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NON-STATIC CHARGED FLUID SPHERE IN GENERAL RELATIVITY

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A class of exact interior solution for charged spherically symmetric distribution of inhomogeneous matter in an empty background is derived and investigated. It is shown that it is possible for the sphere to expand from a singular state to a maximum proper radius and then collapse again to a singular state. Because there is only one proper reversal in the motion, our model does not exhibit oscillatory motion.

1. Introduction

Non-static spherically symmetric fluid spheres consisting of perfect fluid has been discussed in the literature by several workers. Most of the work considered fluids of uniform density. Non-uniform models have also been discussed (Nariai 1967, 1968; Faulkes (1969). Vickers (1973), Banerjee (1975) and others have discussed the charged version of this problem. We give here a new class of exact interior solutions for the radial motion of a charged sphere of fluid with non-uniform density and pressure distributions. Our method consists in generalizing the work of Nariai (1968) and Faulkes (1969) to the charged case and studying in details a particular solvable model of the new differential equation we have derived. From the boundary conditions one can study the behaviour of the model at different instants of time; the only restriction being that the matter density ϱ and the pressure p satisfy the following conditions: $\varrho > 0$ in the region $0 \leqslant r \leqslant r_0$, $p > 0$ in the region $0 \leqslant r \leqslant r_0$ and $p = 0$ at $r = r_0$, so that the model is physically realistic.

2. Field equations and their solutions

We take the line element in the isotropic form

$$ds^2 = e^{\nu(r,t)}dt^2 - e^{\omega(r,t)}(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2). \tag{2.1}$$

Using comoving coordinates we obtain for the four velocity components

$$u^\mu = e^{-\nu/2}\delta_0^\mu, \tag{2.2}$$

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The field equations give the following relations (Nduka 1976, Faulkes 1969)

$$8\pi\varrho - \frac{\varepsilon^2}{s^4} = \tfrac{3}{4}\dot{\omega}^2 e^{-\nu} - e^{-\omega}\left(\omega'' + \frac{\omega'^2}{4} + \frac{2\omega'}{r}\right), \tag{2.3}$$

$$8\pi p + \frac{\varepsilon^2}{s^4} = e^{-\omega}\left[\frac{\omega'+\nu'}{r} + \frac{\omega'}{2}\left(\nu' + \frac{\omega'}{2}\right)\right] - e^{-\nu}\left(\ddot{\omega} + \frac{3\dot{\omega}^2}{4} - \frac{\dot{\omega}\dot{\nu}}{2}\right), \tag{2.4}$$

$$\Psi(r)e^{-\omega/2} - \frac{4\varepsilon^2}{r^4}e^{-\omega} = \omega'' - \frac{\omega'^2}{2} - \frac{\omega'}{r}, \tag{2.5}$$

$$e^{\nu(r,t)} = \dot{\omega}^2(r,t)A(t), \tag{2.6}$$

where $s = re^{\omega/2}$, ε and Ψ are arbitrary functions of r and A an arbitrary function of t. In these equations a prime denotes differentiation with respect to r and a dot denotes differentiation with respect to t.

An exact solution of the interior line element (2.1) may be found by choosing a particular form for the arbitrary functions that appear in equations (2.5) and (2.6). For example, following Faulkes (1969), we may normalize the time coordinate t so that

$$e^{\nu(0,t)} = 1. \tag{2.7}$$

This, according to equation (2.6), is equivalent to choosing $A(t) = 1/\dot{\omega}^2(0,t)$. Then we can make use of equation (2.5) to find the complete solution. In fact on putting

$$R(r,t) = e^{-\omega/2} \quad \text{and} \quad x = r^2, \tag{2.8}$$

we transform equation (2.5) into the partial differential equation

$$\frac{\partial^2 R}{\partial x^2} = \Gamma_1(x)R^2 + \Gamma_2(x)R^3, \tag{2.9}$$

where $\Gamma_1(x) = -\Psi(x^{\frac{1}{2}})/8x$ and $\Gamma_2(x) = \varepsilon^2(x^{\frac{1}{2}})/2x^3$.

It is clear that equation (2.9) together with equations (2.3) and (2.4) represent all solutions for non-static spherically symmetric fluid bodies consistent with the chosen line element (2.1). There is, of course, no reason to expect that all such solutions will be physically reasonable and have, for example, a positive ϱ and p distribution. Only a subclass of these solutions, corresponding to certain choices of $\Gamma_1(x)$ and $\Gamma_2(x)$, will be physically reasonable. A judicious choice of $\Gamma_1(x)$ and $\Gamma_2(x)$ is thus necessary to obtain physically interesting solutions.

3. *Specific analytic solution*

To obtain a solution of the partial differential equation (2.9) which is simple and physically reasonable we take $\Gamma_1(x) = \Gamma_0$, Γ_0 a constant and $\Gamma_2(x) = \sigma/k^4$, a constant also. Then the differential equation (2.9) takes the form

$$\frac{\partial^2 R}{\partial x^2} = \Gamma_0 R^2 + (\sigma/k^4)R^3, \quad 1/k^2 = 2(2\Gamma_0/3)^{1/2}. \tag{3.1}$$

In this paper we assume that the term $(\sigma/k^4)R^3$ is very small, so that it represents a small addition to the equation discussed by Faulkes (1969). Thus we shall solve equation (3.1) by a perturbation approach. We first note that for the uncharged case $\Gamma_2(x) = 0$, since the problem must in this case reduce to that of Faulkes. Thus the parameter σ characterizes the charge of the sphere.

With this choice of $\Gamma_1(x)$ and $\Gamma_2(x)$, the solution to equation (2.9), assuming that the term $(\sigma/k^4)R^3$ is small, is

$$R = R_0 + 2\sigma R_0^2, \tag{3.2}$$

where $R_0 = [E(t) - \mu^2]^{-2}$, with $\mu^2 = r^2/4k^2$, is the solution of equation (2.9) when $\sigma = 0$. Here E is an arbitrary function of time. For this solution the line element (2.1) ,on using equations (3.2), (2.8), (2.7) and (2.6) becomes

$$ds^2 = \frac{E^2}{(E-\mu^2)^2}\left[1 + \frac{4\sigma\mu^2(2E-\mu^2)}{E^2(E-\mu^2)^2}\right]$$
$$-(E-\mu^2)^4\left[1 - \frac{4\sigma}{(E-\mu^2)^2}\right](dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2). \tag{3.3}$$

Then using equations (3.3), (2.4), (2.3) and (2.1) we find the following expressions for the density and pressure

$$4\pi\varrho = \frac{3}{k^2(E-\mu^2)^5} + \frac{6\dot{E}^2}{E^2} + \frac{8\sigma\mu^2}{k^2(E-\mu^2)^8}, \tag{3.4}$$

$$4\pi p = -\frac{1}{2k^2(E-\mu^2)^5} - \frac{2\ddot{E}(E-\mu^2)}{E^2} - \frac{2\dot{E}^2(2E+\mu^2)}{E^3} - \frac{2\sigma(E+3\mu^2)}{k^2(E-\mu^2)^8}. \tag{3.5}$$

The exterior metric is to be taken as the Reissner–Nordström Solution:

$$ds^2 = \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)dt^2 - \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{3.6}$$

where m represents the gravitational mass and e the electric charge of the body. In our particular treatment e is considered to be small.

The metric (3.3) is matched to the Reissner–Nordström metric (3.6) across the moving boundary provided that (Cocke 1966)

$$p(r_0, t) = 0, \quad 2m - \frac{e^2}{r_0 e^{\omega_0/2}} = \tfrac{1}{4}\dot{\omega}_0^2 r_0^3 e^{(3\omega_0/2 - \nu_0)} - \tfrac{1}{4}\omega_0'^2 r_0^3 e^{\omega_0/2} - r_0^2 \omega_0' e^{\omega_0/2}, \tag{3.7}$$

where the subscript "0" indicates the values at the boundary $r = r_0$. These two conditions (3.7) respectively yield

$$\frac{1}{4k^2(E-\mu_0^2)^5} + \frac{\ddot{E}(E-\mu_0^2)}{E^2} + \frac{\dot{E}^2(2E+\mu_0^2)}{E^3} + \frac{\sigma(E+3\mu_0^2)}{k^2(E-\mu_0^2)^8} = 0, \tag{3.8}$$

$$\frac{m}{r_0^3} - \frac{e^2}{2r_0^4(E-\mu_0^2)^2} = \frac{2\dot{E}^2(E-\mu_0^2)^6}{E^2} + \frac{(E-3\mu_0^2)}{k^2} - \frac{4\sigma\mu_0^2}{k^2(E-\mu_0^2)^2}. \tag{3.9}$$

Equations (3.8) and (3.9) may now be used to calculate $(\dot{E}/E)$ and $(\ddot{E}/E^2)$; and then from equations (3.4) and (3.5) we obtain the following expressions for ϱ and p)

$$\frac{4\pi\varrho}{3} = \frac{1}{k^2(E-\mu^2)^5} + (E-\mu_0^2)^{-6}\left\{\frac{m}{r_0^3} - \frac{(E-3\mu_0^2)}{k^2} - \frac{e^2(E-\mu_0^2)^{-2}}{2r_0^4}\right\}$$

$$+\frac{4\sigma}{3k^2}\frac{2\mu^2(E-\mu_0^2)^8+3\mu_0^2(E-\mu^2)^8}{(E-\mu^2)^8(E-\mu_0^2)^8}, \tag{3.10}$$

$$\frac{4\pi p}{3} = \frac{(E-\mu^2)^6-(E-\mu_0^2)^6}{6k^2(E-\mu_0^2)^6(E-\mu^2)^5} + \frac{(\mu_0^2-\mu^2)}{(E-\mu_0^2)^7}\left\{\frac{m}{r_0^3} - \frac{(E-3\mu_0^2)}{k^2} - \frac{e^2(E-\mu_0^2)^{-2}}{2r_0^4}\right\}$$

$$+\sigma\frac{(E+3\mu_0^2)(E-\mu^2)^9-(E+3\mu^2)(E-\mu_0^2)^9}{k^2(E-\mu_0^2)^9(E-\mu^2)^8}. \tag{3.11}$$

4. *Properties of the solution*

We see from equation (3.3) that the line element is singular when $E = \mu^2$, and from equation (3.11) that the pressure is negative when $E < \mu^2$. The solution (3.3) may be considered as describing a sphere in the region $0 \leqslant r \leqslant r_0$ having a proper radius $\xi(r, t)$ with

$$\xi(r,t) = r(E-\mu^2)^2\left[1-\frac{2\sigma}{(E-\mu^2)^2}\right]. \tag{4.1}$$

In order for both density and pressure to be positive in the region $0 \leqslant r \leqslant r_0$ we require that

$$E > \mu_0^2, \tag{4.2}$$

and

$$\frac{m}{r_0^3} \geqslant \frac{(E-3\mu_0^2)}{k^2} + \frac{e^2(E-\mu_0^2)^{-2}}{2r_0^4}. \tag{4.3}$$

It then follows that if equations (4.2) and (4.3) are satisfied, we have the following inequalities for the density and pressure:

$$\varrho(r,t) > 0,\ 0 \leqslant r \leqslant r_0, \quad p(r,t) > 0,\ 0 \leqslant r \leqslant r_0, \quad \text{and} \quad p(r_0,t) = 0. \tag{4.4}$$

Equations (4.2) and (4.3) give the complete range of E in terms of e, μ_0, m and r_0:

$$\mu_0^2 < E \leqslant \frac{m/r_0}{4\mu_0^2} + 3\mu_0^2 - \frac{e^2/8\mu_0^2 r_0^2}{\left(\frac{m/r_0}{4\mu_0^2}+2\mu_0^2\right)^2} \tag{4.5}$$

and equation (4.5), on using equation (4.1), gives the range of possible values of the radius $\xi_0 = \xi(r_0, t)$ of the boundary of the sphere, namely:

$$0 < \xi_0 \leqslant r_0\left(\frac{m/r_0}{4\mu_0^2} + 2\mu_0^2\right)^2 \left\{1 - \frac{e^2/4\mu_0^2 r_0^2}{\left(\dfrac{m/r_0}{4\mu_0^2} + 2\mu_0^2\right)^3}\right\}. \tag{4.6}$$

The range (4.6) shows that it is possible for the sphere to expand from a singular state to a maximum proper radius ξ_{max}, where

$$\xi_{max} = r_0\left(\frac{m/r_0}{4\mu_0^2} + 2\mu_0^2\right)^2 \left\{1 - \frac{e^2/4\mu_0^2 r_0^2}{\left(\dfrac{m/r_0}{4\mu_0^2} + 2\mu_0^2\right)^3}\right\} \tag{4.7}$$

and then collapse again to a singular state. Since there is only one proper reversal in the motion ($\dot{\xi}_0 = 0$), namely when $\xi_0 = \xi_{max}$, there can be no oscillatory motion for this solution.

For the uncharged sphere we must have $\sigma = e \equiv 0$. If in our equations we put $\sigma = e \equiv 0$, the results coincide with those already reported by Faulkes (1969). Thus our equations may be considered as the generalizations of those obtained by Faulkes.

REFERENCES

Barnejee, A., De, U. K., *Acta Phys. Pol.* **B6**, 335 (1975).
Cocke, W. J., *J. Math. Phys.* **7**, 1171 (1966).
Faulkes, M. C., *Prog. Theor. Phys.* **42**, 1139 (1969).
Nariai, H., *Prog. Theor. Phys.* **40**, 1013 (1968).
Nduka, A., *Gen. Relativ. Gravitation* **7**, 493 (1976).
Vickers, P. A., *Ann. Inst. H. Poincaré* **18**, 137 (1973).