu\nu}$ related to the more general $\Gamma^\lambda_{\mu\nu}$ by the (unsolvable) Schrödinger equation (Ref. [4])

$$\tilde{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu}+\tfrac{2}{3}\,\delta^\lambda_\mu\Gamma_\nu, \tag{3}$$

and the Riemann–Christoffel and Ricci tensors constructed from it. A posteriori (Ref. [1]) we also have the metric tensor $a_{\mu\nu}$. The relativistic link between physics and geometry, the UFT analogue of the principle of geometrization, is established through the field equations. There is no room for the equivalence principle. It is completely replaced by the concept of transposition invariance which determines selection of the field equations.

In these circumstances, matter or energy-momentum can only be introduced through the variational principle by the method invented by Weyl. An advantage of this procedure is that the resulting quantity will be automatically conserved without having to postulate equations of state as in general relativity. But then UFT deals only with fields and equations of state do not enter our considerations.

2. *On symmetry of the Riemann–Christoffel tensor*

Before attempting to calculate the energy-momentum expression (it is neither a tensor nor a tensor density) in UFT it is necessary to correct a curious mistake in Einstein's presentation of the symmetries of the generalised Riemann–Christoffel tensor (Ref. [8])

$$\tilde{R}^\lambda_{\mu\nu\kappa} = -\tilde{\Gamma}^\lambda_{\mu\nu,\kappa}+\tilde{\Gamma}^\lambda_{\mu\kappa,\nu}+\tilde{\Gamma}^\varrho_{\mu\kappa}\tilde{\Gamma}^\lambda_{\varrho\nu}-\tilde{\Gamma}^\varrho_{\mu\nu}\tilde{\Gamma}^\lambda_{\varrho\kappa}. \tag{4}$$

With Einstein, I define the covariant tensor

$$\tilde{R}_{\lambda\mu\nu\kappa} = g_{\sigma\lambda}\tilde{R}^\sigma_{\mu\nu\kappa}. \tag{5}$$

Differentiating

$$g_{\mu\nu,\lambda}-\tilde{\Gamma}^\varrho_{\mu\lambda}g_{\varrho\nu}-\tilde{\Gamma}^\varrho_{\lambda\nu}g_{\mu\varrho} = 0,$$

with respect to κ, interchanging κ and λ and subtracting Einstein finds

$$g_{\sigma\nu}(-\tilde{\Gamma}^\sigma_{\mu\lambda,\kappa}+\tilde{\Gamma}^\sigma_{\kappa\mu,\lambda}+\tilde{\Gamma}^\varrho_{\mu\kappa}\tilde{\Gamma}^\sigma_{\varrho\lambda}-\tilde{\Gamma}^\varrho_{\mu\lambda}\tilde{\Gamma}^\sigma_{\varrho\kappa})+g_{\mu\sigma}(-\tilde{\Gamma}^\sigma_{\lambda\nu,\kappa}+\tilde{\Gamma}^\sigma_{\kappa\nu,\lambda}+\tilde{\Gamma}^\varrho_{\kappa\nu}\Gamma^\sigma_{\lambda\varrho}-\tilde{\Gamma}^\varrho_{\lambda\nu}\tilde{\Gamma}^\sigma_{\kappa\varrho}) = 0. \tag{6}$$

Comparing (6) with (4) we get

$$\tilde{R}_{\nu\mu\lambda\kappa}+[g_{\sigma\nu}(-\tilde{\Gamma}^\sigma_{\mu\lambda,\kappa}+\tilde{\Gamma}^\sigma_{\mu\kappa,\lambda}+\tilde{\Gamma}^\varrho_{\mu\kappa}\tilde{\Gamma}^\sigma_{\varrho\lambda}-\tilde{\Gamma}^\varrho_{\mu\lambda}\tilde{\Gamma}^\sigma_{\varrho\kappa})]^\dagger = \tilde{R}_{\nu\mu\lambda\kappa}+\underset{**}{\tilde{R}_{\nu\mu\lambda\kappa}} = 0. \tag{7}$$

The operation denoted by the dagger or, more properly, by the pair of stars, is exactly the operation of hermitian conjugacy or double transposition referred to in the principle of transposition invariance: every $g_{\mu\nu}$ and $\tilde{\Gamma}^{\lambda}_{\mu\nu}$ is transposed with respect to its pair of covariant indices (matrix transposition) and then the starred pair of indices is interchanged (complex conjugation). It follows from (7), contrary to what Einstein asserts, that $R_{\lambda\mu\nu\kappa}$ is not hermitian antisymmetric in the first pair of its indices. Let us define (T referring, without indication, to μ and ν)

$$\tilde{R}^{T}_{\nu\mu\lambda\kappa} = g_{\nu\sigma}(-\tilde{\Gamma}^{\sigma}_{\lambda\mu,\kappa}+\tilde{\Gamma}^{\sigma}_{\kappa\mu,\lambda}+\tilde{\Gamma}^{\varrho}_{\kappa\mu}\tilde{\Gamma}^{\sigma}_{\lambda\varrho}-\tilde{\Gamma}^{\varrho}_{\lambda\mu}\tilde{\Gamma}^{\sigma}_{\kappa\varrho}).$$

Then

$$\underset{**}{\tilde{R}^{T}_{\nu\mu\lambda\kappa}} = g_{\sigma\mu}(-\tilde{\Gamma}^{\sigma}_{\nu\lambda,\kappa}+\tilde{\Gamma}^{\sigma}_{\nu\kappa,\lambda}+\tilde{\Gamma}^{\varrho}_{\nu\kappa}\tilde{\Gamma}^{\sigma}_{\varrho\lambda}-\tilde{\Gamma}^{\varrho}_{\nu\lambda}\tilde{\Gamma}^{\sigma}_{\varrho\kappa}),$$

or

$$\underset{**}{R^{T}_{\mu\nu\lambda\kappa}} = \tilde{R}_{\nu\mu\lambda\kappa} = -\underset{**}{\tilde{R}_{\nu\mu\lambda\kappa}},$$

from (7). Hence, double starring again, $\tilde{R}^{T}_{\mu\nu\lambda\kappa} = -\tilde{R}_{\nu\mu\lambda\kappa}$, but of course, the "$T$" operation is not double transposition. We conclude this brief digression by calculating an expression we shall need presently. We have

$$g^{\mu\lambda}\tilde{R}^{i}_{\mu\lambda\kappa} = g^{\mu\lambda}g^{i\nu}\tilde{R}_{\nu\mu\lambda\kappa} = g^{\mu\lambda}g^{i\nu}\tilde{R}^{T}_{\mu\nu\kappa\lambda} = g^{\mu\lambda}g^{i\nu}g_{\mu\sigma}(-\tilde{\Gamma}^{\sigma}_{\kappa\nu,\lambda}+\tilde{\Gamma}^{\sigma}_{\lambda\nu,\kappa}+\tilde{\Gamma}^{\varrho}_{\lambda\nu}\tilde{\Gamma}^{\sigma}_{\kappa\varrho}-\tilde{\Gamma}^{\varrho}_{\kappa\nu}\tilde{\Gamma}^{\sigma}_{\lambda\varrho})$$

$$= g^{i\nu}(-\tilde{\Gamma}^{\sigma}_{\kappa\nu,\sigma}+\tilde{\Gamma}^{\sigma}_{\sigma\nu,\kappa}+\tilde{\Gamma}^{\varrho}_{\sigma\nu}\tilde{\Gamma}^{\sigma}_{\kappa\varrho}-\tilde{\Gamma}^{\varrho}_{\kappa\nu}\tilde{\Gamma}^{\sigma}_{\sigma\varrho}) = g^{i\nu}\tilde{R}_{\kappa\nu} = g^{i\nu}\tilde{R}_{\underset{\vee}{\kappa\nu}}, \tag{8}$$

in view of the field equations (1).

3. *Conservation law in the generalised UFT*

I shall calculate first the conserved quantity in the generalised non-symmetric theory described in Ref. [4] in terms of the Einstein–Straus connection $\tilde{\Gamma}^{\lambda}_{\mu\nu}$ (for which $\tilde{\Gamma}_{\mu} = 0$). Einstein and Kaufman (Ref. [3]) introduced a variational parameter $U^{\lambda}_{\mu\nu}$ with respect to the Ricci tensor of the non-symmetric theory is automatically transposition invariant and the weak field equations are derived from the action variation

$$\delta \int \mathfrak{g}^{\mu\nu}R_{\mu\nu}(U)d\Omega = 0, \tag{9}$$

carried out with respect to $\mathfrak{g}^{\mu\nu}$ and $U^{\lambda}_{\mu\nu}$. It has been shown in Ref. [4] that the affine connection is most generally expressed in terms of such an $U^{\lambda}_{\mu\nu}$ by

$$\Gamma^{\lambda}_{\mu\nu} = U^{\lambda}_{\mu\nu}+(2\alpha_1+\tfrac{1}{3})\delta^{\lambda}_{\nu}U^{\sigma}_{\underline{\mu\sigma}}-(3\alpha_1+1)\delta^{\lambda}_{\mu}U^{\sigma}_{\underline{\nu\sigma}}+(3\alpha_1+2\alpha_2+1)\delta^{\lambda}_{\mu}U_{\nu}-\tfrac{1}{3}\,\delta^{\lambda}_{\nu}U_{\mu}, \tag{10}$$

where α_1 and α_2 are numerical parameters. Replacing $\Gamma^{\lambda}_{\mu\nu}$ by its Einstein–Straus counterpart $\tilde{\Gamma}^{\lambda}_{\mu\nu}$, we easily find that the contracted symmetric and skew parts of $U^{\lambda}_{\mu\nu}$ are given by

$$pU^{\beta}_{\underline{\alpha\beta}} = \tfrac{3}{2}\,\tilde{\Gamma}^{\beta}_{\alpha\beta}, \qquad qU_{\alpha} = \tfrac{3}{2}\,\tilde{\Gamma}^{\beta}_{\alpha\beta}-2\Gamma_{\alpha}, \tag{11}$$

where $p = 15\alpha_1+4$, $q = 9\alpha_1+6\alpha_2+2$. Hence

$$U^\lambda_{\mu\nu} = \tilde{\Gamma}^\lambda_{\mu\nu}+\tfrac{1}{2}\left(\frac{1}{q}-\frac{6\alpha_1+1}{p}\right)\delta^\lambda_\nu\tilde{\Gamma}^\sigma_{\mu\sigma}+\tfrac{3}{2}\left(\frac{3\alpha_1+1}{p}-\frac{3\alpha_1+2\alpha_2+1}{q}\right)\delta^\lambda_\mu\tilde{\Gamma}^\sigma_{\nu\sigma}+\frac{2}{3q}(\delta^\lambda_\mu\Gamma_\nu-\delta^\lambda_\nu\Gamma_\mu). \tag{12}$$

It has also been shown (in Ref. [4]) that the quantity

$$\mathfrak{T}^\mu_\nu = \mathfrak{g}^{\alpha\beta}U^\mu_{\alpha\beta,\nu}-(1+3\alpha_1)(\mathfrak{g}^{\alpha\mu}U^\beta_{\alpha\beta,\nu}+\mathfrak{g}^{\mu\alpha}U^\beta_{\beta\alpha,\nu})-4\alpha_2\mathfrak{g}^{\underset{\vee}{\alpha\mu}}U_{\alpha,\nu}, \tag{13}$$

is, in consequence of (9), conserved, so that

$$\mathfrak{T}^\mu_{\nu,\mu} = 0. \tag{14}$$

A straightforward substitution from (12) now shows that

$$\mathfrak{T}^\lambda_\kappa = \mathfrak{g}^{\mu\nu}\tilde{\Gamma}^\lambda_{\mu\nu,\kappa}+\frac{1}{q}\mathfrak{g}^{\mu\lambda}\tilde{\Gamma}^\sigma_{\mu\sigma,\kappa}-\tfrac{3}{2}\left(\frac{1+q}{q}\right)\mathfrak{g}^{\lambda\mu}\tilde{\Gamma}^\sigma_{\mu\sigma,\kappa}+\tfrac{4}{3}\left(\frac{2+q}{q}\right)\mathfrak{g}^{\underset{\vee}{\lambda\mu}}\Gamma_{\mu,\kappa}. \tag{15}$$

Introducing now the Riemann–Christoffel and Ricci tensors $\tilde{R}^\lambda_{\nu\mu\kappa}$ and $\tilde{R}^\lambda_{\mu\nu\lambda}$ we readily find that

$$\mathfrak{g}^{\mu\nu}\tilde{\Gamma}^\lambda_{\mu\nu,\kappa} = -\mathfrak{g}^{\mu\nu}\tilde{R}^\lambda_{\mu\nu\kappa}+\mathfrak{g}^{\mu\nu}\tilde{\Gamma}^\varrho_{\mu\kappa}\tilde{\Gamma}^\lambda_{\varrho\nu}+(\mathfrak{g}^{\mu\nu}\tilde{\Gamma}^\lambda_{\mu\kappa})_{,\nu},$$

$$\mathfrak{g}^{\lambda\mu}\tilde{\Gamma}^\sigma_{\mu\sigma,\kappa} = \mathfrak{g}^{\lambda\mu}\tilde{R}_{\mu\kappa}+\mathfrak{g}^{\varrho\mu}\tilde{\Gamma}^\lambda_{\varrho\sigma}\tilde{\Gamma}^\sigma_{\mu\kappa}+(\mathfrak{g}^{\lambda\mu}\tilde{\Gamma}^\sigma_{\mu\kappa})_{,\sigma},$$

so that

$$\mathfrak{T}^\lambda_\kappa = -\tfrac{1}{2}\left(1+\frac{1}{q}\right)(-\mathfrak{g}^{\mu\lambda}+2\mathfrak{g}^{\underset{\vee}{\lambda\mu}})\tilde{R}_{\mu\kappa}$$
$$+\left[\mathfrak{g}^{\mu\nu}\tilde{\Gamma}^\lambda_{\mu\kappa}+\frac{1}{q}(\mathfrak{g}^{\mu\lambda}-\tfrac{3}{2}(1+q)\mathfrak{g}^{\lambda\mu})\tilde{\Gamma}^\nu_{\mu\kappa}+\tfrac{4}{3}\left(\frac{2+q}{q}\right)\mathfrak{g}^{\underset{\vee}{\lambda\nu}}\Gamma_\kappa\right]_{,\nu}$$
$$+\left(1+\frac{1}{q}\right)(\mathfrak{g}^{\mu\nu}\tilde{\Gamma}^\varrho_{\mu\kappa}\tilde{\Gamma}^\lambda_{\varrho\nu}-\tfrac{3}{2}\mathfrak{g}^{\mu\nu}\tilde{\Gamma}^\varrho_{\nu\kappa}\tilde{\Gamma}^\lambda_{\mu\varrho}). \tag{16}$$

In deriving the expression (16), I have used the facts that

$$\mathfrak{g}^{\alpha\beta}{}_{,\sigma} = \mathfrak{g}^{\alpha\beta}\tilde{\Gamma}^\varrho_{\sigma\varrho}-\mathfrak{g}^{\varrho\beta}\tilde{\Gamma}^\alpha_{\varrho\sigma}-\mathfrak{g}^{\alpha\varrho}\tilde{\Gamma}^\beta_{\sigma\varrho},$$

and

$$\tilde{R}_{\underset{\vee}{\mu\nu}} = \tfrac{2}{3}(\Gamma_{\mu,\nu}-\Gamma_{\nu,\mu}), \tag{17}$$

as well as equation (8).

We may observe now that when

$$q = -1, \tag{18}$$

which is equivalent to $3\alpha_1+2\alpha_2+1 = 0$,

$$\mathfrak{T}^{\lambda}_{\kappa} = -\mathfrak{g}^{\mu\nu}\tilde{R}^{\lambda}_{\mu\nu\kappa}-\mathfrak{g}^{\mu\lambda}\tilde{R}_{\mu\kappa}-2\mathfrak{g}^{\lambda\mu}_{\vee}R_{\underset{\vee}{\mu\kappa}}+(\mathfrak{g}^{\mu\nu}\tilde{\Gamma}^{\lambda}_{\mu\kappa}-\mathfrak{g}^{\mu\lambda}\tilde{\Gamma}^{\nu}_{\mu\kappa}-\tfrac{4}{3}\,\mathfrak{g}^{\lambda\nu}_{\vee}\Gamma_{\kappa})_{,\nu}$$

$$= (\mathfrak{g}^{\mu\nu}\tilde{\Gamma}^{\lambda}_{\mu\kappa}-\mathfrak{g}^{\mu\lambda}\tilde{\Gamma}^{\nu}_{\mu\kappa}-\tfrac{4}{3}\,\mathfrak{g}^{\lambda\nu}_{\vee}\Gamma_{\kappa})_{,\nu}, \tag{19}$$

because of equation (8) and of the field equations (1). In this form it is easily seen that $\mathfrak{T}^{\lambda}_{\kappa}$ satisfies identically the conservation equations (14) since the quantity

$$\tau^{\nu\lambda}_{\kappa} = \mathfrak{g}^{\mu\nu}\tilde{\Gamma}^{\lambda}_{\mu\kappa}-\mathfrak{g}^{\mu\lambda}\tilde{\Gamma}^{\nu}_{\mu\kappa}-\tfrac{4}{3}\,\mathfrak{g}^{\lambda\nu}_{\vee}\Gamma_{\kappa} = -\tau^{\lambda\nu}_{\kappa}$$

is skew symmetric in ν and λ.

The condition (17) is satisfied in the Einstein–Kaufman theory which is equivalent to the Einstein–Straus (Ref. [4], but with a much simpler variational principle). The generalised theory which is similarly equivalent in its domain of application and merely exhibits explicitly what may be called α-invariance, reduced to Einstein–Kaufman form when $\alpha_1 = -\frac{1}{3}$ and $\alpha_2 = 0$.

4. An example

Let us now calculate explicitly the components of $\mathfrak{T}^{\lambda}_{\kappa}$ for the spherically symmetric solution obtained in Ref. [1]. The non-zero components of the fundamental tensor $\mathfrak{g}_{\mu\nu}$ are given by

$$g_{11} = -\left[\left(1-\frac{r^2}{r_0^2}\right)\left(1+c\sqrt{\frac{r_0^2}{r^2}-1}\right)\right]^{-1}, \quad g_{22} = g_{33}\operatorname{cosec}^2\theta = -r^2\left(1-\frac{2r^2}{r_0^2}\right),$$

$$g_{44} = 1+c\sqrt{\frac{r_0^2}{r^2}-1}, \quad g_{23} = -g_{32} = \frac{2r^4}{r_0^2}\sqrt{\frac{r_0^2}{r^2}-1}\,\sin\theta. \tag{20}$$

Since I am particularly interested in the material content of the proposed cosmological model (Ref. [9]) we shall use the "Schwarzschild" radial coordinate

$$R = \frac{r_0}{\sqrt{\dfrac{r_0^2}{r^2}-1}}, \tag{21}$$

though, as the results will show, not without reservations. The substitution (20) implies that the new form of the fundamental tensor is

$$g_{11} \to -\left(\frac{dr}{dR}\right)^2 g_{11}, \tag{22}$$

that is the a_{11} of Ref. [9] with the remaining components obtained from (19) by replacing r with R. Similarly, if $\Gamma^{\lambda}_{\mu\nu}$ denotes the Tonnelat (Ref. [8], equations (109)) affine connection expressed in terms of R

$$\tilde{\Gamma}^{1}_{11} = \frac{dr}{dR}\,\Gamma^{1}_{11}+\frac{dR}{dr}\,\frac{d^2r}{dR^2}, \quad \tilde{\Gamma}^{1}_{pq} = \frac{dR}{dr}\,\Gamma^{1}_{pq}, \quad \tilde{\Gamma}^{p}_{1q} = \frac{dr}{dR}\,\Gamma^{p}_{1q}, \tag{23}$$

where $p, q = 2, 3, 4$; the remaining components of $\tilde{\Gamma}^{\lambda}_{\mu\nu}$ being the same as $\Gamma^{\lambda}_{\mu\nu}$. We then find that the non-zero components of $\tilde{\Gamma}^{\lambda}_{\mu\nu}$ are

$$\tilde{\Gamma}^1_{11} = -\left(\frac{2R}{r_0^2+R^2} + \frac{m}{R(R-2m)}\right), \quad \tilde{\Gamma}^1_{22} = \tilde{\Gamma}^1_{33} \operatorname{cosec}^2 \theta = -(R-2m),$$

$$\tilde{\Gamma}^1_{44} = \frac{m(R-2m)(r_0^2+R^2)^2}{r_0^4 R^3}, \quad \tilde{\Gamma}^2_{33} = -\sin\theta\cos\theta, \quad \tilde{\Gamma}^3_{23} = \cot\theta,$$

$$\tilde{\Gamma}^2_{12} = \Gamma^3_{13} = \frac{r_0^2}{R(r_0^2+R^2)}, \quad \tilde{\Gamma}^4_{14} = \frac{m}{R(R-2m)}, \quad \tilde{\Gamma}^1_{23} = \frac{R}{r_0}(R-2m)\sin\theta,$$

$$\tilde{\Gamma}^2_{31} = \tilde{\Gamma}^3_{12} \sin^2\theta = \frac{r_0}{r_0^2+R^2}\sin\theta, \tag{24}$$

where (Ref. [9]) $-cr_0 = 2m$. From the equation (17) we easily find that

$$\Gamma_4 = 0, \quad \Gamma_3 = \tfrac{3}{2}c\cos\theta, \quad \Gamma_2 = \Gamma_2(R, \theta), \quad \Gamma_1 = \Gamma_1(R, \theta), \tag{25}$$

the last two components of Γ_μ being arbitrary functions of R and θ only. We are now ready to calculate $\mathfrak{T}^{\lambda}_{\kappa}$ from equation (19), the components of $\mathfrak{g}^{\mu\nu}$ being given (in view of (22)) by

$$\mathfrak{g}^{11} = -R(R-2m)\sin\theta, \quad \mathfrak{g}^{22} = \mathfrak{g}^{33}\operatorname{cosec}^2\theta = -\frac{r_0^2(r_0^2-R^2)}{(r_0^2+R^2)^2}\sin\theta,$$

$$\mathfrak{g}^{44} = \frac{r_0^4 R^3 \sin\theta}{(r_0^2+R^2)^2(R-2m)}, \quad \mathfrak{g}^{23} = -\mathfrak{g}^{32} = -\frac{2r_0^3 R}{(r_0^2+R^2)^2}, \tag{26}$$

the remaining components vanishing. We now get

$$\mathfrak{T}^1_2 = \frac{2r_0^2(R-2m)(r_0^2-R^2)}{(r_0^2+R^2)^2}\cos\theta, \quad \mathfrak{T}^1_3 = -\frac{8r_0^3 R(R-2m)\sin\theta\cos\theta}{(r_0^2+R^2)^2},$$

$$\mathfrak{T}^2_3 = \frac{8r_0^3(r_0^2 R+3mR^2 = mr_0^2-R^3)}{(r_0^2+R^2)^3}\sin^2\theta, \quad \mathfrak{T}^2_2 = \frac{4r_0^2 R(2r_0^2 R+(r_0^2-R^2)(R-4m))}{(r_0^2+R^2)^3}\sin\theta,$$

$$\mathfrak{T}^3_1 = -\frac{8r_0^3 R}{3(r_0^2+R^2)^2}\frac{\partial\Gamma_1}{\partial\theta}, \quad \mathfrak{T}^3_2 = -\tfrac{2}{3}\operatorname{cosec}^2\theta - \frac{2r_0^3 R}{(r_0^2+R^2)^2}\left(\operatorname{cosec}^2\theta - \tfrac{4}{3}\frac{\partial\Gamma_2}{\partial\theta}\right),$$

$$\mathfrak{T}^3_3 = \frac{r_0^2(r_0^4+6r_0^2R^2-4mr_0^2R+12mR^3-3R^4)}{(r_0^2+R^2)^2}\sin\theta, \tag{27}$$

the remaining components being zero. Particularly interesting, and of course unfortunate is the vanishing of $\mathfrak{T}^4_4$ which one would normally like to interpret as the energy density. I shall comment on this result in more detail in the next section. In the present calculation we have

$$\mathfrak{T}^4_4 = (\mathfrak{g}^{11}\tilde{\Gamma}^4_{14} - \mathfrak{g}^{44}\tilde{\Gamma}^1_{44})_{,1} = -2(m\sin\theta)_{,1} = 0, \tag{28}$$

since m is a constant by hypothesis. In a way this result confirms the apparent "Schwarzschild" or perhaps, rather, "de Sitter" nature of the proposed model. It does not, however, necessarily mean that the universe must be empty, even though, by (27) it would have non-zero stresses present. It would be idle to speculate further on the latter until the problem of energy-density is resolved.

We may note, as might have been expected, that the generalisation of the theory ($q \neq -1$) does not save the situation. $\mathfrak{T}_4^4$ remains zero in the spherically symmetric case.

5. *Discussion*

It is now clear that unless one is willing to accept that the energy density in the universe vanishes there is something wrong. In fact one of three things could be wrong.

We have seen (Ref. [1]) that the definition of the metric $a_{\mu\nu} = a_{\nu\mu}$ by the differential equations

$$a_{\mu\nu,\lambda} - \Gamma^{\sigma}_{\underline{\mu}\lambda} a_{\sigma\nu} - \Gamma^{\sigma}_{\underline{\nu}\lambda} a_{\mu\sigma} = 0 \tag{29}$$

leads to the selection from among all possible, spherically symmetric solutions of the field equations (1) of either the Papapetrou solution (Ref. [10]) or of the solution which leads to the new cosmological model (Ref. [9]). The hypothesis (29) could be incorrect. However, it seems to fit well into the structure of the unified field theory and its advantage of restricting the number of solutions for which one has to find a physical meaning is so great that it should not be lightly discarded.

Secondly, the cosmological interpretation could be wrong. Even if it is though, a local field, presumably of a charged object, would still be energyless and this is difficult to understand. Such a field also would have a cut-off point (at $r = r_0$) for which no empirical evidence exists.

Finally of course, Einstein and Kaufman's identification of the energy-momentum tensor density $\mathfrak{T}_\kappa^\lambda$ need not be right. Einstein and Kaufman themselves observe (Ref. [3]) that $\mathfrak{T}_\kappa^\lambda$ is a tensor density only for linear transformations of coordinates. It is easily seen that this is so either from equations (16) or (19) or from its derivation obtained by subjecting

$$\delta H = \delta \int \mathfrak{g}^{\mu\nu} R_{\mu\nu}(U) d\Omega$$

to the infinitesimal, constant translations

$$x^\lambda \to x^\lambda + \alpha^\lambda, \qquad \alpha^\lambda = \text{constant.} \tag{30}$$

Then, and only then does

$$\delta H = (\mathfrak{g}^{\mu\nu} \delta U^\lambda_{\mu\nu})_{,\lambda} = 0,$$

and

$$\delta U^\lambda_{\mu\nu} = -U^\lambda_{\mu\nu,\kappa} \alpha^\kappa \tag{31}$$

(for the Einstein–Kaufman theory). For any other infinitesimal transformation, the term $(\mathfrak{g}^{\mu\nu} \delta U^\lambda_{\mu\nu})_{,\lambda}$ is integrated and put equal to zero by the condition that $\delta U^\lambda_{\mu\nu}$ should vanish on the 3-dimensional boundary of the space-time. Thus the vanishing or otherwise of $\mathfrak{T}_4^4$ becomes a coordinate dependent result.

If we reject interpretation of $\mathfrak{T}_\kappa^\lambda$ as an energy-momentum tensor (perhaps suitably symmetrised) we must conclude that the problem of matter remains unsolved in the unfied field theory. Papapetrou (Ref. [10]) discussed the problem in the case of weak fields in the first and second approximations (the "zeroth" approximation to $g_{\mu\nu}$ being the Minkowski $\eta_{\mu\nu} = \mathrm{diag}\,(-1, -1, -1, +1)$). This technique is inapplicable here since we are dealing with exact solutions of the field equations. Likewise, his results are inappropriate since they depend on regarding $\underline{g}_{\mu\nu}$ as the metric and, what is worse (Ref. [11]), $\check{g}_{\mu\nu}$ as the electromagnetic field tensor (although Papapetrou is aware that this may not be the case, as indeed it is not).

On the other hand, Papapetrou's investigation does suggest a possible way of solving the problem. The clue lies in the equation (29). We can in fact construct a non-vanishing, symmetric Ricci $R^s_{\mu\nu}$ tensor from $\tilde{\Gamma}^\lambda_{\underline{\mu\nu}}$ and define an energy-momentum tensor $T_{\mu\nu}$ by

$$R^s_{\mu\nu} = -\kappa(T_{\mu\nu} - \tfrac{1}{2}\, a_{\mu\nu} T). \tag{32}$$

We should note that such a tensor $T_{\mu\nu}$ will automatically satisfy the conservation equations

$$T^{\mu\nu}_{\circ\circ;\nu} = 0. \tag{33}$$

The non-vanishing (diagonal) components of the metric $a_{\mu\nu}$ are (Ref. [9])

$$a_{11} = -\frac{r_0^4}{(r_0^2+R^2)^2\left(1-\dfrac{2m}{R}\right)}, \qquad a_{22} = a_{33}\,\mathrm{cosec}^2\,\theta = -\frac{r_0^2 R^2}{r_0^2+R^2},$$

$$a_{44} = 1 - \frac{2m}{R}. \tag{34}$$

From equations (32) and (16) we find, using (34) that

$$T_{11} = -\frac{r_0^2}{\kappa(r_0^2+R^2)^2}, \qquad T_{22} = T_{33} = -\frac{R(R-2m)}{\kappa(r_0^2+R^2)},$$

and

$$T_{44} = \frac{3}{\kappa r_0^2}\left(1-\frac{2m}{R}\right)^2. \tag{35}$$

The only objection against postulating this $T_{\mu\nu}$ to be the energy-momentum tensor of the universe is that the conservation law (33) cannot be readily derived from the variational action principle (9) of the theory. (33) in fact is a consequence of equation (29) and of the assumed structure of the tensor $\tilde{R}^s_{\mu\nu} = \tilde{R}^s_{\nu\mu}$. On the other hand, it satisfies the phenomenological symmetry requirements of an energy-momentum tensor, automatically.

Let us assume then that it represents a tentative solution of the problem of matter in the non-symmetric unified field theory. Granting the above and pending an empirical (or, rather, and as pointed out previously — observational) confirmation of the theory, present investigation leaves the following outstanding problems. If the cosmological interpretation of the spherically symmetric solution is correct it would be both interesting and important to find whether there exist time dependent solutions. Their non-existence of course, would be an analogue of Birkhoff's theorem in the non-symmetric geometry. The second question is to discover axially symmetric solutions, especially those which would reduce to Kerr metric when skew-symmetry is removed. This problem is complicated by the difficulty of defining axial symmetry in the present case. We cannot appeal to Killing equations unless we know the form of the affine connection and this depends on the form of the fundamental tensor we want to find. The finite rotation method of Papapetrou seems to indicate that an axially symmetric skew-tensor $\check{g}_{\mu\nu}$ has, as in the spherical case, only the $\check{g}_{23}$ and $\check{g}_{14}$ components which can be now functions of r and θ.

Finally, and from the point of view of a possible empirical confirmation of the theory, most important is to investigate the correction, if any, to the exact Maxwell equations implied by the theory. Since we have now postulated that one of the electromagnetic field tensors, $f_{\mu\nu}$, is proportional to $\check{R}_{\mu\nu}$ (Ref. [12]), this amounts to finding the corresponding second field tensor, $h^{\mu\nu}$ say, whose divergence would be the electric current density. As previously mentioned, a possible way of tackling this problem may be through the methods of the Born–Infeld non-linear electrodynamics.

REFERENCES

[1] A. H. Klotz, *Acta Phys. Pol.* **B9**, 573 (1978).
[2] A. Einstein, E. G. Straus, *Ann. Math.* **47**, 731 (1946).
[3] A. Einstein, B. Kaufman, *Ann. Math.* **62**, 128 (1955).
[4] A. H. Klotz, G. K. Russell, *Acta Phys. Pol.* **B4**, 579 (1973).
[5] A. H. Klotz, G. K. Russell, *Acta Phys. Pol.* **B4**, 589 (1973).
[6] A. H. Klotz, *Acta Phys. Pol.* **B9**, 595 (1978).
[7] C. Radford, A. H. Klotz, (to be published).
[8] M.-A. Tonnelat, *J. Phys. Rad.* **16**, 21 (1955).
[9] A. H. Klotz, *Acta Phys. Pol.* **B10**, 000 (1979).
[10] A. Papapetrou, *Proc. R. Ir. Acad.* **52A**, 69 (1948).
[11] A. H. Klotz, G. K. Russell, *Acta Phys. Pol.* **B3**, 649 (1972).
[12] L. J. Gregory, A. H. Klotz, *Acta Phys. Pol.* **B8**, 601 (1977).