GENERALIZED GOLDBERG-SACHS THEOREMS IN COMPLEX AND REAL SPACE-TIMES. I

ALGEBRAIC CLASSIFICATIONS OF THE CONFORMAL TENSOR AND THE ENERGY-MOMENTUM TENSOR ON COMPLEX AND REAL SPACE-TIMES

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(*Received August 18, 1978*)

The algebraic classification of the conformal curvature tensor and the energy-momentum tensor on complex space-times are given. These classifications are natural generalizations of the ones well-known in the case of real space times.

1. *Introduction*

The present paper is the first part of the work devoted to generalized Goldberg-Sachs theorems in complex and real space-times. The idea of the complex space-time has attracted much attention in recent years. Complex space-times have appeared as a spaces of "good cones" and then as spaces of asymptotic twistors in the study of asymptotically flat real space-times [1-3]. These spaces, called "heavens", were studied in a series of papers [1-10] (see also [33]). Some other considerations concerning the complex space-times were initiated within our group [11] and then in [12]. It was found that if the conformal curvature tensor of the complex space-time is algebraically degenerated, from (at least) one side, then complex Einstein equations in vacuum may be reduced to a partial differential equation of second order with quadratic nonlinearity only. This result was generalized for the case of nonzero cosmological constant [13] and for the case of complex Maxwell-Einstein

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equations with cosmological constant [14, 31]. In the latter case the Maxwell–Einstein equations are reduced to a pair of equations for two unknown functions.

Since real (physical) space-time may be considered as real cross section [15, 11, 32] of some complex space-time, one hopes that the examination of complex space-times may be very useful in the search for new real (physical) space-times.

The assumption that in a given complex space-time there exists the congruence of "null strings" plays the basic role in the reduction of the complex Einstein and Maxwell–Einstein equations. The null string is a 2-surface which has the tangent space at each point spanned by a pair of mutually orthogonal null vectors. The existence of the null string congruence in the empty complex space-time with the one-side (at least) algebraically degenerated conformal curvature tensor is assured by the Golberg–Sachs theorem [5]. Therefore it is very interesting to generalize the "vacuum" Goldberg–Sachs theorem on the "non-vacuum" case. From such generalized Goldberg–Sachs theorems, one will be able to obtain (by taking real cross sections) the generalized Goldberg–Sachs theorems in the real (physical) space-times with matter.

We hope, these theorems will play a distinguished role in the search for algebraically degenerated real solutions (see [16, 17]) and understanding propagation of gravitational radiation in matter (see [18]). But in order to obtain the generalization of the Goldberg–Sachs theorem in the complex space-time one has to study the algebraic structures of the conformal curvature tensor and the energy-momentum tensor on complex space-time. The aim of this paper is to examine these problems.

In Section 2 we consider some possible approaches to the problem of algebraic classification of the conformal curvature tensor of the complex (oriented) space-time. As it is pointed out, these approaches are equivalent. In Section 3 the algebraic classification of the "energy-momentum" tensor on complex (oriented) space-times is given, which is the natural generalization of the algebraic classification of the energy-momentum tensor for real space-times introduced by one of us (Plebański [19]). The canonical forms of the energy-momentum tensor of the definite type are written. As an example we consider the electromagnetic field (linear and non-linear) in the complex space-time. The notation and technique are adopted from [4, 25].

2. *Algebraic structure of the conformal curvature tensor on complex space-time*

The complex space-time V_4^c is a pair (M_4^c, ds^2), where M_4^c is a four-dimensional complex analytic differential manifold and ds^2 is an analytic metric, that is if $\{z^\alpha\}$ is some local map, then

$$ds^2 = g_{\mu\nu} dz^\mu \otimes dz^\nu, \quad g_{\mu\nu} = g_{(\mu\nu)}, \quad \partial g_{\mu\nu}/\partial \overline{z^\alpha} = 0. \tag{2.1}$$

(In our paper we consider only the holomorphic tensor fields [20].) Let (E^1, E^2, E^3, E^4) be four 1-forms (linearly independent) defined on M_4^c and let the metric be given as follows

$$ds^2 = g_{ab} E^a \otimes E^b, \tag{2.2}$$

where

$$(g_{ab}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{2.3}$$

Every such four 1-forms we call "the null tetrad". If (E_1, E_2, E_3, E_4) are four vector fields dual to 1-forms E^1, E^2, E^3, E^4 respectively, that is

$$E^a E_b = \delta^a{}_b, \qquad a, b = 1, 2, 3, 4, \tag{2.4}$$

then (E_1, E_2, E_3, E_4) will also be referred to as the null tetrad. Let (e^1, e^2, e^3, e^4), (E^1, E^2, E^3, E^4) be two null tetrads. Then

$$E^a = T^a{}_b e^b, \qquad \det(T^a{}_b) \neq 0. \tag{2.5}$$

Furthermore, it is easy to see that

$$\det(T^a{}_b) = \pm 1. \tag{2.6}$$

If $\det(T^a{}_b) = +1$ then we say that (e^1, e^2, e^3, e^4) defines the same orientation on V_4^c as (E^1, E^2, E^3, E^4) does; if $\det(T^a{}_b) = -1$ then we say that (e^1, e^2, e^3, e^4) and (E^1, E^2, E^3, E^4) define the opposite orientations on V_4^c. An equivalence class of null tetrads defining the same orientation on V_4^c as (e^1, e^2, e^3, e^4) does, we denote by $[(e^1, e^2, e^3, e^4)]$. A pair $(V_4^c, [(e^1, e^2, e^3, e^4)].)$ we call "an oriented complex space-time".

Now we can introduce spinors into the complex space-time as follows. Let

$$(B_{s\bar{s}}, \pi, M_4^c, G) \tag{2.7}$$

be the trivial principal fibre bundle over M_4^c, with the projection map π: $B_{s\bar{s}} \to M_4^c$ and the structure group $G := SL(2, \mathscr{C}) \times SL(2, \mathscr{C})$. Then trivially we have $B_{s\bar{s}} \cong M_4^c \times SL(2, \mathscr{C}) \times SL(2, \mathscr{C})$. The fibre bundle so defined we call "the bundle of pairs of spinor bases".

Now, let the group $G := SL(2, \mathscr{C}) \times SL(2, \mathscr{C})$ acts on $\mathscr{C}^2$ to the left as follows

$$((l^{A'}{}_B), (\bar{l}^{\dot{A}'}{}_{\dot{B}})(q)) = (\bar{l}^{\dot{A}'}{}_{\dot{B}}) \cdot (q),$$

where

$$((l^{A'}{}_B), (\bar{l}^{\dot{A}'}{}_{\dot{B}})) \in SL(2, \mathscr{C}) \times SL(2, \mathscr{C}), \qquad A', \dot{A}', B, \dot{B} = 1, 2 \quad \text{and} \quad (q) := \begin{pmatrix} q^1 \\ q^2 \end{pmatrix} \in \mathscr{C}^2.$$

Then we can define, by a well known procedure [20, 21], an associated fibre bundle with the bundle (2.7), which we call "a bundle of undotted contravariant spinors of the first rank".

Likewise, if the group G acts on $\mathscr{C}^2$ to the left as follows

$$((l^{A'}{}_B), (l^{\dot{A}'}{}_{\dot{B}}))(q) = (\bar{l}^{\dot{A}'}{}_{\dot{B}}) \cdot (q)$$

then we obtain an associated fibre bundle with the bundle (2.7), which we call "a bundle of dotted contravariant spinors of the first rank". Now it is evident how one can obtain bundles of spinors of any kind. Spinor fields on V_4^c are sections of these bundles. We denote undotted spinors by k^A, l_A, $m^{AB}{}_C$ etc.; dotted spinors by $\bar{k}^{\dot{A}}$, $\bar{l}_{\dot{A}}$, $\bar{m}^{\dot{A}\dot{B}}{}_{\dot{C}}$ etc. Let (e^1, e^2, e^3, e^4) be some fixed null tetrad on the oriented complex space time $(V_4^c, [(e^1, e^2, e^3, e^4)])$. We define a spinor 1-form, $g^{A\dot{B}}$, which we represent symbolically by the matrix

$$(g^{A\dot{B}}) := \sqrt{2}\begin{pmatrix} e^4 & e^2 \\ e^1 & -e^3 \end{pmatrix}. \tag{2.8}$$

So we have

$$g^{A\dot{B}} = g_a{}^{A\dot{B}} e^a, \tag{2.9}$$

where

$$(g_1{}^{A\dot{B}}) = \sqrt{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad (g_2{}^{A\dot{B}}) = \sqrt{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad (g_3{}^{A\dot{B}}) = \sqrt{2}\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},$$

$$(g_4{}^{A\dot{B}}) = \sqrt{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.10}$$

It is easy to verify that

$$ds^2 = -\tfrac{1}{2}\, \varepsilon_{AB}\bar{\varepsilon}_{\dot{C}\dot{D}} g^{A\dot{C}} \otimes g^{B\dot{D}}, \tag{2.11}$$

where

$$(\varepsilon_{AB}) := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} =: (\bar{\varepsilon}_{\dot{A}\dot{B}}).$$

(We will also consider in the present paper the spinors ε^{AB}, $\bar{\varepsilon}^{\dot{A}\dot{B}}$,

$$(\varepsilon^{AB}) := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} =: (\bar{\varepsilon}^{\dot{A}\dot{B}}).$$

The raising and lowering of spinor indices is defined as follows

$$k_A = \varepsilon_{AB} k^B, \quad \bar{k}_{\dot{A}} = \bar{\varepsilon}_{\dot{A}\dot{B}} k^{\dot{B}}, \quad k^A = \varepsilon^{BA} k_B, \quad \bar{k}^{\dot{A}} = \bar{\varepsilon}^{\dot{B}\dot{A}} \bar{k}_{\dot{B}} \text{ etc.)}$$

Consider two pairs of normalized spinors: (k^A, l^A), $(\bar{k}^A, \bar{l}^A)$; $k^A l_A = \bar{k}^{\dot{A}} \bar{l}_{\dot{A}} = 0$. They can be used to generate a null tetrad (E^1, E^2, E^3, E^4):

$$E^1 := \frac{1}{\sqrt{2}} g^{A\dot{B}} k_A \bar{l}_{\dot{B}}, \quad E^2 := \frac{1}{\sqrt{2}} g^{A\dot{B}} l_A \bar{k}_{\dot{B}}, \quad E^3 := -\frac{1}{\sqrt{2}} g^{A\dot{B}} k_A \bar{k}_{\dot{B}},$$

$$E^4 := \frac{1}{\sqrt{2}} g^{A\dot{B}} l_A \bar{l}_{\dot{B}}. \tag{2.12}$$

One easily finds that $(E^1, E^2, E^3, E^4) \in [(e^1, e^2, e^3, e^4)]$ in Eq. (2.12) is a concrete realization of homomorphism $SL(2, \mathscr{C}) \times SL(2, \mathscr{C})$ $(2 \leftrightarrow 1)$ $SO(3, 1; \mathscr{C})$. Of course

$SO(3, 1; \mathscr{C})(1 \leftrightarrow 1)SO(4, \mathscr{C})$. It is easy to see that (e^1, e^2, e^3, e^4) is generated by: $k^A = (\mp i, 0)$, $l^A = (0, \pm i)$, $\bar{k}^{\dot{A}} = (\pm i, 0)$, $\bar{l}^{\dot{A}} = (0, \mp i)$. An exterior product of the forms $g^{A\dot{B}}$ determines very important spinor 2-forms, S^{AB} and $\bar{S}^{\dot{A}\dot{B}}$:

$$g^{A\dot{B}} \wedge g^{C\dot{D}} =: S^{AC}\bar{\varepsilon}^{\dot{B}\dot{D}} + \bar{S}^{\dot{B}\dot{D}}\varepsilon^{AC}. \tag{2.13}$$

From (2.13)

$$S^{11} = 2e^4 \wedge e^2, \quad S^{12} = S^{21} = e^1 \wedge e^2 + e^3 \wedge e^4, \quad S^{22} = 2e^3 \wedge e^1, \tag{2.14a}$$

$$\bar{S}^{\dot{1}\dot{1}} = 2e^4 \wedge e^1, \quad \bar{S}^{\dot{1}\dot{2}} = \bar{S}^{\dot{2}\dot{1}} = -e^1 \wedge e^2 + e^3 \wedge e^4, \quad \bar{S}^{\dot{2}\dot{2}} = 2e^3 \wedge e^2. \tag{2.14b}$$

The forms S^{AB} and $\bar{S}^{\dot{A}\dot{B}}$ are respectively self-dual and anti-self-dual under Hodge's star operation:

$$*S^{AB} = S^{AB}, \quad *\bar{S}^{\dot{A}\dot{B}} = -\bar{S}^{\dot{A}\dot{B}}. \tag{2.15}$$

(The duality star operation, $*$, is defined as follows. Let $\omega^{A\ldots B\dot{A}\ldots\dot{B}a\ldots b}{}_{C\ldots D\dot{C}\ldots\dot{D}c\ldots d}$ be some spinor-tensor p-form

$$\omega = \frac{1}{p!}\,\omega_{a_1\ldots a_p} e^{a_1} \wedge \ldots \wedge e^{a_p},$$

where for simplicity we have omitted indices $A\ldots B\; C\ldots D$ etc. Then $*\omega$ is a spinor-tensor $(4-p)$-form, defined by

$$*\omega := \frac{1}{p!(4-p)!}\exp\frac{i\pi}{2}\left[p(4-p)-2\right]\cdot \varepsilon^{a_1\ldots a_p}{}_{b_1\ldots b_{4-p}}\omega_{a_1\ldots a_p} e^{b_1} \wedge \ldots \wedge e^{b_{4-p}},$$

where $\varepsilon^{a_1\ldots a_p}{}_{b_1\ldots b_{4-p}} := g^{a_1c_1}\ldots g^{a_pc_p}\varepsilon_{c_1\ldots c_p b_1\ldots b_{4-p}}$ and $\varepsilon_{c_1\ldots c_p b_1\ldots b_{4-p}}$ is the Levi-Civita symbol. From the above definition $*(*\omega) = \omega$.)

There is no place here for developing the theory of spinors on the complex space--time but it is obvious that we can introduce connections on the spinor bundles and then the curvature forms (for details see [4]). Now, let $C^a{}_{bcd}$ be the conformal curvature tensor of $(V_4^c, [(e^1, e^2, e^3, e^4)])$, defined by

$$C^{ab}{}_{cd} := R^{ab}{}_{cd} - \tfrac{1}{2}\,\delta^{abe}_{cdf}C^f{}_e + \frac{R}{12}\,\delta^{ab}_{cd}, \tag{2.16}$$

where $R^a{}_{bcd}$ — the curvature tensor, $R := R^{ab}{}_{ab}$ — the scalar curvature, $C_{ab} := R_{ab} - \frac{1}{4}Rg_{ab}$ — the Ricci tensor with extracted trace, $R_{ab} := R^c{}_{abc}$ — the Ricci tensor. $C^a{}_{bcd}$ is represented by two symmetric spinors C_{ABCD} and $\bar{C}_{\dot{A}\dot{B}\dot{C}\dot{D}}$:

$$C_{ABCD} := \tfrac{1}{16}\, S^{ab}{}_{AB}C_{abcd}S^{cd}{}_{CD}, \tag{2.17a}$$

$$\bar{C}_{\dot{A}\dot{B}\dot{C}\dot{D}} := \tfrac{1}{16}\, \bar{S}^{ab}{}_{\dot{A}\dot{B}}C_{abcd}\bar{S}^{cd}{}_{\dot{C}\dot{D}}, \tag{2.17b}$$

where $S^{ab}{}_{AB}$ and $\bar{S}^{ab}{}_{\dot{A}\dot{B}}$ are defined by formulae

$$S^{AB} =: \tfrac{1}{2} S_{ab}{}^{AB} e^a \wedge e^b, \tag{2.18a}$$

$$\bar{S}^{\dot{A}\dot{B}} =: \tfrac{1}{2} \bar{S}_{ab}{}^{\dot{A}\dot{B}} e^a \wedge e^b. \tag{2.18b}$$

Due to the fundamental theorem of algebra, C_{ABCD} and $\bar{C}_{\dot{A}\dot{B}\dot{C}\dot{D}}$ can be written in the forms

$$C_{ABCD} = \alpha_{(A^{\beta}B^{\gamma}C^{\delta}D)}, \tag{2.19a}$$

$$\bar{C}_{\dot{A}\dot{B}\dot{C}\dot{D}} = \bar{\alpha}_{(\dot{A}^{\bar{\beta}}\dot{B}^{\bar{\gamma}}\dot{C}^{\bar{\delta}}\dot{D})}. \tag{2.19b}$$

Therefore algebraic types of the conformal curvature tensor of the complex oriented space--time can be introduced as elements of the Cartesian product of two Penrose diagrams [4].

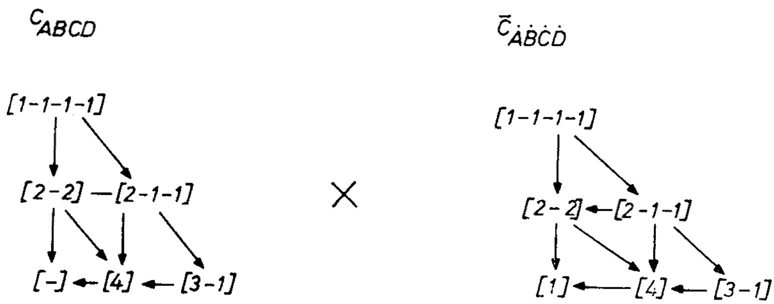


Fig. 2.1

The type ([A], [B]), where [A] corresponds to C_{ABCD} and [B] corresponds to $\bar{C}_{\dot{A}\dot{B}\dot{C}\dot{D}}$, we denote by $[A] \times [B]$. Let (k^A, l^A), $(\bar{k}^{\dot{A}}, \bar{l}^{\dot{A}})$ generate the null tetrad (E^1, E^2, E^3, E^4) according to (2.12). The conformal curvature tensor in the basis (E^1, E^2, E^3, E^4) is determined by the quantities

$$\begin{aligned}
C^{(5)} &:= 2C_{ABCD}k^A k^B k^C k^D, & \bar{C}^{(5)} &:= 2\bar{C}_{\dot{A}\dot{B}\dot{C}\dot{D}}\bar{k}^{\dot{A}}\bar{k}^{\dot{B}}\bar{k}^{\dot{C}}\bar{k}^{\dot{D}},\\
C^{(4)} &:= -2C_{ABCD}k^A k^B k^C l^D, & \bar{C}^{(4)} &:= -2\bar{C}_{\dot{A}\dot{B}\dot{C}\dot{D}}\bar{k}^{\dot{A}}\bar{k}^{\dot{B}}\bar{k}^{\dot{C}}\bar{l}^{\dot{D}},\\
C^{(3)} &:= 2C_{ABCD}k^A k^B l^C l^D, & \bar{C}^{(3)} &:= 2\bar{C}_{\dot{A}\dot{B}\dot{C}\dot{D}}\bar{k}^{\dot{A}}\bar{k}^{\dot{B}}\bar{l}^{\dot{C}}\bar{l}^{\dot{D}},\\
C^{(2)} &:= -2C_{ABCD}k^A l^B l^C l^D, & \bar{C}^{(2)} &:= -2\bar{C}_{\dot{A}\dot{B}\dot{C}\dot{D}}\bar{k}^{\dot{A}}\bar{l}^{\dot{B}}\bar{l}^{\dot{C}}\bar{l}^{\dot{D}},\\
C^{(1)} &:= 2C_{ABCD}l^A l^B l^C l^D, & \bar{C}^{(1)} &:= 2\bar{C}_{\dot{A}\dot{B}\dot{C}\dot{D}}\bar{l}^{\dot{A}}\bar{l}^{\dot{B}}\bar{l}^{\dot{C}}\bar{l}^{\dot{D}}.
\end{aligned} \tag{2.20}$$

The above quantities can be used to characterize types of the conformal curvature tensor. Schematically this characteristic is represented by the Cartesian product of two tables (notice that if the type of the conformal cutvature tensor is $[A] \times [B]$ then we say that the type of C_{ABCD} is [A] and the type of $\bar{C}_{\dot{A}\dot{B}\dot{C}\dot{D}}$ is [B]).

The spinor k^A for which $C^{(5)} = 0$ we call P-spinor; the spinor $\bar{k}^{\dot{A}}$ for which $\bar{C}^{(5)} = 0$ we call $\bar{P}$-spinor. If $C^{(5)} = C^{(4)} = 0$ then we say, that k^A is a multiple P-spinor; analogously if $\bar{C}^{(5)} = \bar{C}^{(4)} = 0$ then we say, that $\bar{k}^{\dot{A}}$ is a multiple $\bar{P}$-spinor.

TABLE 2.1

The type of C_{ABCD} is	if and only if	The type of $\bar{C}_{\dot{A}\dot{B}\dot{C}\dot{D}}$ is	if and only if
[1-1-1-1]	For each pair (k^A, l^A), $\lvert C^{(5)}\rvert + \lvert C^{(4)}\rvert \neq 0$.	[1-1-1-1]	For each pair $(\bar{k}^{\dot{A}}, \bar{l}^{\dot{A}})$, $\lvert \bar{C}^{(5)}\rvert + \lvert \bar{C}^{(4)}\rvert \neq 0$.
[2-1-1]	There exists a pair (k^A, l^A), that $C^{(5)} = C^{(4)} = 0$. For each so pair, is $C^{(3)} \neq 0$, $\lvert C^{(4)}\rvert + \lvert C^{(2)}\rvert \neq 0$.	[2-1-1]	There exists a pair $(\bar{k}^{\dot{A}}, \bar{l}^{\dot{A}})$, that $\bar{C}^{(5)} = \bar{C}^{(4)} = 0$. For each so pair, is $\bar{C}^{(3)} \neq 0$, $\lvert \bar{C}^{(1)}\rvert + \lvert \bar{C}^{(2)}\rvert \neq 0$.
[2-2]	There exists a pair (k^A, l^A), that $C^{(5)} = C^{(4)} = C^{(2)} = C^{(1)} = 0$, $C^{(3)} \neq 0$.	[2-2]	There exists a pair $(\bar{k}^{\dot{A}}, \bar{l}^{\dot{A}})$, that $\bar{C}^{(5)} = \bar{C}^{(4)} = \bar{C}^{(2)} = \bar{C}^{(1)} = 0$, $\bar{C}^{(3)} \neq 0$.
[3-1]	There exists a pair (k^A, l^A), that $C^{(5)} = C^{(4)} = C^{(3)} = 0$, $C^{(2)} \neq 0$.	[3-1]	There exists a pair $(\bar{k}^{\dot{A}}, \bar{l}^{\dot{A}})$, that $\bar{C}^{(5)} = \bar{C}^{(4)} = \bar{C}^{(3)} = 0$, $\bar{C}^{(2)} \neq 0$.
[4]	There exists a pair (k^A, l^A), that $C^{(5)} = C^{(4)} = C^{(3)} = C^{(2)} = 0$, $C^{(1)} \neq 0$.	[4]	There exists a pair $(\bar{k}^{\dot{A}}, \bar{l}^{\dot{A}})$, that $\bar{C}^{(5)} = \bar{C}^{(4)} = \bar{C}^{(3)} = \bar{C}^{(2)} = 0$, $\bar{C}^{(1)} \neq 0$.
[—]	There exists a pair (k^A, l^A), that $C^{(5)} = C^{(4)} = C^{(3)} = C^{(2)} = C^{(1)} = 0$.	[—]	There exists a pair $(\bar{k}^{\dot{A}}, \bar{l}^{\dot{A}})$, that $\bar{C}^{(5)} = \bar{C}^{(4)} = \bar{C}^{(3)} = \bar{C}^{(2)} = \bar{C}^{(1)} = 0$.

An algebraic classification of the conformal curvature tensor can be done by examining the possible algebraic structures of the following linear mappings ([25, 19] for real space-time):

$$C^{AB}{}_{CD}\Phi^{CD} = \Psi^{AB}, \tag{2.21a}$$

$$\bar{C}^{\dot{A}\dot{B}}{}_{\dot{C}\dot{D}}\bar{\Phi}^{\dot{C}\dot{D}} = \bar{\Psi}^{\dot{A}\dot{B}}, \tag{2.21b}$$

where $\Phi^{CD} = \Phi^{(CD)}$, $\bar{\Phi}^{\dot{C}\dot{D}} = \bar{\Phi}^{(\dot{C}\dot{D})}$. By using (2.20) we can easily see that, in a given pair of spinor bases the mappings (2.21a) (2.21b) are determined by matrices (notice, that the space of undotted or dotted symmetric contravariant spinors of the second rank is three-dimensional):

$$C := \begin{Bmatrix} \frac{1}{2}C^{(3)} & C^{(2)} & \frac{1}{2}C^{(1)} \\ -\frac{1}{2}C^{(4)} & -C^{(3)} & -\frac{1}{2}C^{(2)} \\ \frac{1}{2}C^{(5)} & C^{(4)} & \frac{1}{2}C^{(3)} \end{Bmatrix}, \tag{2.22a}$$

and

$$\bar{C} := \begin{Bmatrix} \frac{1}{2}\bar{C}^{(3)} & \bar{C}^{(2)} & \frac{1}{2}\bar{C}^{(1)} \\ -\frac{1}{2}\bar{C}^{(4)} & -\bar{C}^{(3)} & -\frac{1}{2}\bar{C}^{(2)} \\ \frac{1}{2}\bar{C}^{(5)} & \bar{C}^{(4)} & \frac{1}{2}\bar{C}^{(3)} \end{Bmatrix}, \tag{2.22b}$$

respectively. Now from the linear algebra [22, 23, 28] it follows, that the algebraic structures of the mappings (2.21a), (2.21b) are determined by characteristics of λ-matrices:

$$C-\lambda I = \begin{Bmatrix} \frac{1}{2}C^{(3)}-\lambda & C^{(2)} & \frac{1}{2}C^{(1)} \\ -\frac{1}{2}C^{(4)} & -C^{(3)}-\lambda & -\frac{1}{2}C^{(2)} \\ \frac{1}{2}C^{(5)} & C^{(4)} & \frac{1}{2}C^{(3)}-\lambda \end{Bmatrix}, \tag{2.23a}$$

$$\bar{C}-\lambda I = \begin{Bmatrix} \frac{1}{2}\bar{C}^{(3)}-\lambda & \bar{C}^{(2)} & \frac{1}{2}\bar{C}^{(1)} \\ -\frac{1}{2}\bar{C}^{(4)} & -\bar{C}^{(3)}-\lambda & -\frac{1}{2}\bar{C}^{(2)} \\ \frac{1}{2}\bar{C}^{(5)} & \bar{C}^{(4)} & \frac{1}{2}\bar{C}^{(3)}-\lambda \end{Bmatrix}, \tag{2.23b}$$

here

$$I := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The possible characteristics of the matrix $C-\lambda I$ are: (1, 1, 1), ((1, 1), 1), ((1, 1, 1)), (2, 1), ((2, 1)), (3) (and analogously for the matrix $\bar{C}-\lambda I$). Now using the results of the Table 2.1 one can easily find (writing (2.23a) and (2.23b) in the suitable bases) that there is the following one to one correspondence (for details see [24]):

The type of C_{ABCD}		The characteristic of $C-\lambda I$
[1−1−1−1]	$\leftrightarrow$	(1, 1, 1)
[2−2]	$\leftrightarrow$	((1, 1), 1)
[−]	$\leftrightarrow$	((1, 1, 1))
[2−1−1]	$\leftrightarrow$	(2,1)
[4]	$\leftrightarrow$	((2, 1))
[3−1]	$\leftrightarrow$	(3)

and analogously for $\bar{C}_{ABCD}$ and $\bar{C}-\lambda I$. So one finds, that the study of the algebraic structures of the linear mappings (2.21a), (2.21b) leads to the same types of the conformal curvature tensor as the Penrose classification does. Eigenvalues of the mappings (2.21a) and (2.21b) are the solutions of equations

$$\det(C-\lambda I) = 0, \tag{2.24a}$$

and

$$\det(\bar{C}-\lambda I) = 0, \tag{2.24b}$$

respectively. (2.24a) and (2.24b) respectively lead to the equations

$$\lambda^3-\tfrac{1}{2}\overset{2}{C}\lambda-\tfrac{1}{3}\overset{3}{C} = 0, \tag{2.25a}$$

$$\lambda^3-\tfrac{1}{2}\overset{2}{\bar{C}}\lambda-\tfrac{1}{3}\overset{3}{\bar{C}} = 0, \tag{2.25b}$$

where

$$\overset{2}{C} := C^{AB}{}_{CD}C^{CD}{}_{AB}, \quad \overset{2}{\bar{C}} := \bar{C}^{\dot{A}\dot{B}}{}_{\dot{C}\dot{D}}\bar{C}^{\dot{C}\dot{D}}{}_{\dot{A}\dot{B}}, \quad \overset{3}{C} := C^{AB}{}_{CD}C^{CD}{}_{EF}C^{EF}{}_{AB},$$

$$\overset{3}{\bar{C}} := \bar{C}^{\dot{A}\dot{B}}{}_{\dot{C}\dot{D}}\bar{C}^{\dot{C}\dot{D}}{}_{\dot{E}\dot{F}}\bar{C}^{\dot{E}\dot{F}}{}_{\dot{A}\dot{B}}.$$

From the theory of algebraic equations of the third order [22, 23] it follows, that (2.25a) has multiple roots if and only if invariant Δ defined by $\Delta := \frac{1}{2}(\overset{2}{C})^3 - 3(\overset{3}{C})^2$ is equal to 0: likewise (2.25b) has multiple roots if and only if invariant $\bar{\Delta}$ defined by $\bar{\Delta} := \frac{1}{2}(\overset{2}{\bar{C}})^3 - 3(\overset{3}{\bar{C}})^2$ is equal to 0.

Now one can easily show that the type of the conformal curvature tensor is determined by: $\Delta, \overset{2}{C}, \bar{\Delta}, \overset{2}{\bar{C}}$ and the orders of minimal polynomials of matrices C and $\bar{C}$ (for details in real space-times see [19, 25]). One obtains characteristics of types of the conformal curvature tensor in the form of the Cartesian product of two diagrams (figure 2.2). We

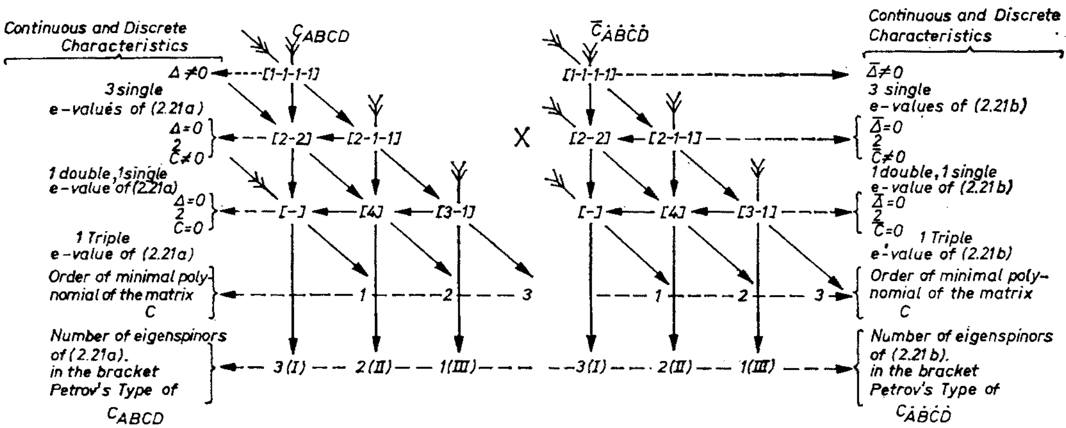


Fig. 2.2. The lines under 45° indicate the order of the minimal polynomial of C and $\bar{C}$, respectively. The vertical lines indicate the number of eigenspinors of (2.21a) and (2.21b) and Petrov's Type of C_{ABCD} and $\bar{C}_{\dot{A}\dot{B}\dot{C}\dot{D}}$, respectively

will call C_{ABCD} "the heavenly part of the conformal curvature" or "the left conformal curvature"; $\bar{C}_{\dot{A}\dot{B}\dot{C}\dot{D}}$ will be called "the hellish part of the conformal curvature or "the right conformal curvature". Note that the left conformal curvature determines the self-dual part of the conformal curvature tensor, $\frac{1}{2}(C_{abcd} + {}^*C_{abcd})$: $\frac{1}{2}(C_{abcd} + {}^*C_{abcd}) = \frac{1}{4} S_{ab}{}^{AB}C_{ABCD}S_{cd}{}^{CD}$, where ${}^*C_{abcd}$ is defined by ${}^*(\frac{1}{2} C_{abcd}e^c \wedge e^d) =: \frac{1}{2}({}^*C_{abcd})e^c \wedge e^d$ and the right conformal curvature determines the anti-self-dual part of the conformal curvature tensor, $\frac{1}{2}(C_{abcd} - {}^*C_{abcd})$: $\frac{1}{2}(C_{abcd} - {}^*C_{abcd}) = \frac{1}{4}\bar{S}_{ab}{}^{\dot{A}\dot{B}}\bar{C}_{\dot{A}\dot{B}\dot{C}\dot{D}}\bar{S}_{cd}{}^{\dot{C}\dot{D}}$.

Now we are going to generalize the notion of the Debever–Penrose vector on the case of the complex oriented space-time. If k^A is P-spinor then each vector l^μ of the form

$$l^\mu := z g^{\mu A\dot{B}} k_A \bar{m}_{\dot{B}}, \tag{2.26}$$

where z — the complex number $\neq 0$, $\bar{m}_{\dot{B}}$ is any dotted spinor $\neq 0$, we call "the left (or heavenly) Debever–Penrose (briefly D–P) vector". If k^A is multiple P-spinor then the corresponding left D–P vector we call the multiple one. The direction defined by l^μ we call "the left (or heavenly) D–P direction". Now if $\bar{k}^{\dot{A}}$ is P-spinor then each vector r^μ of the form

$$r^\mu := z g^{\mu A\dot{B}} m_A \bar{k}_{\dot{B}}, \tag{2.27}$$

where z is the complex number $\neq 0$, m_A — any undotted spinor $\neq 0$, we call "the right (or hellish) D–P vector". If $\bar{k}^{\dot{A}}$ is multiple $\bar{P}$-spinor then we will say that r^μ is the multiple right (or hellish) D–P vector. The direction of r^μ we call "the right (or hellish) D–P direction".

Each P-spinor determines the two-dimensional vector space such that each vector ($\neq 0$) belonging to this space is the left D–P vector. Analogously, each $\bar{P}$-spinor determines the two-dimensional vector space such that each vector ($\neq 0$) belonging to this space is the right D–P vector. The vector spaces determined by P-spinor and $\bar{P}$-spinor possess the common one-dimensional vector space, the elements ($\neq 0$) of which we will call "the generalized D–P vectors". From the above definition, each generalized D–P vector k^μ is of the form

$$k^\mu := z g^{\mu A\dot{B}} k_A \bar{k}_{\dot{B}}, \tag{2.28}$$

here z is the complex number $\neq 0$, k^A — P-spinor, $\bar{k}^{\dot{A}}$ — $\bar{P}$-spinor. The direction of k^μ we call "the generalized D–P direction". If k^A is P-spinor and $C^{(5)} = \ldots = C^{(5-m+1)} = 0$, $C^{(5-m)} \neq 0$ (see 2.20) then we say that k^A is "m-fold P-spinor" and corresponding left D–P vector (direction) is m-fold. Similarly if $\bar{k}^{\dot{A}}$ is $\bar{P}$-spinor and $\bar{C}^{(5)} = \ldots = \bar{C}^{(5-n+1)} = 0$, $C^{(5-n)} \neq 0$ then we say that $\bar{k}^{\dot{A}}$ is "n-fold $\bar{P}$-spinor" and corresponding right D–P vector (direction) is n-fold. Let k^A and $\bar{k}^{\dot{A}}$ in (2.28) respectively be m-fold P-spinor and n-fold $\bar{P}$-spinor, then generalized D–P vector k^μ (the direction of k^μ) will be called (m, n)-fold one. For example, there exist 16 (1, 1)-fold D–P directions for the type $[1-1-1-1] \otimes [1-1-1-1]$ and only one generalized (4, 4)-fold D–P direction for the type $[4] \otimes [4]$.

Now let K^μ be an arbitrary null vector. In the space of tensor of the same algebraic properties as the conformal curvature tensor we define the following linear mappings: $D(K_\mu)$, $D^+(K_\mu)$, $D^-(K_\mu^-)$ by

$$K_{\alpha\beta\gamma\delta} := (D(K_\mu) C_{\tau\nu\varrho\sigma} dz^\tau \otimes dz^\nu \otimes dz^\varrho \otimes dz^\sigma)_{\alpha\beta\gamma\delta} := K^\mu K_{[\alpha} C_{\beta]\mu\nu[\gamma} K_{\delta]} K^\nu, \tag{2.29}$$

$$K^+_{\alpha\beta\gamma\delta} := (D^+(K_\mu) C_{\tau\nu\varrho\sigma} dz^\tau \otimes dz^\nu \otimes dz^\varrho \otimes dz^\sigma)_{\alpha\beta\gamma\delta}$$

$$:= (D(K_\mu) \tfrac{1}{2} (C_{\tau\nu\varrho\sigma} + {}^*C_{\tau\nu\varrho\sigma}) dz^\tau \otimes dz^\nu \otimes dz^\varrho \otimes dz^\sigma)_{\alpha\beta\gamma\delta}, \tag{2.30}$$

$$K^-{}_{\alpha\beta\gamma\delta} := (D^-(K_\mu) C_{\tau\nu\varrho\sigma} dz^\tau \otimes dz^\nu \otimes dz^\varrho \otimes dz^\sigma)_{\alpha\beta\gamma\delta}$$

$$:= (D(K_\mu) \tfrac{1}{2} (C_{\tau\nu\varrho\sigma} - {}^*C_{\tau\nu\varrho\sigma}) dz^\tau \otimes dz^\nu \otimes dz^\varrho \otimes dz^\sigma)_{\alpha\beta\gamma\delta}. \tag{2.31}$$

Of course $D(K_\mu) = D^+(K_\mu)+D^-(K_\mu)$, One can easily find that the spinorial images of the objects $K_{\alpha\beta\gamma\delta}$, ${K^+}_{\alpha\beta\gamma\delta}$, ${K^-}_{\alpha\beta\gamma\delta}$ are given by

$$K_{ABCD} = \tfrac{1}{4} K_A K_B K_C K_D \bar{C}_{\dot{P}\dot{Q}\dot{R}\dot{S}} \bar{K}^{\dot{P}} \bar{K}^{\dot{Q}} \bar{K}^{\dot{R}} \bar{K}^{\dot{S}},$$

$$\bar{K}_{\dot{A}\dot{B}\dot{C}\dot{D}} = \tfrac{1}{4} \bar{K}_{\dot{A}} \bar{K}_{\dot{B}} \bar{K}_{\dot{C}} \bar{K}_{\dot{D}} C_{\dot{P}\dot{Q}\dot{R}\dot{S}} K^P K^Q K^R K^S, \tag{2.32}$$

$${K^+}_{ABCD} = 0, \quad {\bar{K}^+}_{\dot{A}\dot{B}\dot{C}\dot{D}} = \tfrac{1}{4} \bar{K}_{\dot{A}} \bar{K}_{\dot{B}} \bar{K}_{\dot{C}} \bar{K}_{\dot{D}} C_{PQRS} K^P K^Q K^R K^S, \tag{2.33}$$

$${K^-}_{ABCD} = \tfrac{1}{4} K_A K_B K_C K_D \bar{C}_{\dot{P}\dot{Q}\dot{R}\dot{S}} \bar{K}^{\dot{P}} \bar{K}^{\dot{Q}} \bar{K}^{\dot{R}} \bar{K}^{\dot{S}}, \quad {\bar{K}^-}_{\dot{A}\dot{B}\dot{C}\dot{D}} = 0, \tag{2.34}$$

where K^A, $\bar{K}^{\dot{A}}$ are defined by

$$K^\mu =: -\frac{1}{\sqrt{2}} g^{\mu A\dot{B}} K_A \bar{K}_{\dot{B}}. \tag{2.35}$$

From (2.32), (2.33), (2.34) and using the results of Table 2.1 one can easily deduce.

Proposition 2.1

$K_{\alpha\beta\gamma\delta} = 0 \Leftrightarrow K^A$ is P-spinor and $\bar{K}^{\dot{A}}$ is $\bar{P}$-spinor $\Leftrightarrow K^\mu$ is generalized D–P vector. □

Proposition 2.2

${K^+}_{\alpha\beta\gamma\delta} = 0 \Leftrightarrow K^A$ is P-spinor $\Leftrightarrow K^\mu$ is the left D–P vector. □

Proposition 2.3

${K^-}_{\alpha\beta\gamma\delta} = 0 \Leftrightarrow \bar{K}^{\dot{A}}$ is $\bar{P}$-spinor $\Leftrightarrow K^\mu$ is the right D–P vector. □

The mappings: $D(\cdot)$, $D^+(\cdot)$, $D^-(\cdot)$ possess some interesting properties (for details see [4, 24]). Furthermore it is not hard to generalize the well-known in the case of real space-time. Sachs–Debever's [26, 27] characteristics of algebraic types of the conformal curvature tensor (Table 2.2).

One can easily show that for each type $[A]\otimes[B]$ one can choose common set of directions mentioned in Table 2.2. This set contains the number of directions equal to: max (number of l^μ directions; number of r^μ directions).

Now for the sake of completness we consider the generalization of the Petrov approach to the problem of classification of the conformal curvature tensor [28] on the case of the complex oriented space-time. Let us introduce the following base in the space of 2-forms

$$\tilde{e}^1 := e^{1'} \wedge e^{4'}, \quad \tilde{e}^2 := e^{2'} \wedge e^{4'}, \quad \tilde{e}^3 := e^{3'} \wedge e^{4'},$$

$$\tilde{e}^4 := e^{2'} \wedge e^{3'}, \quad \tilde{e}^5 := e^{3'} \wedge e^{1'}, \quad \tilde{e}^6 := e^{1'} \wedge e^{2'}, \tag{2.36}$$

where $(e^{1'}, e^{2'}, e^{3'}, e^{4'})$ is "the rightly oriented Lorentz tetrad" defined by

$$e^{1'} := \frac{1}{\sqrt{2}}(e^1+e^2), \quad e^{2'} := -\frac{i}{\sqrt{2}}(e^1-e^2),$$

$$e^{3'} := \frac{1}{\sqrt{2}}(e^3+e^4), \quad e^{4'} := \frac{1}{\sqrt{2}}(e^3-e^4). \tag{2.37}$$

TABLE 2.2

The type of C_{ABCD} is	if and only if	The type of $\bar{C}_{\dot{A}\dot{B}\dot{C}\dot{D}}$ is	if and only if
[1–1–1–1]	There exist four non-orthogonal null directions such that $D^{+}(l_{\mu})C_{\alpha\beta\gamma\delta} = 0$ (here l^{μ} is an arbitrary vector defining one of these directions).	[1–1–1–1]	There exist four non-orthogonal null directions such that $D^{-}(r_{\mu})C_{\alpha\beta\gamma\delta} = 0$ (here r^{μ} is an arbitrary vector defining one of these directions).
[2–1–1]	There exists the null direction defined by the vector l^{μ}, such that (a) $l^{\mu}(C_{\alpha\mu\nu[\gamma} + {}^{*}C_{\alpha\mu\nu[\gamma})\, l_{\delta]}l^{\nu} = 0$, (b) $(C_{\alpha\mu\nu[\gamma} + {}^{*}C_{\alpha\mu\nu[\gamma})\, l_{\delta]}l^{\nu} \neq 0$. If null vector L^{μ} fulfills (a), then $l^{\mu}L_{\mu} = 0$.	[2–1–1]	There exists the null direction defined by the vector r^{μ}, such that (a) $r^{\mu}(C_{\alpha\mu\nu[\gamma} - {}^{*}C_{\alpha\mu\nu[\gamma})\, r_{\delta]}r^{\nu} = 0$, (b) $(C_{\alpha\mu\nu[\gamma} - {}^{*}C_{\alpha\mu\nu[\gamma})\, r_{\delta]}\, r^{\nu} \neq 0$. If null vector R^{μ} fulfills (a), then $r^{\mu}R_{\mu} = 0$.
[2–2]	There exist two non-orthogonal null directions defined by vectors l_{1}^{μ} and l_{2}^{μ}, such that $l_{1}^{\mu}(C_{\alpha\mu\nu[\gamma} + {}^{*}C_{\alpha\mu\nu[\gamma})l_{\vert 1 \vert \delta]}l_{1}^{\nu} = 0$, $l_{2}^{\mu}(C_{\alpha\mu\nu[\gamma} + {}^{*}C_{\alpha\mu\nu[\gamma})l_{\vert 2 \vert \delta]}l_{2}^{\nu} = 0$, and $C_{\alpha\mu\nu\gamma} + {}^{*}C_{\alpha\mu\nu\gamma} \neq 0$.	[2–2]	There exist two non-orthogonal null directions defined by vectors r_{1}^{μ} and r_{2}^{μ}, such that $r_{1}^{\mu}(C_{\alpha\mu\nu[\gamma} - {}^{*}C_{\alpha\mu\nu[\gamma})r_{\vert 1 \vert \delta]}r_{1}^{\nu} = 0$, $r_{2}^{\mu}(C_{\alpha\mu\nu[\gamma} - {}^{*}C_{\alpha\mu\nu[\gamma})r_{\vert 2 \vert \delta]}r_{2}^{\nu} = 0$, and $C_{\alpha\mu\nu\gamma} - {}^{*}C_{\alpha\mu\nu\gamma} \neq 0$.
[3–1]	There exists the null direction defined by the vector l^{μ}, such that $(C_{\alpha\mu\nu[\gamma} + {}^{*}C_{\alpha\mu\nu[\gamma})l_{\delta]}l^{\nu} = 0$, $(C_{\alpha\mu\nu\gamma} + {}^{*}C_{\alpha\mu\nu\gamma})l^{\nu} \neq 0$.	[3–1]	There exists the null direction defined by the vector r^{μ}, such that $(C_{\alpha\mu\nu[\gamma} - {}^{*}C_{\alpha\mu\nu[\gamma})r_{\delta]}r^{\nu} = 0$, $(C_{\alpha\mu\nu\gamma} - {}^{*}C_{\alpha\mu\nu\gamma})r^{\nu} \neq 0$.
[4]	There exists the null direction defined by the vector l^{μ}, such that $(C_{\alpha\mu\nu\gamma} + {}^{*}C_{\alpha\mu\nu\gamma})l^{\nu} = 0$, $C_{\alpha\mu\nu\gamma} + {}^{*}C_{\alpha\mu\nu\gamma} \neq 0$.	[4]	There exists the null direction defined by the vector r^{μ}, such that $(C_{\alpha\mu\nu\gamma} - {}^{*}C_{\alpha\mu\nu\gamma})r^{\nu} = 0$, $C_{\alpha\mu\nu\gamma} - {}^{*}C_{\alpha\mu\nu\gamma} \neq 0$.
[—]	$C_{\alpha\mu\nu\gamma} + {}^{*}C_{\alpha\mu\nu\gamma} = 0$.	[—]	$C_{\alpha\mu\nu\gamma} - {}^{*}C_{\alpha\mu\nu\gamma} = 0$.

If ω is 2-form then one can write

$$\omega = \omega_{\tilde{A}}\tilde{e}^{\tilde{A}}, \quad \tilde{A} = 1, 2, 3, 4, 5, 6. \tag{2.38}$$

The conformal curvature tensor at some point of the complex oriented space-time is represented by the 6×6 symmetric matrix $(C_{\tilde{A}\tilde{B}})$ according to

$$\tfrac{1}{4}\, C_{a'b'c'd'}(e^{a'} \wedge e^{b'})\otimes(e^{c'} \wedge e^{d'}) =: C_{\tilde{A}\tilde{B}}\tilde{e}^{\tilde{A}}\otimes\tilde{e}^{\tilde{B}}. \tag{2.39}$$

Furthermore, we introduce, in the space of 2-forms, the symmetric tensor of the second rank, "the metric tensor" as follows

$$g_{\tilde{A}\tilde{B}}\tilde{e}^{\tilde{A}}\otimes\tilde{e}^{\tilde{B}} := \tfrac{1}{4}\,(g_{a'c'}g_{b'd'} - g_{a'd'}g_{b'c'})\,(e^{a'} \wedge e^{b'})\otimes(e^{c'} \wedge e^{d'}). \tag{2.40}$$

One can define the tensor $g^{\tilde{A}\tilde{B}}$ reciprocal to $g_{\tilde{A}\tilde{B}}$, by

$$g_{\tilde{A}\tilde{B}}g^{\tilde{B}\tilde{C}} = \delta_{\tilde{A}}{}^{\tilde{C}}. \tag{2.41}$$

In the base which we consider, this tensor has the form

$$(g_{\tilde{A}\tilde{B}}) = (g^{\tilde{A}\tilde{B}}) = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.42}$$

Now consider the linear mapping $\overset{2}{\Lambda}L_4 \to \Lambda^2$ (here L_4 denotes the space of tangent vector fields and Λ^2 the space of 2-forms) defined by

$$C_{\tilde{A}\tilde{B}}\Phi^{\tilde{B}} = \Psi_{\tilde{A}}. \tag{2.43}$$

The structure of this mapping at a given point of the complex oriented space-time is determined by the λ-matrix

$$(C_{\tilde{A}\tilde{B}} - \lambda g_{\tilde{A}\tilde{B}}). \tag{2.44}$$

The Petrov approach consists in studying the matrix (2.44). It can be shown that by elementary transformations the matrix (2.44), can be brought to the form:

$$\begin{Bmatrix} M - iN + \lambda I & 0 \\ 0 & M + iN + \lambda I \end{Bmatrix}, \tag{2.45}$$

where M, N are 3×3 symmetric matrices and

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

One can easily verify that $M - iN = (C_{\tilde{A}\tilde{B}} + {}^*C_{\tilde{A}\tilde{B}})$, $M + iN = (C_{\tilde{A}\tilde{B}} - {}^*C_{\tilde{A}\tilde{B}})$. (Notice that in the complex space-time $(M + iN) \neq (M - iN)^*$, where $*$ denotes the complex conjugation.)

The matrices $M-iN+\lambda I$ and $M+iN+\lambda I$ possess one of the characteristics: (1, 1, 1), ((1, 1), 1), ((1, 1, 1)), (2, 1), ((2, 1)), (3). Therefore we obtain the Petrov classification of the conformal curvature tensor of the complex oriented space-time:

Petrov's type of $M-iN = C_{\tilde{A}\tilde{B}}+{}^*C_{\tilde{A}\tilde{B}}$		Petrov's type of $M+iN = C_{\tilde{A}\tilde{B}}-{}^*C_{\tilde{A}\tilde{B}}$
I		I
II	×	II
III		III

Fig. 2.3

where Petrov's Type I of $M-iN$ $(M+iN)$ is defined by characteristics of $M-iN+\lambda I$ $(M+iN+\lambda I)$: (1, 1, 1,) ((1, 1), 1), ((1, 1, 1)), Petrov's Type II by characteristics (2,1), ((2, 1)) and Petrov's Type III by the characteristic (3). It is meaningful that the conformal curvature tensor of the oriented complex space-time is of Petrov's Type: (I, II), (II, III) etc. (Notice that we will write I⊗II: = (I, II), II⊗III: = (II, III) etc.)

Naturally one may ask now, what is the connection between the spinor approach and Petrov's approach to the algebraic classification of the conformal curvature tensor. The answer is that the matrix $(iN-M)$ is similar to the matrix C (see 2.22a) and the matrix $(-iN-M)$ is similar to $\bar{C}$ (see 2.22b). Therefore the characteristic of $(C-\lambda I)$ is the same as that of $(M-iN+\lambda I)$ and the characteristic of $(\bar{C}-\lambda I)$ is the same as that of $(M+iN+\lambda I)$ (for the real space-time see similar consideration in [29]). Indeed

$$C^{AB}{}_{CD} = \tfrac{1}{16}\, S^{a'b'AB}C_{a'b'c'd'}S^{c'd'}{}_{CD} = \tfrac{1}{8}\, S^{\tilde{A}AB}(C_{\tilde{A}\tilde{B}}+{}^*C_{\tilde{A}\tilde{B}})S^{\tilde{B}}{}_{CD}$$

$$= \tfrac{1}{2} \sum_{\tilde{A},\tilde{B}=1}^{3} S_{\tilde{A}}{}^{AB}(C^{\tilde{A}}{}_{\tilde{B}}+{}^*C^{\tilde{A}}{}_{\tilde{B}})S^{\tilde{B}}{}_{CD}. \tag{2.46}$$

Substituting

AB	11	12	22
X, Y, Z	1	2	3

$$L^X{}_{\tilde{A}} := \frac{1}{\sqrt{2}}\, S_{\tilde{A}}{}^X, \quad P^{\tilde{A}}{}_1 := \frac{1}{\sqrt{2}}\, S^{\tilde{A}}{}_1, \quad P^{\tilde{A}}{}_2 := \sqrt{2}\, S^{\tilde{A}}{}_2, \quad P^{\tilde{A}}{}_3 := \frac{1}{\sqrt{2}}\, S^{\tilde{A}}{}_3,$$

$$\tilde{A}, \tilde{B} = 1, 2, 3,$$

and taking into account relation $(C^{\tilde{A}}{}_{\tilde{B}}+{}^*C^{\tilde{A}}{}_{\tilde{B}}) = iN-M$ for $\tilde{A}, \tilde{B} = 1, 2, 3$ one obtains

$$C = \Big(\sum_{\tilde{A},\tilde{B}=1}^{3} L^X{}_{\tilde{A}}(iN-M)^{\tilde{A}}{}_{\tilde{B}}P^{\tilde{B}}{}_Y\Big). \tag{2.47}$$

Now taking into account the algebraic properties of the spin-tensor $S_{ab}{}^{AB}$ (see [25]) we can easily verify $(L^X{}_{\tilde{A}})^{-1} = (P^{\tilde{B}}{}_Y)$. Hence

$$C = L\cdot(iN-M)L^{-1}; \tag{2.48}$$

here $L := (L^{X}{}_{\dot{A}})$. Similar arguments lead to similar results for the matrices $(-iN-M)$ and $\bar{C}$.

Therefore we have proved that the spinor approach and the Petrov approach to the problem of algebraic classification of the conformal curvature tensor of the oriented complex space-time are equivalent and we have justified the presence of Petrov notations in Fig. 2.2.

At this stage, few remarks about improper transformations of null tetrads are in order. Let $('e^1, 'e^2, 'e^3, 'e^4)$ be a null tetrad "conjugated" to (e^1, e^2, e^3, e^4) on V_4^c, that is:

$$'e^1 := e^2, \quad 'e^2 := e^1, \quad 'e^3 := e^3, \quad 'e^4 := e^4, \tag{2.49}$$

and let $('E^1, 'E^2, 'E^3, 'E^4)$ be a null tetrad "conjugated" to $(E^1, E^2, E^3, E^4) \in [(e^1, e^2, e^3, e^4)]$:

$$'E^1 := E^2, \quad 'E^2 := E^1, \quad 'E^3 := E^3, \quad 'E^4 := E^4. \tag{2.50}$$

If

$$E^1 = \frac{1}{\sqrt{2}} g_a{}^{A\dot{B}} k_A \bar{l}_{\dot{B}} e^a, \quad E^3 = -\frac{1}{\sqrt{2}} g_a{}^{A\dot{B}} k_A \bar{k}_{\dot{B}} e^a,$$

$$E^2 = \frac{1}{\sqrt{2}} g_a{}^{A\dot{B}} l_A \bar{k}_{\dot{B}} e^a, \quad E^4 = \frac{1}{\sqrt{2}} g_a{}^{AB} l_A \bar{l}_{\dot{B}} e^a, \quad k^A l_A = \bar{k}^{\dot{A}} \bar{l}_{\dot{A}} = 1, \tag{2.51}$$

and

$$'E^1 = \frac{1}{\sqrt{2}} g_a{}^{A\dot{B}\prime} k_A{}' \bar{l}_{\dot{B}}{}' e^a, \quad 'E^3 = -\frac{1}{\sqrt{2}} g_a{}^{A\dot{B}\prime} k_A{}' \bar{k}_{\dot{B}}{}' e^a,$$

$$'E^2 = \frac{1}{\sqrt{2}} g_a{}^{A\dot{B}\prime} l_A{}' \bar{k}_{\dot{B}}{}' e^a, \quad 'E^4 = \frac{1}{\sqrt{2}} g_a{}^{A\dot{B}\prime} l_A{}' \bar{l}_{\dot{B}}{}' e^a, \quad 'k^{A\prime} l_A = '\bar{k}^{\dot{A}\prime} \bar{l}_{\dot{A}} = 1, \tag{2.52}$$

then one can easily verify that with the precision to the sign:

$$'k^A = \bar{k}^{\dot{A}}, \quad 'l^A = \bar{l}^{\dot{A}}, \quad '\bar{k}^{\dot{A}} = k^A, \quad '\bar{l}^{\dot{A}} = l^A. \tag{2.53}$$

Further one finds

$$'S^{AB} = \bar{S}^{\dot{A}\dot{B}}, \quad '\bar{S}^{\dot{A}\dot{B}} = S^{AB}, \tag{2.54}$$

$$'C_{ABCD} = \bar{C}_{\dot{A}\dot{B}\dot{C}\dot{D}}, \quad '\bar{C}_{\dot{A}\dot{B}\dot{C}\dot{D}} = C_{ABCD}. \tag{2.55}$$

Then from (2.53), (2.55), (2.20) it follows

$$'C^{(a)} = \bar{C}^{(a)}, \quad '\bar{C}^{(a)} = C^{(a)}, \quad a = 1, 2, 3, 4, 5. \tag{2.56}$$

Therefore one concludes

(1) If $(V_4^c, [(e^1, e^2, e^3, e^4)])$ is of the type $[A]\otimes[B]$ then $(V_4^c, [('e^1, 'e^2, 'e^3, 'e^4)])$ is of the type $[B]\otimes[A]$.

(2) P-spinors ($\bar{P}$-spinors) on $(V_4^c, [(e^1, e^2, e^3, e^4)])$ are in some sense equal to respective $\bar{P}$-spinors (P-spinors) on $(V_4^c, [('e^1, 'e^2, 'e^3, 'e^4)])$.

(3) Left (right) D–P vectors on $(V_4^c, [(e^1, e^2, e^3, e^4)])$ are right (left) D–P vectors on $(V_4^c, [('e^1, 'e^2, 'e^3, 'e^4)])$.

(4) Generalized D–P vectors overlap in these two cases.

(5) Irrespective of the orientation of V_4^c one can only say that V_4^c is of the type $\{[A], [B]\}$; where $\{[A], [B]\} := [A]\otimes[B]$ or $[B]\otimes[A]$.

One can easily pass to the case of the real space-time considered as a real cross section of the respective complex space-time. We have to remember that for the real space-time $E^1 = (E^2)^*$; $E^3 = (E^3)^*$; $E^4 = (E^4)^*$, dotted spinors = (respective undotted spinors)* (here $*$ denotes the complex conjugation). Of course the type of the real space-time is $[A]\otimes[A]$.

3. *Algebraic structure of the energy-momentum tensor on complex space-time*

In this section we shall generalize the algebraic classification of the energy-momentum tensor given by one of us (Plebański [19]) for the real space-time to the case of the complex space-time. Einstein equations on the complex space-time are of the form

$$R_{\alpha\beta} - \tfrac{1}{2} R g_{\alpha\beta} = -8\pi\tau_{\alpha\beta} + \Lambda g_{\alpha\beta}, \tag{3.1}$$

where: Λ — "the cosmological constant", $\tau_{\alpha\beta}$ — "the energy-momentum tensor". Now introduce the trace-less tensor $C_{\alpha\beta}$

$$C_{\alpha\beta} := R_{\alpha\beta} - \tfrac{1}{4} R g_{\alpha\beta}. \tag{3.2}$$

From (3. 1), (3.2) it follows

$$\tau_{\alpha\beta} = -\frac{1}{8\pi} C_{\alpha\beta} + \tfrac{1}{4}\tau \cdot g_{\alpha\beta}, \tag{3.3}$$

here $\tau := \tau^{\alpha}{}_{\alpha}$. Due to (3.3) the algebraic structure of $\tau_{\alpha\beta}$ is determined by the algebraic structure of $C_{\alpha\beta}$. The tensor $C_{\alpha\beta}$ allows one to define the linear mapping L_4 into L_4 as follows

$$C^{\alpha}{}_{\beta} X^{\beta} = Y^{\alpha}; \quad X^{\alpha}\frac{\partial}{\partial z^{\alpha}}, \quad Y^{\beta}\frac{\partial}{\partial z^{\beta}} \in L_4. \tag{3.4}$$

Therefore the algebraic structure of $C_{\alpha\beta}$ at some point of the complex space-time is determined by the algebraic structure of the mapping (3.4) at the point p. Let C_1, C_2, ... denote different eigenvalues of (3.4) (at the point p); then $C^{\alpha}{}_{\beta}$ can be represented in the Jordanian base, by one and only one of the matrices

$$\begin{pmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & C_3 & 0 \\ 0 & 0 & 0 & C_4 \end{pmatrix}, \begin{pmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & C_3 & 0 \\ 0 & 0 & 0 & C_3 \end{pmatrix}, \begin{pmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_1 & 0 & 0 \\ 0 & 0 & C_2 & 0 \\ 0 & 0 & 0 & C_2 \end{pmatrix}, \begin{pmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & C_2 & 0 \\ 0 & 0 & 0 & C_2 \end{pmatrix}, \begin{pmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_1 & 0 & 0 \\ 0 & 0 & C_1 & 0 \\ 0 & 0 & 0 & C_1 \end{pmatrix} \rightarrow \text{I},$$

(1, 1, 1, 1), (1, 1, (1, 1)), ((1, 1), (1, 1)), (1, (1, 1, 1)), ((1, 1, 1, 1)),

$$\begin{pmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & C_3 & 1 \\ 0 & 0 & 0 & C_3 \end{pmatrix}, \begin{pmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_1 & 0 & 0 \\ 0 & 0 & C_2 & 1 \\ 0 & 0 & 0 & C_2 \end{pmatrix}, \begin{pmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & C_2 & 1 \\ 0 & 0 & 0 & C_2 \end{pmatrix}, \begin{pmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_1 & 0 & 0 \\ 0 & 0 & C_1 & 1 \\ 0 & 0 & 0 & C_1 \end{pmatrix} \rightarrow \text{II},$$

(2, 1, 1) (2, (1, 1)), ((2, 1), 1), ((2, 1, 1)),

$$\begin{pmatrix} C_1 & 1 & 0 & 0 \\ 0 & C_1 & 0 & 0 \\ 0 & 0 & C_2 & 1 \\ 0 & 0 & 0 & C_2 \end{pmatrix}, \begin{pmatrix} C_1 & 1 & 0 & 0 \\ 0 & C_1 & 0 & 0 \\ 0 & 0 & C_1 & 1 \\ 0 & 0 & 0 & C_1 \end{pmatrix} \rightarrow \text{III}_\text{N},$$

(2, 2), ((2, 2)),

$$\begin{pmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_2 & 1 & 0 \\ 0 & 0 & C_2 & 1 \\ 0 & 0 & 0 & C_2 \end{pmatrix}, \begin{pmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_1 & 1 & 0 \\ 0 & 0 & C_1 & 1 \\ 0 & 0 & 0 & C_1 \end{pmatrix} \rightarrow \text{III}_\text{C},$$

(3, 1), ((3, 1)),

$$\begin{pmatrix} C_1 & 1 & 0 & 0 \\ 0 & C_1 & 1 & 0 \\ 0 & 0 & C_1 & 1 \\ 0 & 0 & 0 & C_1 \end{pmatrix} \rightarrow \text{IV},$$

(4).

By I, II, III_N, III_C, IV we have denoted Types (capital T!) of $C_{\alpha\beta}$. Beneath each matrix the characteristic of the λ-matrix: $(C^\alpha{}_\beta - \lambda\delta^\alpha{}_\beta)$ is given. From $C^\alpha{}_\alpha = 0$ it follows, that the sum of diagonal elements of the matrices displayed above is equal to 0. There is the following correspondence between the number of eigenvectors of the mapping (3.4) and Type of $C_{\alpha\beta}$:

Type of $C_{\alpha\beta}$	→	Number of eigenvectors
I	→	4
II	→	3
III_N	→	2
III_C	→	2
IV	→	1

Fig. 3.1

The possible Jordanian forms define the possible "types" of $C_{\alpha\beta}$ (and of $\tau_{\alpha\beta}$). By some analysis one can conclude that the type of $C_{\alpha\beta}$ is determined by the symbol

$$ {}^{(r)}[n_1C_1 - n_2C_2 - \dots - n_{N_0}C_{N_0}]_{(q_1 - q_2 - \dots - q_{N_0})}, \tag{3.5}$$

where r is the number of eigenvectors of the mapping (3.4), $C_1, C_2, \dots C_{N_0}$ — different eigenvalues of the mapping (3.4), $n_1, n_2, \dots n_{N_0}$ — multiplicities of the corresponding eigenvalues, $q_1, q_2, \dots q_{N_0}$ — are defined as follows: if $\tilde{D}(\lambda)$ denotes the minimal polynomial of the matrix $(C^\alpha{}_\beta)$ then

$$\tilde{D}(\lambda) = \sum_{l=1}^{N_0} (\lambda - C_l)^{q_l}. \tag{3.6}$$

Now, if $n_1, n_2, \dots, n_{N_0}$ and $q_1, q_2, \dots, q_{N_0}$ determine the number r, then we will omit in (3.5) the index (r); if the order $\tilde{N} = q_1 + q_2 + \dots + q_{N_0}$ of the minimal equation of matrix $(C^\alpha{}_\beta)$ and numbers $n_1, n_2, \dots n_{N_0}$ determine uniquely $q_1, q_2, \dots q_{N_0}$ and the number r then we will write briefly

$$[n_1C_1 - n_2C_2 - \dots - n_{N_0}C_{N_0}]_{\tilde{N}}. \tag{3.7}$$

If an eigenspace corresponding to some eigenvalue contains a null vector then such eigenvalue will be denoted by one of the symbols: N, N_1, N_2 ... etc. Notice that of course

$$n_1C_1 + n_2C_2 + \dots + n_{N_0}C_{N_0} = 0, \tag{3.8}$$

$$n_1 + n_2 + \dots + n_{N_0} = 4. \tag{3.9}$$

We shall now transform $C_{\alpha\beta}$ of the definite type to the canonical form. But before we do this we prove some auxiliary propositions:

Proposition 3.1

Let $\{X^\alpha, Y^\alpha, Z^\alpha\}$ be the set of three linearly independent, mutually orthogonal tangent vectors at some point of V_4^c, and let $X^\alpha X_\alpha = 0$, then $Y^\alpha Y_\alpha \neq 0$ and $Z^\alpha Z_\alpha \neq 0$.

Proof

Choose the orthonormal base such that

$$X^{a''} = (0, 0, X^{3''}, iX^{3''}), \qquad X^{3''} \neq 0. \tag{3.10}$$

From $X^{a''}Y_{a''} = X^{a''}Z_{a''} = 0$ it follows

$$Y^{a''} = (Y^{1''}, Y^{2''}, Y^{3''}, iY^{3''}), \tag{3.11}$$

$$Z^{a''} = (Z^{1''}, Z^{2''}, Z^{3''}, iZ^{3''}). \tag{3.12}$$

Suppose

$$Y^{a''}Y_{a''} = 0, \tag{3.13}$$

then from (3.11) (changing eventually the orthonormal base but with fixed (3.10)) we obtain

$$Y^{a''} = (Y^{1''}, iY^{1''}, Y^{3''}, iY^{3''}), \tag{3.14}$$

and by independence of X^α, Y^α one concludes $Y^{1''} \neq 0$. Now

$$Y^{a''}Z_{a''} = 0 \Leftrightarrow Z^{a''} = (Z^{1''}, iZ^{1''}, Z^{3''}, iZ^{3''}). \tag{3.15}$$

But vectors: (3.10), (3.14), (3.15) are linearly dependent and this result contradicts the assumptions of our proposition. Similarly one obtains the contradiction assuming $Z^{a''}Z_{a''} = 0$. So the proposition is proved. □

Proposition 3.2

Let $\{X^a, V^a, W^\alpha, Y^\alpha\}$ be the set of arbitrary four tangent vectors at some point of V_4^c, fulfilling relations

$$X^\alpha X_\alpha = X^\alpha V_\alpha = X^\alpha W_\alpha = X^\alpha Y_\alpha = 0. \tag{3.16}$$

Then these vectors aie linearly dependent.

Proof

If $X^\alpha = 0$, then there is nothing to prove. Let $X^\alpha \neq 0$. From (3.16) it follows that in a suitable orthonormal base our vectors are of the forms

$$X^{a''} = (0, 0, X^{3''}, iX^{3''}), \quad X^{3''} \neq 0, \quad V^{a''} = (V^{1''}, V^{2''}, V^{3''}, iV^{3''}),$$
$$W^{a''} = (W^{1''}, W^{2''}, W^{3''}, iW^{3''}), \quad Y^{a''} = (Y^{1''}, Y^{2''}, Y^{3''}, iY^{3''}),$$

but

$$\det\begin{Bmatrix} 0 & 0 & X^{3''} & iX^{3''} \\ V^{1''} & V^{2''} & V^{3''} & iV^{3''} \\ W^{1''} & W^{2''} & W^{3''} & iW^{3''} \\ Y^{1''} & Y^{2''} & Y^{3''} & iY^{3''} \end{Bmatrix} = 0,$$

therefore our four vector are linearly dependent. □

Now we are going to write the types of $C_{\alpha\beta}$ in the canonical forms, that is we are going to express $C_{\alpha\beta}$ in terms of the rightly oriented null tetrad (we assume that the complex space-time is oriented one!) (E^1, E^2, E^3, E^4) and in terms of the "rightly oriented orthonormal tetrad" $(E^{1''}, E^{2''}, E^{3''}, E^{4''})$ defined as follows

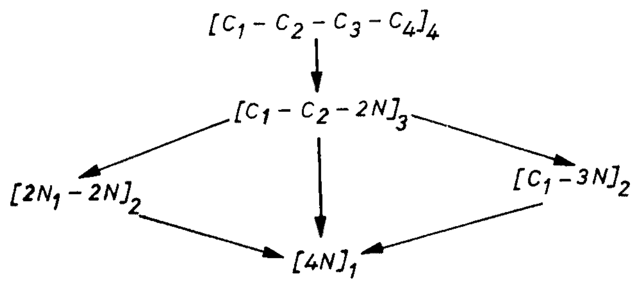
$$E^{1''} := \frac{1}{\sqrt{2}}(E^1+E^2), \quad E^{2''} := -\frac{i}{\sqrt{2}}(E^1-E^2),$$

$$E^{3''} := \frac{1}{\sqrt{2}}(E^3+E^4), \quad E^{4''} := -\frac{i}{\sqrt{2}}(E^3-E^4). \tag{3.17}$$

Using some results of linear algebra [22, 23] and Propositions 2.1, 2.2 one can obtain the following results:

Type I $\rightarrow$ 4 eigenvectors

The types belonging to Type I:



The canonical forms of "the parent type" $[C_1\text{-}C_2\text{-}C_3\text{-}C_4]_4$:

$$C_{\alpha\beta} = C_1 E^{1''}_\alpha E^{1''}_\beta + C_2 E^{2''}_\alpha E^{2''}_\beta + C_3 E^{3''}_\alpha E^{3''}_\beta + C_4 E^{4''}_\alpha E^{4''}_\beta, \tag{3.18}$$

and

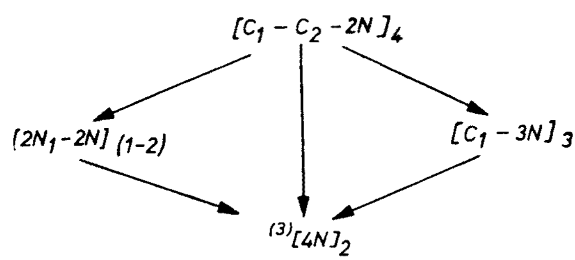
$$C_{\alpha\beta} = \tfrac{1}{2}(C_3 - C_4)(E^3_\alpha E^3_\beta + E^4_\alpha E^4_\beta) + \tfrac{1}{2}(C_3 + C_4)(E^3_\alpha E^4_\beta + E^4_\alpha E^3_\beta - E^1_\alpha E^2_\beta - E^2_\alpha E^1_\beta) + \tfrac{1}{2}(C_1 - C_2)(E^1_\alpha E^1_\beta + E^2_\alpha E^2_\beta). \tag{3.19}$$

Eigenvalues and eigenvectors:

$$C_1 \leftrightarrow E^\alpha_{1''}, \quad C_2 \leftrightarrow E^\alpha_{2''}, \quad C_3 \leftrightarrow E^\alpha_{3''}, \quad C_4 \leftrightarrow E^\alpha_{4''}, \quad C_1 + C_2 + C_3 + C_4 = 0.$$

Type II $\rightarrow$ 3 eigenvectors

The types belonging to Type II:



The canonical forms of the parent type $[C_1\text{-}C_2\text{-}2N]_4$:

$$C_{\alpha\beta} = C_1 E^{1''}_\alpha E^{1''}_\beta + C_2 E^{2''}_\alpha E^{2''}_\beta + N(E^{3''}_\alpha E^{3''}_\beta + E^{4''}_\alpha E^{4''}_\beta) + \tfrac{1}{2}(E^{3''}_\alpha + iE^{4''}_\alpha)(E^{3''}_\beta + iE^{4''}_\beta), \tag{3.20}$$

and

$$C_{\alpha\beta} = E^3_\alpha E^3_\beta + N(E^3_\alpha E^4_\beta + E^4_\alpha E^3_\beta - E^1_\alpha E^2_\beta - E^2_\alpha E^1_\beta) + \tfrac{1}{2}(C_1 - C_2)(E^1_\alpha E^1_\beta + E^2_\alpha E^2_\beta). \tag{3.21}$$

Eigenvalues and eigenvectors:

$$C_1 \leftrightarrow E^\alpha_{1''}, \quad C_2 \leftrightarrow E^\alpha_{2''}, \quad N \leftrightarrow E^\alpha_4, \quad C_1+C_2+2N = 0.$$

Type $\mathrm{III_N} \to 2$ eigenvectors

The types belonging to Type $\mathrm{III_N}$:

$$[2N_1\text{-}2N]_4$$
$$\downarrow$$
$$^{(2)}[4N]_2$$

In the case of oriented complex space-time each type possess two sub-types, "a" and "b". The canonical forms of parent sub-types

a) $[2N_1-2N]^a_4$

$$C_{\alpha\beta} = N(E^{3''}_\alpha E^{3''}_\beta + E^{4''}_\alpha E^{4''}_\beta) + N_1(E^{1''}_\alpha E^{1''}_\beta + E^{2''}_\alpha E^{2''}_\beta)$$
$$+\tfrac{1}{2}(E^{1''}_\alpha - iE^{2''}_\alpha)(E^{1''}_\beta - iE^{2''}_\beta) + \tfrac{1}{2}(E^{3''}_\alpha + iE^{4''}_\alpha)(E^{3''}_\beta + iE^{4''}_\beta), \tag{3.22}$$

and (using relation $N_1 = -N$)

$$C_{\alpha\beta} = N(E^3_\alpha E^4_\beta + E^4_\alpha E^3_\beta - E^1_\alpha E^2_\beta - E^2_\alpha E^1_\beta) + E^2_\alpha E^2_\beta + E^3_\alpha E^3_\beta, \tag{3.23}$$

b) $[2N_1-2N)^b_4$

$$C_{\alpha\beta} = N(E^{3''}_\alpha E^{3''}_\beta + E^{4''}_\alpha E^{4''}_\beta) + N_1(E^{1''}_\alpha E^{1''}_\beta + E^{2''}_\alpha E^{2''}_\beta)$$
$$+\tfrac{1}{2}(E^{1''}_\alpha + iE^{2''}_\alpha)(E^{1''}_\beta + iE^{2''}_\beta) + \tfrac{1}{2}(E^{3''}_\alpha + iE^{4''}_\alpha)(E^{3''}_\beta + iE^{4''}_\beta), \tag{3.24}$$

and (using relation $N_1 = -N$)

$$C_{\alpha\beta} = N(E^3_\alpha E^4_\beta + E^4_\alpha E^3_\beta - E^1_\alpha E^2_\beta - E^2_\alpha E^1_\beta) + E^1_\alpha E^1_\beta + E^3_\alpha E^3_\beta. \tag{3.25}$$

Eigenvalues and eigenvectors:

a) $N \leftrightarrow E^\alpha_4, \quad N_1 = -N \leftrightarrow E^\alpha_1,$

b) $N \leftrightarrow E^\alpha_4, \quad N_1 = -N \leftrightarrow E^\alpha_2.$

Type $\mathrm{III_C}$

The types belonging to Type $\mathrm{III_C}$:

$$[C_1\text{-}3N]_4$$
$$\downarrow$$
$$[4N]_3$$

The canonical forms of the parent type $[C_1\text{-}3N]_4$:

$$C_{\alpha\beta} = C_1 E^{1''}_\alpha E^{1''}_\beta + N(E^{2''}_\alpha E^{2''}_\beta + E^{3''}_\alpha E^{3''}_\beta + E^{4''}_\alpha E^{4''}_\beta)$$
$$+ \frac{1}{\sqrt{2}}(E^{3''}_\alpha + iE^{4''}_\alpha)E^{2''}_\beta + \frac{1}{\sqrt{2}}E^{2''}_\alpha(E^{3''}_\beta + iE^{4''}_\beta), \tag{3.26}$$

and (using relation $C_1 = -3N$)

$$C_{\alpha\beta} = \frac{i}{\sqrt{2}}[E^3_\alpha(E^2_\beta - E^1_\beta) + (E^2_\alpha - E^1_\alpha)E^3_\beta] + N(E^3_\alpha E^4_\beta + E^4_\alpha E^3_\beta - E^1_\alpha E^2_\beta - E^2_\alpha E^1_\beta)$$
$$-2N(E^1_\alpha E^1_\beta + E^2_\alpha E^2_\beta). \tag{3.27}$$

Eigenvalues and eigenvectors:

$$N \leftrightarrow E^\alpha_4, \quad C_1 = -3N \leftrightarrow E^\alpha_{1''}.$$

Type IV

The types belonging to Type IV:

There exists one type only: $[4N]_4$. In the case of oriented complex space-time this class possess two sub-types a and b.

The canonical forms of the sub-types:

a) $[4N]^a_4$

$$C_{\alpha\beta} = \tfrac{1}{2}[(E^{3''}_\alpha + iE^{4''}_\alpha)(E^{1''}_\beta + iE^{2''}_\beta) + (E^{1''}_\alpha + iE^{2''}_\alpha)(E^{3''}_\beta + iE^{4''}_\beta)$$
$$+ (E^{1''}_\alpha - iE^{2''}_\alpha)(E^{1''}_\beta - iE^{2''}_\beta)], \tag{3.28}$$

and

$$C_{\alpha\beta} = E^1_\alpha E^3_\beta + E^3_\alpha E^1_\beta + E^2_\alpha E^2_\beta, \tag{3.29}$$

b) $[4N]^b_4$

$$C_{\alpha\beta} = \tfrac{1}{2}[(E^{3''}_\alpha + iE^{4''}_\alpha)(E^{1''}_\beta - iE^{2''}_\beta) + (E^{1''}_\alpha - iE^{2''}_\alpha)(E^{3''}_\beta + iE^{4''}_\beta)$$
$$+ (E^{1''}_\alpha + iE^{2''}_\alpha)(E^{1''}_\beta + iE^{2''}_\beta)], \tag{3.30}$$

and

$$C_{\alpha\beta} = E^2_\alpha E^3_\beta + E^3_\alpha E^2_\beta + E^1_\alpha E^1_\beta. \tag{3.31}$$

Eigenvalues and eigenvectors:

a) $N = 0 \leftrightarrow E^\alpha_4$,

b) $N = 0 \leftrightarrow E^\alpha_4$.

The spinor image of $C_{\alpha\beta}$ is defined by

$$C_{AB\dot{C}\dot{D}} := \tfrac{1}{4} C_{\alpha\beta} g^\alpha{}_{A\dot{C}} g^\beta{}_{B\dot{D}} \leftrightarrow C_{\alpha\beta} = g_\alpha{}^{A\dot{C}} g_\beta{}^{B\dot{D}} C_{AB\dot{C}\dot{D}}. \tag{3.32}$$

Using the canonical forms of $C_{\alpha\beta}$ and taking into account the formulae (2.12) one can obtain the canonical forms of $C_{AB\dot{C}\dot{D}}$:

Type I

The type $[C_1\text{-}C_2\text{-}C_3\text{-}C_4]_4$

$$C_{AB\dot{C}\dot{D}} = \tfrac{1}{2}\left[\tfrac{1}{2}(C_3-C_4)(a\times\bar{a}+b\times\bar{b})-(C_3+C_4)c\times\bar{c}+\tfrac{1}{2}(C_1-C_2)(a\times\bar{b}+b\times\bar{a})\right]. \tag{3.33}$$

Type II

The type $[C_1\text{-}C_2\text{-}2N]_4$

$$C_{AB\dot{C}\dot{D}} = \tfrac{1}{2}\left[a\times\bar{a}-2Nc\times\bar{c}+\tfrac{1}{2}(C_1-C_2)(a\times\bar{b}+b\times\bar{a})\right]. \tag{3.34}$$

Type $\mathrm{III_N}$

a) The sub-type $[2N_1-2N]_4^{\mathrm{a}}$

$$C_{AB\dot{C}\dot{D}} = \tfrac{1}{2}(-2Nc\times\bar{c}+a\times\bar{a}+b\times\bar{a}), \tag{3.35}$$

b) The sub-type $[2N_1-2N]_4^{\mathrm{b}}$

$$C_{AB\dot{C}\dot{D}} = \tfrac{1}{2}(-2Nc\times\bar{c}+a\times\bar{a}+a\times\bar{b}). \tag{3.36}$$

Type $\mathrm{III_C}$

The type $[C_1\text{-}3N]_4$

$$C_{AB\dot{C}\dot{D}} = \tfrac{1}{2}\left[-2N(a\times\bar{b}+b\times\bar{a}+c\times\bar{c})-(a\times\bar{c}+c\times\bar{a})\right]. \tag{3.37}$$

Type IV

a) The sub-type $[4N]_4^{\mathrm{a}}$

$$C_{AB\dot{C}\dot{D}} = \tfrac{1}{2}(-\sqrt{2}\,ia\times\bar{c}+b\times\bar{a}), \tag{3.38}$$

b) The sub-type $[4N]_4^{\mathrm{b}}$

$$C_{AB\dot{C}\dot{D}} = \tfrac{1}{2}(\sqrt{2}\,ic\times\bar{a}+a\times\bar{b}), \tag{3.39}$$

where $a\times\bar{b} := a_{AB}\bar{b}_{\dot{C}\dot{D}}$, $a\times\bar{a} := a_{AB}\bar{a}_{\dot{C}\dot{D}}$, $a\times\bar{c} := a_{AB}\bar{c}_{\dot{C}\dot{D}}$ etc., $a_{AB} := k_Ak_B$, $\bar{a}_{\dot{C}\dot{D}} := \bar{k}_{\dot{C}}\bar{k}_{\dot{D}}$, $b_{AB} := l_Al_B$, $\bar{b}_{\dot{C}\dot{D}} := \bar{l}_{\dot{C}}\bar{l}_{\dot{D}}$, $c_{AB} := \sqrt{2}\,ik_{(A}l_{B)}$, $\bar{c}_{\dot{C}\dot{D}} := -\sqrt{2}\,i\bar{k}_{(\dot{C}}\bar{l}_{\dot{D})}$, and pairs of normalized spinors: (k^A, l^A), $(\bar{k}^{\dot{A}}, \bar{l}^{\dot{A}})$ determine the canonical null tetrad according to (2.12). Further one can define two undotted and two dotted four-spinors as follows

$$U_{ABCD} := 4C_{AB}{}^{\dot{R}\dot{S}}C_{CD\dot{R}\dot{S}}, \tag{3.40}$$

$$V_{ABCD} := U_{(ABCD)}, \tag{3.41}$$

$$\bar{U}_{\dot{A}\dot{B}\dot{C}\dot{D}} := 4C^{RS}{}_{\dot{A}\dot{B}}C_{RS\dot{C}\dot{D}}, \tag{3.42}$$

$$\bar{V}_{\dot{A}\dot{B}\dot{C}\dot{D}} := \bar{U}_{(\dot{A}\dot{B}\dot{C}\dot{D})}. \tag{3.43}$$

Using (3.33)–(3.39) one easily finds U_{ABCD}, V_{ABCD}, $\bar{U}_{\dot{A}\dot{B}\dot{C}\dot{D}}$ and $\bar{V}_{\dot{A}\dot{B}\dot{C}\dot{D}}$. Spinors V_{ABCD} and $\bar{V}_{\dot{A}\dot{B}\dot{C}\dot{D}}$ possess the symmetry of the spinorial images of the conformal curvature tensor,

C_{ABCD} and $\bar{C}_{\dot{A}\dot{B}\dot{C}\dot{D}}$ respectively, therefore one can consider them from the point of view of Petrov-Penrose classification (see Fig. 3.2).

Now we shall describe briefly our algebraic classification of $C_{\alpha\beta}$ in terms of continuous characteristics expressible in term of invariants of the tensor $C_{\alpha\beta}$ (for details see [19, 24, 25]). The characteristic polynomial of the matrix $(C^{\alpha}{}_{\beta})$ is:

$$D(\lambda) := \det(C^{\alpha}{}_{\beta} - \lambda\delta^{\alpha}{}_{\beta}) = \prod_{l=0}^{4} (-1)^l \underset{[l]}{\mathscr{C}} \lambda^{4-l} = \prod_{i=1}^{4} (\lambda - C'_i), \tag{3.44}$$

where

$$\underset{[0]}{\mathscr{C}} := 1, \quad \underset{[k]}{\mathscr{C}} := C^{\alpha_1}{}_{[\alpha_1} \ldots C^{\alpha_k}{}_{\alpha_k]}, \quad k = 1, 2, 3, 4, \tag{3.45}$$

and C'_i, $i = 1, 2, 3, 4$ are eigenvalues of the mapping (3.4) (at a given point p of the complex space-time, of course).

$$C^{\alpha}{}_{\alpha} = 0 \Leftrightarrow \underset{[l]}{\mathscr{C}} = 0. \tag{3.46}$$

Besides $\underset{[l]}{\mathscr{C}}$ we introduce also the second sequence of invariants

$$\overset{0}{\mathscr{C}} := 4, \quad \overset{p}{\mathscr{C}} := \operatorname{Tr}[(C^{\alpha}{}_{\beta})^p], \quad p = 1, 2 \ldots . \tag{3.47}$$

One can verify the following relations:

$$\underset{[l]}{\mathscr{C}} = \sum_{\substack{\text{comb} \\ i_1 \neq \ldots \neq i_l}} C'_{i_1} \ldots C'_{i_l} \quad \overset{p}{\mathscr{C}} = \sum_{i=1}^{4} (C'_i)^p, \tag{3.48}$$

$$\overset{1}{\mathscr{C}} = 0, \quad \underset{[2]}{\mathscr{C}} = -\tfrac{1}{2}\overset{2}{\mathscr{C}}, \quad \underset{[3]}{\mathscr{C}} = \tfrac{1}{3}\overset{3}{\mathscr{C}}, \quad \underset{[4]}{\mathscr{C}} = -\tfrac{1}{4}\overset{4}{\mathscr{C}} + \tfrac{1}{8}(\overset{2}{\mathscr{C}})^2. \tag{3.49}$$

From the theory of algebraic equations of the fourth order [22, 23] one easily deduces that the eigenvalues C'_i, $i = 1, 2, 3, 4$, are expressible in terms of the roots (x_1, x_2, x_3) of the following third order algebraic equation

$$x^3 - \underset{[1]}{U} x^2 + \underset{[2]}{U} x - \underset{[3]}{U} = 0, \tag{3.50}$$

here

$$\underset{[1]}{U} := -2\underset{[2]}{\mathscr{C}} = \overset{2}{\mathscr{C}}, \tag{3.51}$$

$$\underset{[2]}{U} := (\underset{[2]}{\mathscr{C}})^2 - 4\underset{[4]}{\mathscr{C}} = \overset{4}{\mathscr{C}} - \tfrac{1}{4}(\overset{2}{\mathscr{C}})^2, \tag{3.52}$$

$$\underset{[3]}{U} := (\underset{[3]}{\mathscr{C}})^2 = \tfrac{1}{9}(\overset{3}{\mathscr{C}})^2. \tag{3.53}$$

One obtains

$$C_1' = \frac{\varepsilon_1\sqrt{x_1}+\varepsilon_2\sqrt{x_2}+\varepsilon_3\sqrt{x_3}}{2}, \quad C_3' = \frac{-\varepsilon_1\sqrt{x_1}+\varepsilon_2\sqrt{x_2}-\varepsilon_3\sqrt{x_3}}{2},$$

$$C_2' = \frac{\varepsilon_1\sqrt{x_1}-\varepsilon_2\sqrt{x_2}-\varepsilon_3\sqrt{x_3}}{2}, \quad C_4' = \frac{-\varepsilon_1\sqrt{x_1}-\varepsilon_2\sqrt{x_2}+\varepsilon_2\sqrt{x_3}}{2}, \tag{3.54}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1$, and the signs are chosen so that:

$$\varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3 \cdot \sqrt{x_1} \cdot \sqrt{x_2} \cdot \sqrt{x_3} = \underset{[3]}{\mathscr{C}}. \tag{3.55}$$

Substituting

$$V' := x + \tfrac{2}{3}\underset{[2]}{\mathscr{C}} \tag{3.56}$$

one transforms the equation (3.50) to the canonical form

$$V'^3 - 3pV' - 2q = 0, \tag{3.57}$$

where

$$p := (\tfrac{1}{3}\underset{[1]}{U})^2 - \tfrac{1}{3}\underset{[2]}{U} = (\tfrac{1}{3}\underset{[2]}{\mathscr{C}})^2 + \tfrac{4}{3}\underset{[4]}{\mathscr{C}}, \tag{3.58}$$

$$q := (\tfrac{1}{3}\underset{[1]}{U})^3 - \tfrac{1}{6}\underset{[1]}{U} \cdot \underset{[2]}{U} + \tfrac{1}{2}\underset{[3]}{U} = (\tfrac{1}{3}\underset{[2]}{\mathscr{C}})^3 - \tfrac{4}{3}\underset{[2]}{\mathscr{C}}\underset{[4]}{\mathscr{C}} + \tfrac{1}{2}(\underset{[3]}{\mathscr{C}})^2. \tag{3.59}$$

(It is interesting to note that the equation (3.50) is precisely the characteristic equation of the eigenvalue problems:

$$U^{AB}{}_{CD}\Phi^{CD} = x\Phi^{AB}, \quad \Phi^{AB} = \Phi^{(AB)}, \quad \overline{U}^{\dot{A}\dot{B}}{}_{\dot{C}\dot{D}}\overline{\Phi}^{\dot{C}\dot{D}} = x\overline{\Phi}^{\dot{A}\dot{B}}, \quad \overline{\Phi}^{\dot{A}\dot{B}} = \overline{\Phi}^{(\dot{A}\dot{B})},$$

and the equation (3.57) is the characteristic equation of the eigenvalue problems

$$V^{AB}{}_{CD}\Phi^{CD} = V'\Phi^{AB}, \quad \Phi^{AB} = \Phi^{(AB)}, \quad \overline{V}^{\dot{A}\dot{B}}{}_{\dot{C}\dot{D}}\overline{\Phi}^{\dot{C}\dot{D}} = V'\overline{\Phi}^{\dot{A}\dot{B}}, \quad \overline{\Phi}^{\dot{A}\dot{B}} = \overline{\Phi}^{(\dot{A}\dot{B})}.)$$

Finally we introduce invariants:

$$\Delta' := 3^3 \cdot 2^2(p^2 - q^2) = [(C_4'-C_1')(C_4'-C_2')(C_4'-C_3')(C_3'-C_2')(C_3'-C_1')(C_2'-C_1')]^2 \tag{3.60}$$

and

$$J := x_1(x_2-x_3)^2 + x_2(x_1-x_3)^2 + x_3(x_1-x_2)^2 = \underset{[1]}{U}\underset{[2]}{U} - 9\underset{[3]}{U}$$

$$= -2(\underset{[2]}{\mathscr{C}})^3 + 8\underset{[2]}{\mathscr{C}}\underset{[4]}{\mathscr{C}} - 9(\underset{[3]}{\mathscr{C}})^2 \tag{3.61}$$

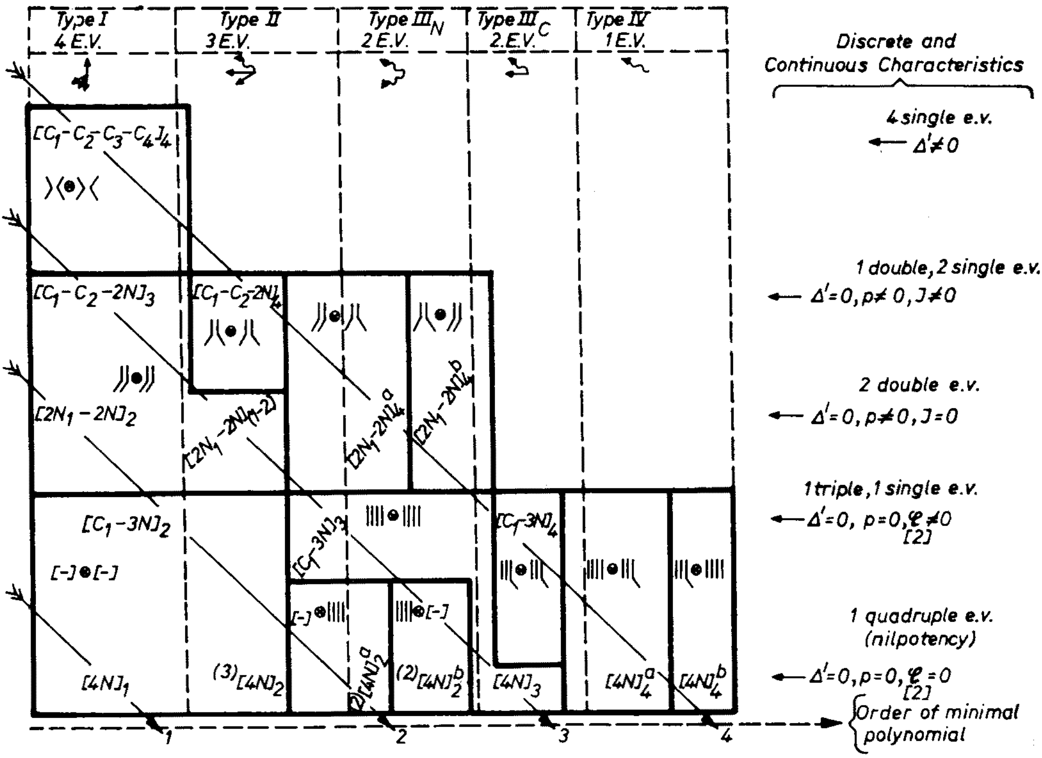
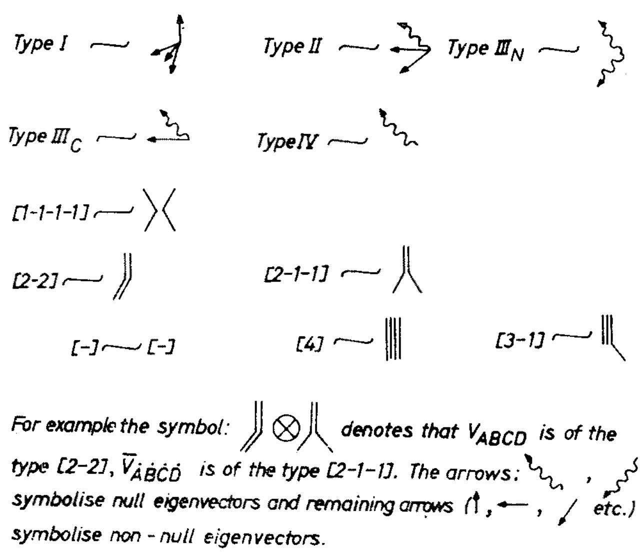


Fig. 3.2. Algebraic classification of energy-momentum tensor in complex space-time

After some analysis one can describe the algebraic types of $C_{\alpha\beta}$ in terms of continuous characteristics. The results are tabulated in the figure 3.2. In the figure we introduce the graphical symbols of $C_{\alpha\beta}$ Types and Petrov–Penrose's types of V_{ABCD} and $\bar{V}_{\dot{A}\dot{B}\dot{C}\dot{D}}$ as follows:



An abbreviation "E. V." in our table means "eigenvector(s)"; abbreviation "e. v." refers to "eigenvalue(s)". The vertical lines --- divide algebraic types of $C_{\alpha\beta}$ into Types. The lines under 45° indicate the order of the minimal polynomial of the matrix $(C^{\alpha}{}_{\beta})$.

Now, as a "physical" example consider the electromagnetic field in the complex oriented space-time.

Linear electrodynamics (see [25]). The energy-momentum tensor

$$\tau_{\alpha\beta} = \frac{1}{4\pi}(f^{\lambda}{}_{\alpha}f_{\lambda\beta} - \tfrac{1}{4}\, g_{\alpha\beta}f^{\mu\nu}f_{\mu\nu}), \tag{3.62}$$

where of course $f_{\alpha\beta} = f_{[\alpha\beta]}$ is the tensor of an electromagnetic field.

$$C_{\alpha\beta} = -2(f_{\alpha}^{\cdot\lambda}\, f_{\lambda\beta} - \tfrac{1}{4}\, g_{\alpha\beta}f^{\mu\nu}f_{\mu\nu}). \tag{3.63}$$

The spinor images of $f_{\alpha\beta}$ are:

$$f_{AB} := \tfrac{1}{8} f_{\mu\nu}S^{\mu\nu}{}_{AB}, \tag{3.64}$$

$$\bar{f}_{\dot{A}\dot{B}} := \tfrac{1}{8} f_{\mu\nu}\bar{S}^{\mu\nu}{}_{\dot{A}\dot{B}}. \tag{3.65}$$

Hence

$$f_{\mu\nu} = f_{AB}S_{\mu\nu}{}^{AB} + \bar{f}_{\dot{A}\dot{B}}\bar{S}_{\mu\nu}{}^{\dot{A}\dot{B}}. \tag{3.66}$$

Furthermore one can easily verify that

$$C_{AB\dot{C}\dot{D}} = -8f_{AB}\bar{f}_{\dot{C}\dot{D}}. \tag{3.67}$$

The invariants of $f_{\mu\nu}$ are

$$F := \tfrac{1}{4} f_{\mu\nu}f^{\mu\nu} = 2f_{AB}f^{AB} + 2\bar{f}_{\dot{A}\dot{B}}\bar{f}^{\dot{A}\dot{B}}, \tag{3.68}$$

$$\check{G} := \tfrac{1}{4} f_{\mu\nu}\check{f}^{\mu\nu} = 2f_{AB}f^{AB} - 2\bar{f}_{\dot{A}\dot{B}}\bar{f}^{\dot{A}\dot{B}}, \tag{3.69}$$

(where $\check{f}_{\mu\nu}$ is defined by $*(\tfrac{1}{2} f_{\mu\nu}dz^{\mu} \wedge dz^{\nu}) =: \tfrac{1}{2}\check{f}_{\mu\nu}dz^{\mu} \wedge dz^{\nu}$). One has to consider four cases:

a) General electromagnetic field is characterized by:

$$\left.\begin{matrix} f_{AB}f^{AB} \neq 0 \\ \bar{f}_{\dot{A}\dot{B}}\bar{f}^{\dot{A}\dot{B}} \neq 0 \end{matrix}\right\} \Leftrightarrow F \neq 0 \text{ and } \check{G} \neq 0. \tag{3.70}$$

In this case

$$f_{AB} = \tfrac{1}{2}(E + i\check{B})k_{(A}l_{B)}, \tag{3.71}$$

$$\bar{f}_{\dot{A}\dot{B}} = \tfrac{1}{2}(\bar{E} - i\bar{\check{B}})\bar{k}_{(\dot{A}}\bar{l}_{\dot{B})}, \tag{3.72}$$

where E, $\check{B}$, $\bar{E}$, $\bar{\check{B}}$ are some real quantities and $k^{A}l_{A} = \bar{k}^{\dot{A}}\bar{l}_{\dot{A}} = 1$.

Therefore (using (3.67), (3.32) and (2.12)) we obtain

$$C_{\alpha\beta} = (E+i\check{B})(\bar{E}-i\bar{\check{B}})\,[2E^3{}_{(\alpha}E^4{}_{\beta)}-2E^1{}_{(\alpha}E^2{}_{\beta)}]. \tag{3.73}$$

Consequently $C_{\alpha\beta}$ is of the type $[2N_1-2N]_2$ and $N = -N_1 = (E+i\check{B})(\bar{E}-i\bar{\check{B}})$.

b) Null electromagnetic field is characterized by:

$$f_{AB}f^{AB} = \bar{f}_{\dot{A}\dot{B}}\bar{f}^{\dot{A}\dot{B}} = 0 \Leftrightarrow F = 0 \text{ and } \check{G} = 0. \tag{3.74}$$

In this case:

$$f_{AB} = \tfrac{1}{4}\,k_A k_B, \tag{3.75}$$

$$\bar{f}_{\dot{A}\dot{B}} = \tfrac{1}{4}\,\bar{k}_{\dot{A}}\bar{k}_{\dot{B}}. \tag{3.76}$$

Therefore

$$C_{\alpha\beta} = -E^3_\alpha E^3_\beta \tag{3.77}$$

and we conclude that $C_{\alpha\beta}$ is of the type ${}^{(3)}[4N]_2$.

c) Right-null electromagnetic field is characterized by:

$$\left.\begin{aligned} \bar{f}_{\dot{A}\dot{B}}\bar{f}^{\dot{A}\dot{B}} &= 0 \\ f_{AB}f^{AB} &\neq 0 \end{aligned}\right\} \Leftrightarrow F-\check{G} = 0 \text{ and } F+\check{G} \neq 0. \tag{3.78}$$

Now one can put

$$f_{AB} = \tfrac{1}{2}(E+i\check{B})k_{(A}l_{B)}, \tag{3.79}$$

$$\bar{f}_{\dot{A}\dot{B}} = \frac{1}{4(E+i\check{B})}\,\bar{k}_{\dot{A}}\bar{k}_{\dot{B}}, \tag{3.80}$$

and then one obtains

$$C_{\alpha\beta} = 2E^3{}_{(\alpha}E^2{}_{\beta)} \tag{3.81}$$

and it is easy to see that $C_{\alpha\beta}$ is of the type ${}^{(2)}[4N]_2$ and of the sub-type ${}^{(2)}[4N]^{a}_2$.

d) Left-electromagnetic field is characterized by:

$$\left.\begin{aligned} f_{AB}f^{AB} &= 0 \\ \bar{f}_{\dot{A}\dot{B}}\bar{f}^{\dot{A}\dot{B}} &\neq 0 \end{aligned}\right\} \Leftrightarrow F+\check{G} = 0 \text{ and } F-\check{G} \neq 0, \tag{3.82}$$

one can put

$$f_{AB} = \frac{1}{4(\bar{E}-i\bar{\check{B}})}\,k_A k_B, \tag{3.83}$$

$$\bar{f}_{\dot{A}\dot{B}} = \tfrac{1}{2}(\bar{E}-i\bar{\check{B}})\bar{k}_{(\dot{A}}\bar{l}_{\dot{B})}. \tag{3.84}$$

Therefore

$$C_{\alpha\beta} = 2E^3{}_{(\alpha}E^1{}_{\beta)}, \tag{3.85}$$

and $C_{\alpha\beta}$ is of the type ${}^{(2)}[4N]_2$ and of the sub-type ${}^{(2)}[4N]_2^{\mathrm{b}}$.

Non-linear electrodynamics (see [30]). The energy-momentum tensor is of the form:

$$\tau_{\alpha\beta} = \frac{1}{4\pi}\left[\frac{\partial L}{\partial F} f^{\lambda}{}_{\alpha} f_{\lambda\beta} + g_{\alpha\beta}\left(\frac{\partial L}{\partial \check{G}}\check{G} - L\right)\right], \tag{3.86}$$

where $L = L(F, \check{G}) := (-4\pi)$ (Lagrangian of the field). From (3.86) one obtains:

$$C_{\alpha\beta} = -2\frac{\partial L}{\partial F}(f^{\lambda}{}_{\alpha} f_{\lambda\beta} - \tfrac{1}{4} g_{\alpha\beta} f^{\mu\nu} f_{\mu\nu}). \tag{3.87}$$

Therefore, comparing (3.63) and (3.87) one concludes that the possible types of $C_{\alpha\beta}$ in the case of non-linear electrodynamics are the same as in the case of linear electrodynamics.

Finally we should like to obtain the canonical forms of $C_{\alpha\beta}$ in the case of the real oriented space-time. We choose the standard null tetrad:

$$E^1 = (E^2)^*, \quad E^3 = (E^3)^*, \quad E^4 = (E^4)^* \tag{3.88}$$

(here $*$ denotes the complex conjugation). The algebraic classification of the energy-momentum tensor in the real space-time had been given by one of us (Plebański [19]). In fact the method of that classification is very similar to the one given in the present paper. Since in the real space-time it is meaningful to distinguish time-like vectors, null vectors, space-like vectors and complex vectors, we denote the complex eigenvalues of the mapping (3.4) (at some point of the real space-time) by $Z, Z_1, Z_2, \ldots$; (a real eigenvalue whose eigenspace contains a time-like vector is denoted by T; a real eigenvalue with eigenspace without time-like vectors but with at least one null vector (of course real one) is denoted by N; the real eigenvalues with eigenspaces spanned only by space-like vectors are denoted by $S, S_1, S_2, \ldots$).

Type I → 4 eigenvectors

Sub-Type $\mathrm{I_R}$: All eigenvectors are real. "The parent type" is: $[S_1\text{-}S_2\text{-}S_3\text{-}T]_4$. The canonical forms of $C_{\alpha\beta}$ and $C_{AB\dot{C}\dot{D}}$ can be obtained from (3.19) and (3.33) by changing: $C_1 \to S_1$, $C_2 \to S_2$, $C_3 \to S_3$, $C_4 \to T$;

Sub-Type $\mathrm{I_Z}$: Not all eigenvectors are real. The parent type is: $[S_1\text{-}S_2\text{-}Z\text{-}\bar{Z})_4$ (here $\bar{Z}$ denotes the complex conjugation to Z). The canonical forms of $C_{\alpha\beta}$ and $C_{AB\dot{C}\dot{D}}$ can be obtained from (3.19) and (3.33) by changing: $C_1 \to S_1$, $C_2 \to S_2$, $C_3 \to Z$, $C_4 \to \bar{Z}$, $E^3_\alpha E^3_\beta + E^4_\alpha E^4_\beta \to i(E^4_\alpha E^4_\beta - E^3_\alpha E^3_\beta)$, $a \times \bar{a} + b \times \bar{b} \to i(b \times \bar{b} - a \times \bar{a})$.

Type II → 3 eigenvectors

The parent type is: $[S_1\text{-}S_2\text{-}2N]_4$. The canonical forms of $C_{\alpha\beta}$ and $C_{AB\dot{C}\dot{D}}$ can be obtained from (3.21) and (3.34) by changing: $C_1 \to S_1$, $C_2 \to S_2$, $E^3_\alpha E^3_\beta \to \varepsilon \cdot E^3_\alpha E^3_\beta$, $a \times \bar{a} \to \varepsilon a \times \bar{a}$, where $\varepsilon = \pm 1$.

Type III → 2 eigenvectors

The parent type is: $[S\text{-}3N]_4$. The canonical forms of $C_{\alpha\beta}$ and $C_{AB\dot{C}\dot{D}}$ are (3.27) and (3.37). Notice that in the real space-time Types: III_N and IV do not exist: In the case of real space-time the dotted spinors are complex conjugated to the corresponding undotted spinors.

One of us (M. Przanowski) is grateful to Dr. M. Demiański for valuable discussions and for his assistance. The authors are also indebted to Dr. Bogdan Mielnik for his helpful and critical discussions.

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