GENERALIZED GOLDBERG–SACHS THEOREMS IN COMPLEX AND REAL SPACE-TIMES (II)

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Using results concerning the algebraic classification of curvature quantities in complex and real space-times (*Acta Phys. Pol.* **B10**, 485 (1979)) a sequence of generalizations of Goldberg–Sachs Theorem is enumerated.

1. *Introduction*

It is well known that the assumption of the existence of so called null-string congruences in the complex space-time (or the existence of a shear-free geodesic null congruence in the real space-time) plays special role for algebraically degenerate solutions of complex Einstein equations (real Einstein equations, respectively) [2–5].

In turn the existence of null string congruences (shear-free geodesic null congruence) in vacuum, $G_{\alpha\beta} = 0$, is guaranteed by the Goldberg–Sachs theorem [6, 7]. Therefore, there appears a natural problem: to generalize Goldberg–Sachs theorems (in the complex and real space-times) to the cases when $G_{\alpha\beta} \neq 0$. We are going to give some propositions and theorems aimed to provide such a generalizations.

In Section 2 we formulate the so called Generalized Goldberg–Sachs theorems and then we extend to the case of complex space-time some interesting results concerning the algebraic types of $C_{\alpha\beta}$ for which there exists a null tetrad (E^1, E^2, E^3, E^4) such that in it $C_{44} = C_{42} = C_{22} = 0$ (for the real space-time see [8] p. 178).

In Section 3 we list propositions which are further generalizations of Goldberg–Sachs theorems. They are obtained by assuming that $C_{\alpha\beta}$ is of definite type and (for the most part) that the null eigenvector of $C^{\alpha}{}_{\beta}$ is a multiple (left, right) Deveber–Penrose vector.

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In the Appendix we write the explicit form of Bianchi identities. Techniques and notations are as in [1].

We hope that the considerations of the present paper will be useful for understanding of the interaction of gravitational radiation with matter, see also [9].

2. *Goldberg–Sachs theorems and their generalizations in complex and real space-times*

The classical Goldberg–Sachs theorem [6] was extended to complex space-time by Plebański and Hacyan [7]. They proved that:

Theorem 2.1. The existence of a null tetrad such that $\Gamma_{424} = 0 = \Gamma_{422}$ ($\Gamma_{414} = 0 = \Gamma_{411}$) is a necessary and sufficient condition for the complex oriented empty space-time ($G_{ab} = 0$) to have the undotted (dotted) conformal curvature spinor algebraically degenerate with $C^{(5)} = 0 = C^{(4)}$ ($\bar{C}^{(5)} = 0 = \bar{C}^{(4)}$) and with E^3 defining some (at least double) left (right) Debever–Penrose direction. □

Then it was shown that $\Gamma_{424} = 0 = \Gamma_{422} \Leftrightarrow$ there exists a congruence of null strings determined by the vector fields E_4, E_2; and analogously, $\Gamma_{414} = 0 = \Gamma_{411} \Leftrightarrow$ there exists a congruence of null strings determined by the vector fields E_4, E_1. In the first case the 2-form

$$2E^3 \wedge E^1, \tag{2.1}$$

which represents an element of area of a null string is self-dual, and therefore it defines a congruence of the left (heavenly) null strings. In the second case the 2-form

$$2E^3 \wedge E^2, \tag{2.2}$$

which represents an element of area of a null string is anti-self-dual, and therefore it defines a congruence of the right (hellish) null strings. Now we are going to formulate (without proofs) some Generalized Goldberg–Sachs Theorems in which it is not assumed that $G_{ab} = 0$ on our complex oriented space-time.

Theorem 2.2. Let (a), (b), (c) mean the following statements:

(a) C_{ABCD} is algebraically degenerate and k^A is a multiple P-spinor,

(b) $k^A k^B \nabla_{A\dot{C}} k_B = 0$ ($\Leftrightarrow$ 2-form $k_A k_B S^{AB}$ defines the congruence of the left null strings),

(c)
$$k^A k^B k^C \nabla_{(A}{}^{\dot{S}} C_{BC)\dot{D}\dot{S}} = 0 \tag{2.3}$$

if C_{ABCD} is of types [2–1–1] or [2–2],
or $k^B k^C \nabla_{(A}{}^{\dot{S}} C_{BC)\dot{D}\dot{S}} = 0$ if C_{ABCD} is of the type [3–1],
or $k^C \nabla_{(A}{}^{\dot{S}} C_{BC)\dot{D}\dot{S}} = 0$ if C_{ABCD} is of the type [4],
then: (a) and (b) $\Rightarrow$ (c), (a) and (c) $\Rightarrow$ (b), (b) and (2.3) $\Rightarrow$ (a). □

Theorem 2.3. Let ($\bar{\text{a}}$), ($\bar{\text{b}}$), ($\bar{\text{c}}$) mean:

($\bar{\text{a}}$) $\bar{C}_{\dot{A}\dot{B}\dot{C}\dot{D}}$ is algebraically degenerate and $\bar{k}^{\dot{A}}$ is a multiple $\bar{P}$-spinor,

($\bar{\text{b}}$) $\bar{k}^{\dot{A}} \bar{k}^{\dot{B}} \nabla_{C\dot{A}} \bar{k}_{\dot{B}} = 0$ ($\Leftrightarrow$ 2-form $\bar{k}_{\dot{A}} \bar{k}_{\dot{B}} \bar{S}^{\dot{A}\dot{B}}$ defines the congruence of the right null strings),

($\bar{\text{c}}$)
$$\bar{k}^{\dot{A}} \bar{k}^{\dot{B}} \bar{k}^{\dot{C}} \nabla^{S}{}_{(\dot{A}} C_{|DS|\dot{B}\dot{C})} = 0 \tag{2.4}$$

if $\bar{C}_{\dot{A}\dot{B}\dot{C}\dot{D}}$ is of the types [2–1–1] or [2–2],
or $\bar{k}^{\dot{B}}\bar{k}^{\dot{C}}\nabla^{S}{}_{(\dot{A}}C_{|DS|\dot{B}\dot{C})} = 0$ if $\bar{C}_{\dot{A}\dot{B}\dot{C}\dot{D}}$ is of the type [3–1],
or $\bar{k}^{\dot{C}}\nabla^{S}{}_{(\dot{A}}C_{|DS|\dot{B}\dot{C})} = 0$ if $\bar{C}_{\dot{A}\dot{B}\dot{C}\dot{D}}$ is of the type [4],
then: ($\bar{a}$) and ($\bar{b}$) ⇒ ($\bar{c}$), ($\bar{a}$) and ($\bar{c}$) ⇒ ($\bar{b}$), ($\bar{b}$) and (2.4) ⇒ ($\bar{a}$). □

(In the above theorems $\nabla_{A\dot{B}} := g^{a}{}_{A\dot{B}}\nabla_{a}$.)

Theorems 2.2, 2.3 may be formulated in the tensor terms as follows:

Theorem 2.2(t). Let (a), (b), (c) mean:

(a) C_{abcd} is algebraically degenerate and E_4 is the multiple left D–P vector,

(b) $\Gamma_{424} = 0 = \Gamma_{422}$ (⇔ 2-form $2E^3 \wedge E^1$ defines the congruence of the left null strings),

(c) $$C_{4[4;2]} = 0, \quad C_{2[2;4]} = 0 \tag{2.5}$$

if C_{abcd} is of types [2–1–1] ⊗ [something], [2–2] ⊗ [something],
or $C_{a[4;2]} + \frac{1}{3} g_{a[4}C^{d}{}_{2];d} = 0$ if C_{abcd} is of types [3–1] ⊗ [something],
or $C_{a[4;2]} + \frac{1}{3} g_{a[4}C^{d}{}_{2];d} = 0$, $C_{4[3;1]} + \frac{1}{3} g_{4[3}C^{d}{}_{1];d} = 0$, $C_{2[3;1]} + \frac{1}{3} g_{2[3}C^{d}{}_{1];d} = 0$, if C_{abcd} is of types [4] ⊗ [something];
then: (a) and (b) ⇒ (c), (a) and (c) ⇒ (b), (b) and (2.5) ⇒ (a). □

Theorem 2.3(t). Let ($\bar{a}$), ($\bar{b}$), ($\bar{c}$) mean:

($\bar{a}$) C_{abcd} be algebraically degenerate and E_4 be a multiple right D–P vector,

($\bar{b}$) $\Gamma_{414} = 0 = \Gamma_{411}$ (⇔ 2-form $2E^3 \wedge E^2$ defines the congruence of the right null strings),

($\bar{c}$) $$C_{4[4;1]} = 0, \quad C_{1[1;4]} = 0 \tag{2.6}$$

if C_{abcd} is of types [something] ⊗ [2–1–1], [something] ⊗ [2–2],
or $C_{a[4;1]} + \frac{1}{3} g_{a[4}C^{d}{}_{1];d} = 0$ if C_{abcd} is of types [something] ⊗ [3–1],
or $C_{a[4;1]} + \frac{1}{3} g_{a[4}C^{d}{}_{1];d} = 0$, $C_{4[3;2]} + \frac{1}{3} g_{4[3}C^{d}{}_{2];d} = 0$, $C_{1[3;2]} + \frac{1}{3} g_{1[3}C^{d}{}_{2];d} = 0$, if C_{abcd} is of types [something] ⊗ [4];
then: ($\bar{a}$) and ($\bar{b}$) ⇒ ($\bar{c}$), ($\bar{a}$) and ($\bar{c}$) ⇒ ($\bar{b}$), ($\bar{b}$) and (2.6) ⇒ ($\bar{a}$). □

In theorems: 2.2(t), 2.3(t), all tensors are expressed in some null tetrad (E^1, E^2, E^3, E^4). (Notice, that one can easy obtain the theorem 2.3(t) from 2.2(t) by interchanges: left → right, 2 ↔ 1.) For completeness one has to add the following theorem:

Theorem 2.4. If the complex oriented space-time is of the type [–] ⊗ [–] then there exists congruence of left and right null strings. □

(It appears a natural question, whether there exist congruences of left (right) null strings in the oriented complex space-times of types:

$$[-] \otimes \begin{matrix} [1\text{–}1\text{–}1\text{–}1] \\ [2\text{–}1\text{–}1] \\ [2\text{–}2] \\ [3\text{–}1] \\ [4] \end{matrix} \qquad \left(\begin{matrix} [1\text{–}1\text{–}1\text{–}1] \\ [2\text{–}1\text{–}1] \\ [2\text{–}2] \\ [3\text{–}1] \\ [4] \end{matrix} \otimes [-] \right)$$

This problem is unsolved, even for $C_{\alpha\beta} = 0$, $R = -4\Lambda \neq 0$.)

If one considers the real space-time as the real cross section of a suitable complex space-time then one can easily deduce the Goldberg–Sachs theorem and the Generalized

Goldberg–Sachs theorem in the real spacetime [6, 10, 11, 12, 8], out of the same theorems in the complex case. Since on the real cross section one can have $E^1 = (E^2)^*$, $E^3 = (E^3)^*$, $E^4 = (E^4)^*$, (the dotted spinors) = (corresponding undotted spinors)*, $\bar{C}^{(a)} = (C^{(a)})^*$, $\Gamma_{414} = (\Gamma_{424})^*$, $\Gamma_{411} = (\Gamma_{422})^*$ etc., therefore on the real cross section the complex space-time is of the type $[A] \otimes [A]$.

Taking into account all these facts one finds the Generalized Goldberg–Sachs theorem in the real space-time:

Theorem 2.5. Let

(a) C_{ABCD} be algebraically degenerate and k^A be a multiple P-spinor,

(b) $k^A k^B \nabla_{A\dot{C}} k_B = 0$ ($\Leftrightarrow$ the vector field $E_4{}^\alpha := -\frac{1}{\sqrt{2}} g^\alpha{}_{A\dot{B}} k^A \bar{k}^{\dot{B}}$ defines the shear--free geodesic null congruence),

(c)
$$k^A k^B k^C \nabla_{(A}{}^{\dot{S}} C_{BC)\dot{D}\dot{S}} = 0 \tag{2.7}$$
if C_{ABCD} is of types [2–1–1] or [2–2], $k^B k^C \nabla_{(A}{}^{\dot{S}} C_{BC)\dot{D}\dot{S}} = 0$ if C_{ABCD} is of the type [3–1], $k^C \nabla_{(A}{}^{\dot{S}} C_{BC)\dot{D}\dot{S}} = 0$ if C_{ABCD} is of the type [4];

then: (a) and (b) $\Rightarrow$ (c), (a) and (c) $\Rightarrow$ (b), (b) and (2.7) $\Rightarrow$ (a), □

or in tensor terms:

Theorem 2.5(t). Let

(a) C_{abcd} be algebraically degenerate and E_4 be a multiple D–P vector,

(b) $\Gamma_{424} = 0 = \Gamma_{422}$ ($\Leftrightarrow E_4$ defines the shear-free geodesic null congruence),

(c)
$$C_{4[4;2]} = 0, \quad C_{2[2;4]} = 0 \tag{2.8}$$
if C_{abcd} is of types [2–1–1] or [2–2]),
or $C_{a[4;2]} + \frac{1}{3} g_{a[4} C^d{}_{2];d} = 0$ if C_{abcd} is of the type [3–1]),
or $C_{a[4;2]} + \frac{1}{3} g_{a[4} C^d{}_{2];d} = 0$, $C_{4[3;1]} + \frac{1}{3} g_{4[3} C^d{}_{1];d} = 0$, $C_{2[3;1]} + \frac{1}{3} g_{2[3} C^d{}_{1];d} = 0$, if C_{abcd} is of the type [4];

then: (a) and (b) $\Rightarrow$ (c), (a) and (c) $\Rightarrow$ (b), (b) and (2.8) $\Rightarrow$ (a). □

And for completeness:

Theorem 2.6. If the real space time is of the type [–] then there exists a shear-free geodesic null congruence. □

The proofs of theorems: 2.5, 2.5(t), 2.6 are given in [10, 11], see also [8]. The proofs of the corresponding theorems in the complex (oriented) space-time are very similar (see also [7]). The statements (c) in our theorems correspond to the vanishing of the right sides of some Bianchi identities (Appendix). The Goldberg–Sachs theorem in the real space-time [6, 8] is a consequence of theorems: 2.5 (or 2.5(t)) and 2.6, when $G_{\alpha\beta} = 0$ ($\Rightarrow C_{\alpha\beta} = 0$).

Now we are going to extend to the case of complex (oriented) space-time some interesting lemmas known in the case of the real space-time (see [8] p. 178).

Lemma 2.1. In complex oriented space-time there exists (rightly oriented) null tetrad (E^1, E^2, E^3, E^4) such that
$$C_{44} = C_{42} = C_{22} = 0, \tag{2.9}$$
$$(C_{44} = C_{41} = C_{11} = 0) \tag{2.10}$$

if and only if the tensor C_{ab} is one of (sub-)types:

$$[4N]_4{}^{b}, [4N]_3, {}^{(2)}[4N]_2{}^{a}, {}^{(2)}[4N]_2{}^{b}, [2N_1-2N]_4{}^{b}, {}^{(3)}[4N]_2, [2N_1-2N]_{(1-2)}, [4N]_1, [2N_1-2N]_2, \tag{2.11}$$

$$([4N]_4{}^{a}, [4N]_3, {}^{(2)}[4N]_2{}^{a}, {}^{(2)}[4N]_2{}^{b}, [2N_1-2N]_4{}^{a}, {}^{(3)}[4N]_2, [2N_1-2N]_{(1-2)}, [4N]_1, [2N_1-2N]_2). \tag{2.12}$$

If C_{ab} is one of these (sub-)types one can select a (rightly oriented) null tetrad so that (2.9), ((2.10)) holds and moreover E_4 is the eigenvector of $C^a{}_b$.

Proof: Let $C_{44} = C_{42} = C_{22} = 0$; then from characteristic equation $\det(C^a{}_b - \lambda\delta^a{}_b) = 0$ one easily finds

$$\lambda = \pm\sqrt{(C_{12})^2 + C_{41}C_{32}}\,. \tag{2.13}$$

Hence $C^a{}_b$ has two double eigenvalues or one quadruple eigenvalue and therefore may be one of types [1]:

$$[4N]_4, [4N]_3, {}^{(2)}[4N]_2, [2N_1-2N]_4, {}^{(3)}[4N]_2, [2N_1-2N]_{(1-2)}, [4N]_1, [2N_1-2N]_2.$$

Now, using the possible canonical forms of $C^a{}_b$ ([1] Section 3) one can easily find by some algebraic manipulations that a (rightly oriented) null tetrad (E^1, E^2, E^3, E^4) such that $C_{44} = C_{42} = C_{22} = 0$ exists only for (sub-)types (2.11). Moreover this tetrad may be so selected that E_4 is the eigenvector of $C^a{}_b$. Similar considerations can be given under assumption $C_{44} = C_{41} = C_{11} = 0$. □

As a consequence of lemma 2.1 and the Generalized Goldberg–Sachs theorems one finds:

Lemma 2.2. Let C_{ab} be one of the (sub-)types (2.11) ((2.12)); then one can select a (rightly oriented) null tetrad so that $C_{44} = C_{42} = C_{22} = 0$, $(C_{44} = C_{41} = C_{11} = 0)$ and E_4 is an eigenvector of $C^a{}_b$. If moreover $\Gamma_{424} = 0 = \Gamma_{422}(\Gamma_{414} = 0 = \Gamma_{411})$ E_4 is a multiple left (right) D–P vector.

Proof: Notice that

$$C_{44} = C_{42} = C_{22} = 0, \Gamma_{424} = \Gamma_{422} = 0 \Rightarrow (2.5)$$

$$(C_{44} = C_{41} = C_{11} = 0, \Gamma_{414} = \Gamma_{411} = 0 \Rightarrow (2.6)).$$

(Notice that in the case of the (sub-)types:

$$[4N]_4{}^{b}, [4N]_3, {}^{(2)}[4N]_2{}^{b}, [2N_1-2N]_4{}^{b}, {}^{(3)}[4N]_2, [2N_1-2N]_{(1-2)}, [4N]_1, [2N_1-2N]_2,$$

the null tetrad (E^1, E^2, E^3, E^4) for which $C_{44} = C_{42} = C_{22} = 0$ and E_4 is the eigenvector of $C^a{}_b$, is exactly the tetrad in [1] (Section 3); analogously in the case of the (sub-)types:

$$[4N]_4{}^{a}, [4N]_3, {}^{(2)}[4N]_2{}^{a}, [2N_1-2N]_4{}^{a}, {}^{(3)}[4N]_2, [2N_1-2N]_{(1-2)}, [4N]_1, [2N_1-2N]_2$$

the null tetrad (E^1, E^2, E^3, E^4) for which $C_{44} = C_{41} = C_{11} = 0$ and E_4 is the eigenvector of $C^a{}_b$, is exactly the one in [1] (Section 3).) □

3. *Some other generalizations of Goldberg–Sachs theorems in complex and real space-times*

One can observe (Section 2) that the Generalized Goldberg–Sachs theorems concern the Ricci coefficients Γ_{424}, Γ_{422} (or Γ_{414}, Γ_{411}) only; moreover, the assumptions about $C_{\alpha\beta}$ ($C_{AB\dot{C}\dot{D}}$) in these theorems are somewhat complicated. Now, we will list propositions concerning other Ricci coefficients (and other objects, as the curvature scalar R or the eigenvalues of $C^{\alpha}{}_{\beta}$). We assume that $C_{\alpha\beta}$ is of a definite algebraic type and that the corresponding mapping $L_4 \to L_4$: $X^{\alpha} \to C^{\alpha}{}_{\beta}X^{\beta}$, has a multiple (left or right) D–P eigenvector. Thus, we shall give generalizations of Goldberg–Sachs theorems for the energy-momentum tensor of a definite algebraic type. Because of their number, our propositions will be most conveniently presented in form of tables which will be ordered according to the types of $C_{\alpha\beta}$. The propositions valid for the real space-time can be easily obtained from the respective propositions in the complex case. It suffices to notice that for the real space-time:

$$(E^1)^* = E^2, (E^3)^* = E^3, (E^4)^* = E^4$$

and then

$$\bar{C}^{(a)} = (C^{(a)})^*, (R)^* = R, (T)^* = T, (N)^* = N,$$

$$\Gamma_{414} = (\Gamma_{424})^*, \quad \Gamma_{411} = (\Gamma_{422})^*, \quad \Gamma_{344} = (\Gamma_{344})^* \text{ etc.}$$

Some of our propositions are consequences of the theorems of Section 2. In the case of real space-times, some of our propositions are known in the literature (e.g. [8, 9]).

The propositions listed in the tables have been obtained mainly by the analysis of Bianchi identities (see Appendix).

Tables of propositions:

For all types of $C_{\alpha\beta}$ except of $^{(2)}[4N]_2$ the components C_{ab} are given in null tetrads introduced in [1] (Section 3). For the type $[4N]_2$ we introduce in our tables the null tetrad suitable for the energy-momentum tensors of electromagnetic fields (see [1] Section 3). In the tables, comma (,) denotes the directional derivative; $[A]$ or $[B]$ denote any type of C_{ABCD} or $C_{\dot{A}\dot{B}\dot{C}\dot{D}}$ ($[A]$ = [Anything]).

As an example of an application of our results one can consider the electromagnetic field (for details see [13]). From the previous paper [1] it follows that in the case of linear electrodynamics, $C_{\alpha\beta}$ is one of the (sub-)types:

complex space-time	real space-time
$[2N_1-2N]_2$, $N = (E+i\check{B})(\bar{E}-i\bar{\check{B}})$	$[2S_1-2T]_2$, $T = E^2+\check{B}^2$
$^{(3)}[4N]_2$	$[4N]_2$
$^{(2)}[4N]_2{}^{a}$	—
$^{(2)}[4N]_2{}^{b}$	—

Notice that
$$R = -4\Lambda = \text{const.} \tag{3.1}$$

Now one can easily obtain the propositions valid in the case of the presence of a linear electromagnetic field from our tables of propositions. But for completeness one has to consider the Maxwell equations:

$$d(f_{AB}S^{AB}) = 0, \tag{3.2a}$$

$$d(\bar{f}_{\dot{A}\dot{B}}\bar{S}^{\dot{A}\dot{B}}) = 0. \tag{3.2b}$$

Equations (3.2a) are equivalent to (see [8, 13]) the following equations:

$$(-\partial_1+\Gamma_{121}+\Gamma_{341}-\Gamma_{314})(f_{AB}k^Ak^B)+(-\partial_4+2\Gamma_{421})(f_{AB}k^Al^B)+\Gamma_{424}(f_{AB}l^Al^B) = 0, \tag{3.3}$$

$$\Gamma_{311}(f_{AB}k^Ak^B)+(\partial_1+2\Gamma_{314})(f_{AB}k^Al^B)+(\partial_4+\Gamma_{124}+\Gamma_{344}+\Gamma_{421})(f_{AB}l^Al^B) = 0, \tag{3.4}$$

$$(\partial_3-\Gamma_{123}-\Gamma_{343}-\Gamma_{312})(f_{AB}k^Ak^B)+(-\partial_2-2\Gamma_{423})(f_{AB}k^Al^B)+\Gamma_{422}(f_{AB}l^Al^B) = 0, \tag{3.5}$$

$$(-\Gamma_{313})(f_{AB}k^Ak^B)+(-\partial_3+2\Gamma_{312})(f_{AB}k^Al^B)+(\partial_2+\Gamma_{122}+\Gamma_{342}+\Gamma_{423})(f_{AB}l^Al^B) = 0, \tag{3.6}$$

and equations equivalent to (3.2b) can be obtained from (3.3)-(3.6) by the interchanges: $1 \leftrightarrow 2, f_{AB} \to \bar{f}_{\dot{A}\dot{B}}, k^A \to \bar{k}^{\dot{A}}, l^A \to \bar{l}^{\dot{A}}$.

(Of course $k^Al_A = \bar{k}^{\dot{A}}\bar{l}_{\dot{A}} = 1$ and (k^A, l^A), $(\bar{k}^{\dot{A}}, \bar{l}^{\dot{A}})$ define the null tetrad (E^1, E^2, E^3, E^4) according to the formulae:

$$E^1 = \frac{1}{\sqrt{2}}\, g^{A\dot{B}}k_A\bar{l}_{\dot{B}}, \quad E^2 := \frac{1}{\sqrt{2}}\, g^{A\dot{B}}l_A\bar{k}_{\dot{B}}, \quad E^3 := -\frac{1}{\sqrt{2}}\, g^{A\dot{B}}k_A\bar{k}_{\dot{B}}, \quad E^4 := \frac{1}{\sqrt{2}}\, g^{A\dot{B}}l_A\bar{l}_{\dot{B}}.\Big)$$

Now in order to obtain the propositions valid in the case of the presence of a non-linear electromagnetic field from our tables of propositions, one has to notice that $C_{\alpha\beta}$ is one of the (sub-)types [1]:

complex space-time	real space-time
$[2N_1-2N]_2,\ N = \frac{\partial L}{\partial F}(E+i\breve{B})(\bar{E}-i\bar{\breve{B}})$	$[2S_1-2T]_2,\ T = \frac{\partial L}{\partial F}(E^2+\breve{B}^2)$
$^{(3)}[4N]_2$	$[4N]_2$
$^{(2)}[4N]_2{}^a$	—
$^{(2)}[4N]_2{}^b$	—

and

$$R = 8\left(\frac{\partial L}{\partial F}F+\frac{\partial L}{\partial \breve{G}}\breve{G}-L\right)-4\Lambda. \tag{3.7}$$

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TABLE I

Sub-type $[4N]_4{}^a$ (complex space-time)

No	C_{ab}	Type of C_{abcd}	Other assumptions	Proposition
1	$C_{22} = 1$ $C_{31} = 1$ (other components vanish)	$[A] \otimes \begin{matrix}[2\text{-}1\text{-}1]\\ [2\text{-}2]\end{matrix}$	E_4 is a multiple right D–P vector	$\Gamma_{414} = \Gamma_{411} = 0.$
		$[A] \otimes \begin{matrix}[3\text{-}1]\\ [4]\\ [-]\end{matrix}$	E_4 is a multiple right D–P vector	$\Gamma_{414} = \Gamma_{411}+\Gamma_{424} = 0;$ $R_{,4} = 0.$
2	$C_{22} = 1$ $C_{31} = 1$	$\begin{matrix}[2\text{-}1\text{-}1]\\ [2\text{-}2]\\ [4]\\ [-]\end{matrix} \otimes \begin{matrix}[2\text{-}1\text{-}1]\\ [2\text{-}2]\\ [3\text{-}1]\\ [4]\\ [-]\end{matrix}$	E_4 is a multiple right and left D-P vector	$\Gamma_{414} = \Gamma_{411} = \Gamma_{424} = 0;$ $R_{,4} = 0.$
3	$C_{22} = 1$ $C_{31} = 1$	$[3\text{-}1] \otimes \begin{matrix}[2\text{-}1\text{-}1]\\ [2\text{-}2]\end{matrix}$	E_4 is a multiple right and left D–P vector, $C^{(2)} \neq 1/3$	$\Gamma_{414} = \Gamma_{411} = \Gamma_{424} = 0;$ $R_{,4} = 0.$
		$[3\text{-}1] \otimes \begin{matrix}[3\text{-}1]\\ [4]\\ [-]\end{matrix}$	E_4 is a multiple right and left D–P vector, $C^{(2)} \neq 1/2$	$\Gamma_{414} = \Gamma_{411} = \Gamma_{424} = 0;$ $R_{,4} = 0.$
4	$C_{22} = 1$ $C_{31} = 1$	$\begin{matrix}[4]\\ [-]\end{matrix} \otimes [A]$	E_4 is a multiple left D–P vector, $R_{,4} = 0$	$\Gamma_{414} = \Gamma_{411} = \Gamma_{424} = 0;$ $R_{,1} = 0 \Rightarrow \Gamma_{421} = 0.$
		$[3\text{-}1] \otimes [A]$	E_4 is a multiple left D–P vector, $R_{,4} = 0, C^{(2)} \neq 1/2$	$\Gamma_{414} = \Gamma_{411} = \Gamma_{424} = 0;$ $R_{,1} = 0 \Rightarrow \Gamma_{421} = 0.$
5	$C_{22} = 1$ $C_{31} = 1$	$[A] \otimes [3\text{-}1]$	E_4 is a multiple right D–P vector, $R_{,1} = 0$	$\Gamma_{414} = \Gamma_{411}+\Gamma_{424} = 0;$ $R_{,4} = 0;$ $\Gamma_{412} = 0 \Leftrightarrow \Gamma_{411} = 0;$
		$[A] \otimes \begin{matrix}[4]\\ [-]\end{matrix}$	E_4 is a multiple right D–P vector, $R_{,1} = 0$	$\Gamma_{414} = \Gamma_{411}+\Gamma_{424} = \Gamma_{412} = 0;$ $R_{,4} = 0.$
6	$C_{22} = 1$ $C_{31} = 1$	$[A] \otimes [B]$	—	$R_{,4} 0 \Leftrightarrow \Gamma_{411}+\Gamma_{424} = 0.$
7	$C_{22} = 1$ $C_{31} = 1$	$[A] \otimes [B]$	$\Gamma_{414} = \Gamma_{411} = 0$	E_4 is a multiple right D-P vector.

Sub-type $[4N]_4{}^b$ (complex). All propositions are obtained from those concerning $[4N]_4{}^a$ by interchanging right $\leftrightarrow$ left; $1 \leftrightarrow 2$; $[A] \otimes [B] \leftrightarrow [B] \otimes [A]$; $C^{(a)} \rightarrow \bar{C}^{(a)}$. $[4N]_4{}^a$ and $[4N]_4{}^b$ in the real space time do not exist.

TABLE II

Type $[4N]_3$ (complex)

1	$C_{31} = -\frac{i}{\sqrt{2}}$ $C_{32} = \frac{i}{\sqrt{2}}$ (other vanish)	Any type of $C_{ABCD} \otimes \bar{C}_{\dot{A}\dot{B}\dot{C}\dot{D}}$	E_4 is a multiple left D–P vector	$\Gamma_{424} = 0$.
2	$C_{31} = -\frac{i}{\sqrt{2}}$ $C_{32} = \frac{i}{\sqrt{2}}$	[2–1–1] [2–2] $\otimes$ [A]	E_4 is a multiple left D–P vector	$\Gamma_{424} = \Gamma_{422} = 0$; $R_{,4} = 0 \Rightarrow \Gamma_{414} = 0$.
3	$C_{31} = -\frac{i}{\sqrt{2}}$ $C_{32} = \frac{i}{\sqrt{2}}$	[4] [-] $\otimes$ [A]	E_4 is a multiple left D–P vector, $R_{,1} = R_{,2} = 0$	$\Gamma_{424} = \Gamma_{414} = \Gamma_{411} = \Gamma_{421} = 0$; $\Gamma_{412} = \frac{1}{2}(\Gamma_{124}+\Gamma_{344})$; $\Gamma_{422} = -\frac{1}{2}(3\Gamma_{214}+\Gamma_{344})$; $R_{,4} = 0$.
4	$C_{31} = -\frac{i}{\sqrt{2}}$ $C_{32} = \frac{i}{\sqrt{2}}$	[3–1] $\otimes$ [A]	E_4 is a multiple left D–P vector	$\Gamma_{424} = \Gamma_{414} = 0$; $R_{,4} = 0$; $R_{,2} = 0$ and $\Gamma_{421} = 0 \Rightarrow \Gamma_{422} = 0$ and $\Gamma_{412} = \Gamma_{214}+\Gamma_{344}$; $R_{,2} = 0$ and $\Gamma_{422} = 0 \Rightarrow \Gamma_{421} = 0$ and $\Gamma_{412} = \Gamma_{214}+\Gamma_{344}$.
5	$C_{31} = -\frac{i}{\sqrt{2}}$ $C_{32} = \frac{i}{\sqrt{2}}$	[A] $\otimes$ [B]	—	$R_{,4} = 0 \Leftrightarrow \Gamma_{414}-\Gamma_{424} = 0$.
6	$C_{31} = -\frac{i}{\sqrt{2}}$ $C_{32} = \frac{i}{\sqrt{2}}$	[A] $\otimes$ [B]	$\Gamma_{424} = \Gamma_{422} = 0$	E_4 is multiple left D–P vector.

Analogous propositions concerning the right D–P vectors are obtained by interchanging: $2 \leftrightarrow 1$, left $\rightarrow$ right, $[A] \otimes [B] \leftrightarrow [B] \otimes [A]$.

TABLE III

Type $[4N]_3$ (real space-time)

1	$C_{31} = -\frac{i}{\sqrt{2}}$ $C_{32} = \frac{i}{\sqrt{2}}$ (other vanish)	$[A]$	E_4 is a multiple D–P vector	$\Gamma_{424} = 0$; $R_{,4} = 0$.
2	$C_{31} = -\frac{i}{\sqrt{2}}$ $C_{32} = \frac{i}{\sqrt{2}}$	[2–1–1] [2–2]	E_4 is a multiple D–P vector	$\Gamma_{424} = \Gamma_{422} = 0$; $R_{,4} = 0$.
3	$C_{31} = -\frac{i}{\sqrt{2}}$ $C_{32} = \frac{i}{\sqrt{2}}$	[4]	E_4 is a multiple D–P vector and $R_{,1} = 0$	$\Gamma_{424} = \Gamma_{422} = \Gamma_{421} = \Gamma_{124}$ $= \Gamma_{434} = 0$; $R_{,4} = 0$.
4	$C_{31} = -\frac{i}{\sqrt{2}}$ $C_{32} = \frac{i}{\sqrt{2}}$	[3–1]	E_4 is a multiple D–P vector, $R_{,1} = 0$	$\Gamma_{424} = 0$; $R_{,4} = 0$; $\Gamma_{421} = 0 \Rightarrow \Gamma_{422} = \Gamma_{124} = \Gamma_{434}$ $= 0$; $\Gamma_{422} = 0 \Rightarrow \Gamma_{421} = \Gamma_{124} = \Gamma_{434}$ $= 0$.
5	$C_{31} = -\frac{i}{\sqrt{2}}$ $C_{32} = \frac{i}{\sqrt{2}}$	$[A]$	—	$R_{,4} = 0 \Leftrightarrow \Gamma_{414} = \Gamma_{424}$.
6	$C_{31} = -\frac{i}{\sqrt{2}}$ $C_{32} = \frac{i}{\sqrt{2}}$	$[A]$	$\Gamma_{424} = \Gamma_{422} = 0$	E_4 is a multiple D–P vector.

TABLE IV

Type $[C_1-3N]_4$ (complex)

1	$C_{11}=-2N$ $C_{22}=-2N$ $C_{12}=-N$ $C_{31}=-\frac{i}{\sqrt{2}}$ $C_{32}=\frac{i}{\sqrt{2}}$ $C_{34}=N$ (other compo- nents vanish)	$[A]\otimes[B]$	E_4 is a multiple left and right D–P vector $\frac{C^{(3)}+\bar{C}^{(3)}}{2}\neq\frac{3C^{(3)}\cdot\bar{C}^{(3)}}{4N}$	$\Gamma_{424}=\Gamma_{414}=0.$
2	$C_{11}=-2N$ $C_{22}=-2N$ $C_{12}=-N$ $C_{31}=-\frac{i}{\sqrt{2}}$ $C_{32}=\frac{i}{\sqrt{2}}$ $C_{34}=N$	[3–1] [4] ⊗ [A] [–]	E_4 is a multiple left D–P vector	$\Gamma_{424}+\Gamma_{414}=0.$
		[3–1] [A] ⊗ [4] [–]	E_4 is a multiple right D–P vector	$\Gamma_{424}+\Gamma_{414}=0.$
3	$C_{11}=-2N$ $C_{22}=-2N$ $C_{12}=-N$ $C_{31}=-\frac{i}{\sqrt{2}}$ $C_{32}=\frac{i}{\sqrt{2}}$ $C_{34}=N$	[4] ⊗ [4] [–] [–]	E_4 is a multiple right and left D–P vector	$\Gamma_{424}=\Gamma_{414}=0.$
			E_4 is a multiple right and left D–P vector, $R_{,4}=0$	$\Gamma_{424}=\Gamma_{414}=0;$ $\Gamma_{412}=0\Leftrightarrow\Gamma_{422}=0;$ $\Gamma_{421}=0\Leftrightarrow\Gamma_{411}=0.$
4	$C_{11}=-2N$ $C_{22}=-2N$ $C_{12}=-N$ $C_{31}=-\frac{i}{\sqrt{2}}$ $C_{32}=\frac{i}{\sqrt{2}}$ $C_{34}=N$	[3–1] [3–1] ⊗ [4] [–] or [3–1] [4] ⊗ [3–1] [–]	E_4 is a multiple right and left D–P vector, $C^{(2)}-\bar{C}^{(2)}\neq-\sqrt{2}\,i$	$\Gamma_{424}=\Gamma_{414}=0.$
			E_4 is a multiple right and left D–P vector, $C^{(2)}-\bar{C}^{(2)}\neq-\sqrt{2}\,i,$ $R_{,4}=0$	$\Gamma_{424}=\Gamma_{414}=0;$ $\Gamma_{412}=0\Leftrightarrow\Gamma_{422}=0;$ $\Gamma_{421}=0\Leftrightarrow\Gamma_{411}=0.$

TABLE V

Type $[S_1-3N]_4$ (real)

1	$C_{11}=-2N$ $C_{22}=-2N$ $C_{12}=-N$ $C_{34}=N$ $C_{31}=-\frac{i}{\sqrt{2}}$ $C_{32}=\frac{i}{\sqrt{2}}$ (other components vanish)	[A]	E_4 is a multiple D–P vector and $\mathrm{Re}\, C^{(3)} \neq 3\lvert C^{(3)}\rvert^2/4N$	$\Gamma_{424}=0.$
2	$C_{11}=-2N$ $C_{22}=-2N$ $C_{12}=-N$ $C_{34}=N$ $C_{31}=-\frac{i}{\sqrt{2}}$ $C_{32}=\frac{i}{\sqrt{2}}$	[3–1] [4] [–]	E_4 is a multiple D–P vector	$\mathrm{Re}\, \Gamma_{424}=0.$
3	$C_{11}=-2N$ $C_{22}=-2N$ $C_{12}=-N$ $C_{34}=N$ $C_{31}=-\frac{i}{\sqrt{2}}$ $C_{32}=\frac{i}{\sqrt{2}}$	[4] [–]	E_4 is a multiple D–P vector	$\Gamma_{424}=0.$
			E_4 is a multiple D–P vector and $R_{,4}=0$	$\Gamma_{424}=0;$ $\Gamma_{412}=0 \Leftrightarrow \Gamma_{422}=0.$
4	$C_{11}=-2N$ $C_{22}=-2N$ $C_{12}=-N$ $C_{34}=N$ $C_{31}=-\frac{i}{\sqrt{2}}$ $C_{32}=\frac{i}{\sqrt{2}}$	[3–1]	E_4 is a multiple D–P vector, $\mathrm{Im}\, C^{(2)} \neq -\frac{1}{\sqrt{2}}$	$\Gamma_{424}=0.$
			E_4 is a multiple D–P vector, $\mathrm{Im}\, C^{(2)} \neq -\frac{1}{\sqrt{2}}$ $R_{,4}=0$	$\Gamma_{424}=0;$ $\Gamma_{412}=0 \Leftrightarrow \Gamma_{422}=0.$

TABLE VI

Sub-type ${}^{(2)}[4N]_2{}^{a}$ (complex)

1	$C_{32} = 1$ (other components vanish)	$\begin{matrix}[2\text{–}1\text{–}1]\\ [2\text{–}2]\end{matrix} \otimes [A]$	E_4 is a multiple left D–P vector	$\Gamma_{424} = \Gamma_{422} = 0.$
		$[3\text{–}1] \otimes [A]$	E_4 is a multiple left D–P vector	$\Gamma_{424} = \Gamma_{414} = 0;$ $R_{,4} = 0.$
		$[4] \otimes [A]$	E_4 is a multiple left D–P vector	$\Gamma_{424} = \Gamma_{414} = \Gamma_{411} = 0;$ $R_{,4} = R_{,1} = 0.$
		$[-] \otimes [A]$	E_4 is a multiple left D–P vector	$\Gamma_{424} = \Gamma_{414} = \Gamma_{411} = \Gamma_{311} = 0;$ $R_{,4} = R_{,1} = 0.$
2	$C_{32} = 1$	$[3\text{–}1] \otimes [A]$	E_4 is a multiple left D–P vector, $R_{,2} = 0$	$\Gamma_{424} = \Gamma_{414} = 0;$ $R_{,4} = 0;$ $\Gamma_{422} = 0 \Leftrightarrow \Gamma_{421} = 0.$
3	$C_{32} = 1$	$[4] \otimes [A]$	E_4 is a multiple left D–P vector, $R_{,2} = R_{,3} = 0$	$\Gamma_{424} = \Gamma_{414} = \Gamma_{411} = \Gamma_{421} = 0;$ $\Gamma_{422} = 0 \Leftrightarrow \Gamma_{314} = 0;$ $R = \text{const.}$
4	$C_{32} = 1$	$[-] \otimes [A]$	$R_{,2} = R_{,3} = 0$	$\Gamma_{424} = \Gamma_{414} = \Gamma_{411} = \Gamma_{311} = \Gamma_{421} = \Gamma_{314} = 0;$ $R = \text{const.}$
5	$C_{32} = 1$	$[A] \otimes [B]$	—	$R_{,1} = 0 \Leftrightarrow \Gamma_{411} = 0;$ $R_{,4} = 0 \Leftrightarrow \Gamma_{414} = 0.$
6	$C_{32} = 1$	$[A] \otimes \begin{matrix}[2\text{–}1\text{–}1]\\ [2\text{–}2]\\ [4]\\ [-]\end{matrix}$	E_4 is a multiple right D–P vector	$\Gamma_{414} = \Gamma_{411} = 0;$ $R_{,4} = R_{,1} = 0.$
		$[A] \otimes [3\text{–}1]$	E_4 is a multiple right D–P vector, $\bar{C}^{(2)} \neq \pm\frac{1}{3}$	$\Gamma_{414} = \Gamma_{411} = 0;$ $R_{,4} = R_{,1} = 0.$
7	$C_{32} = 1$	$[4] \otimes \begin{matrix}[A]\\ [-]\end{matrix}$	E_4 is a multiple right D–P vector	$\Gamma_{41} = 0;$ $R = 0.$
8	$C_{32} = 1$	$[-] \otimes \begin{matrix}[4]\\ [-]\end{matrix}$	E_4 is a multiple right D–P vector, $R_{,2} = R_{,3} = 0$	$\Gamma_{41} = 0;$ $\Gamma_{424} = \Gamma_{421} = \Gamma_{311} = \Gamma_{314} = \Gamma_{214}+\Gamma_{344} = \Gamma_{211}+\Gamma_{341} = 0;$ $R = 0.$
9	$C_{32} = 1$	$[A] \otimes [B]$	$\Gamma_{424} = \Gamma_{422} = 0$	E_4 is a multiple left D–P vector.
10	$C_{32} = 1$	$[A] \otimes [B]$	$\Gamma_{414} = \Gamma_{411} = 0$	E_4 is a multiple right D–P vector.
11	$C_{32} = 1$	$[A] \otimes [B]$	$R_{,4} = R_{,1} = 0$	E_4 is a multiple right D–P vector.
12	$C_{32} = 1$	$\begin{matrix}[4]\\ [-]\end{matrix} \otimes [A]$	E_4 is a multiple left D–P vector	E_4 is also a multiple right D–P vector.

Sub-type$^{(2)}[4N]_2{}^{b}$ (complex). All the propositions concerning this sub-type can be obtained from those concerning $^{(2)}[4N]_2{}^{a}$ by interchanging: $2 \leftrightarrow 1$, left $\leftrightarrow$ right, $[A] \otimes [B] \leftrightarrow [B] \otimes [A]$, $\bar{C}^{(a)} \to C^{(a)}$. $^{(2)}[4N]_2{}^{a}$ and $^{(2)}[4N]_2{}^{b}$ in the real space-time do not exist.

TABLE VII

Sub-type $[2N_1-2N]_4{}^{a}$ (complex)

1	$C_{33} = 1$ $C_{22} = 1$ $C_{34} = N$ $C_{12} = -N$ (other components vanish)	$[3\text{–}1]$ $[A] \otimes [4]$ $[-]$	E_4 is a multiple right D–P vector	$\Gamma_{414} = \Gamma_{411} = 0$.
		$[A] \otimes \begin{matrix}[2\text{–}1\text{–}1]\\ [2\text{–}2]\end{matrix}$	E_4 is a multiple right D–P vector	$\bar{C}^{(3)} \neq \frac{2}{3}N \Rightarrow \Gamma_{414} = 0$; $\bar{C}^{(3)} \neq -\frac{2}{3}N \Rightarrow \Gamma_{411} = 0$.
2	$C_{33} = 1$ $C_{22} = 1$ $C_{22} = N$ $C_{12} = -N$	$[3\text{–}1]$ $[A] \otimes [4]$ $[-]$	E_4 is a multiple right D–P vector, $R_{,4} = 0$	$\Gamma_{414} = \Gamma_{411} = \Gamma_{412} = 0$; $N_{,4} = 0 \Leftrightarrow \Gamma_{421} = 0$.
			E_4 is a multiple right D–P vector, $R_{,4} = R_{,1} = 0$	$\Gamma_{414} = \Gamma_{411} = \Gamma_{412}$ $= \Gamma_{413} = 0$; $N_{,4} = 0 \Leftrightarrow \Gamma_{421} = 0$; $N_{,1} = 0 \Leftrightarrow \Gamma_{314} = 0$; $R = 0$.
3	$C_{33} = 1$ $C_{22} = 1$ $C_{34} = N$ $C_{12} = -N$	$[3\text{–}1]\ \ [3\text{–}1]$ $[4] \otimes [4]$ $[-]\ \ [-]$	E_4 is a multiple left and right D–P vector, $R_{,4} = R_{,1} = 0$	$\Gamma_{41} = 0$; $\Gamma_{424} = \Gamma_{421} = 0$; $\Gamma_{422} = 0 \Leftrightarrow \Gamma_{124} = 0$; $R = 0$; $N_{,4} = 0$.
		$[3\text{–}1]$ $[4] \otimes [4]$ $[-]\ \ [-]$	E_4 is a multiple left and right D–P vector, $R_{,4} = R_{,1} = 0$	$\Gamma_{41} = 0$; $\Gamma_{424} = \Gamma_{421} = 0$; $\Gamma_{422} = 0 \Leftrightarrow \Gamma_{124} = 0$; $R = 0$, $N_{,4} = 0$; $\Gamma_{314} = 0$; $N_{,1} = 0$.
4	$C_{33} = 1$ $C_{22} = 1$ $C_{34} = N$ $C_{12} = -N$	$[-] \otimes \begin{matrix}[4]\\ [-]\end{matrix}$	E_4 is a multiple right D–P vector, $R_{,1} = R_{,4} = 0$	$\Gamma_{41} = 0$; $\Gamma_{424} = \Gamma_{421} = \Gamma_{314}$ $= \Gamma_{311} = \Gamma_{312} - \Gamma_{321}$ $= \Gamma_{423} - \Gamma_{324} = 0$; $\Gamma_{124} = 0 \Leftrightarrow \Gamma_{422} = 0$; $R = 0$; $N_{,1} = N_{,4} = 0$.
5	$C_{33} = 1$ $C_{22} = 1$ $C_{34} = N$ $C_{12} = -N$	$[A] \otimes [B]$	$\Gamma_{414} = \Gamma_{411} = 0$	E_4 is a multiple right D–P vector.

Sub-type $[2N_1-2N]_4{}^{b}$ (complex). All the propositions concerning this type are obtained from those concerning $[2N_1-2N]_4{}^{a}$ (complex) interchanging: $2 \leftrightarrow 1$, left $\leftrightarrow$ right, $\bar{C}^{(a)} \to C^{(a)}$, $[A] \otimes [B] \leftrightarrow [B] \otimes [A]$.$[2N_1-2N]_4$ real does not exist.

TABLE VIII

Type $^{(3)}[4N]_2$ (complex)

1	$C_{33} = 1$ (other components vanish)	$[A] \otimes [B]$	E_4 is a multiple left D–P vector	$\Gamma_{424} = 0$; $R_{,4} = R_{,2} = 0$.
		[2–1–1] [2–2] $\otimes$ [A] [3–1]	E_4 is a multiple left D–P vector	$\Gamma_{424} = \Gamma_{422} = 0$; $R_{,4} = R_{,2} = 0$.
		$[-] \otimes [A]$	E_4 is a multiple left D–P vector	$\Gamma_{424} = \Gamma_{414} = \Gamma_{411} = 0$; $R_{,4} = R_{,2} = R_{,1} = 0$; $\Gamma_{413} = 0 \Leftrightarrow \Gamma_{341} = 0$.
		$[4] \otimes [A]$	E_4 is a multiple left D–P vector	$\Gamma_{424} = \Gamma_{414} = 0$; $R_{,4} = R_{,2} = R_{,1} = 0$.
2	$C_{33} = 1$	$[-] \otimes [A]$	$R_{,3} = 0$	$\Gamma_{424} = \Gamma_{414} = \Gamma_{411}$ $\Gamma_{421} = 0$; $\Gamma_{413} = 0 \Leftrightarrow \Gamma_{341} = 0$; $R = \text{const.}$
		$[-] \otimes [-]$	—	$\Gamma_{424} = \Gamma_{422} = \Gamma_{414}$ $= \Gamma_{411} = \Gamma_{421} - \Gamma_{412} = 0$; $\Gamma_{413} = 0 \Leftrightarrow \Gamma_{341} = 0$; $\Gamma_{423} = 0 \Leftrightarrow \Gamma_{342} = 0$; $\Gamma_{421} + \Gamma_{412} = 0 \Leftrightarrow \Gamma_{434} = 0$; $R_{,4} = R_{,2} = R_{,1} = 0$; $R_{,3} = 0 \Rightarrow (R = \text{const}$, $\Gamma_{421} = \Gamma_{412} = \Gamma_{434} = 0)$.
3	$C_{33} = 1$	$[A] \otimes [B]$	—	$R_{,4} = 0$; $R_{,1} = 0 \Leftrightarrow \Gamma_{414} = 0$; $R_{,2} = 0 \Leftrightarrow \Gamma_{424} = 0$.
4	$C_{33} = 1$	$[A] \otimes [B]$	$\Gamma_{424} = \Gamma_{422} = 0$	E_4 is a multiple left D–P vector.
5	$C_{33} = 1$	$[A] \otimes [4]$	E_3 is a multiple right D–P vector,	E_4 is a multiple left D–P vector, $\Gamma_{424} = \Gamma_{422} = 0$.
		$[A] \otimes [3–1]$	E_3 is a multiple right D–P vector, $R_{,2} = 0$	E_4 is a multiple left D–P vector, $\Gamma_{424} = \Gamma_{411} = 0$.
		$[A] \otimes$ [2–1–1] [2–2]	E_3 is a multiple right D–P vector, $\Gamma_{322} = 0$, $R_{,2} = 0$	E_4 is a multiple left D–P vector, $\Gamma_{424} = \Gamma_{422} = 0$.
6	$C_{33} = 1$	$[A] \otimes [-]$	—	E_4 is a multiple left D–P vector, $\Gamma_{424} = \Gamma_{422} = 0$.

The remaining propositions are found by interchanging: $2 \leftrightarrow 1$, left $\leftrightarrow$ right, $[A] \otimes [B] \leftrightarrow [B] \otimes [A]$.

TABLE IX

Type $[4N]_2$ (real)

1	$C_{33} = \pm 1$ (other components vanish)	$[A]$	E_4 is a multiple D–P vector	$\Gamma_{424} = 0$; $R_{,4} = R_{,2} = 0$.
		[2–1–1] [2–2] [3–1] [–]	E_4 is a multiple D–P vector	$\Gamma_{424} = \Gamma_{422} = 0$; $R_{,4} = R_{,2} = 0$.
2	$C_{33} = \pm 1$	[–]	—	$\Gamma_{424} = \Gamma_{422} = \Gamma_{421} - \Gamma_{412} = 0$; $\Gamma_{423} = 0 \Leftrightarrow \Gamma_{342} = 0$; $\Gamma_{412} + \Gamma_{421} = 0 \Leftrightarrow \Gamma_{434} = 0$; $R_{,4} = R_{,2} = 0$; $R_{,3} = 0 \Rightarrow (\Gamma_{421} = \Gamma_{434} = 0$ and $R = \text{const})$.
3	$C_{33} = \pm 1$	$[A]$	—	$R_{,4} = 0$; $R_{,2} = 0 \Leftrightarrow \Gamma_{424} = 0$.
4	$C_{33} = \pm 1$	$[A]$	$\Gamma_{424} = \Gamma_{422} = 0$	E_4 is a multiple D–P vector.
5	$C_{33} = \pm 1$	[4]	—	E_4 is a 4-fold D–P vector, $\Gamma_{424} = \Gamma_{422} = 0$.
		[3–1]	$R_{,2} = 0$	E_4 is a 3-fold D–P vector.
		[2–1–1] [2–2]	$R_{,2} = 0$, E_3 is a multiple D–P vector and is shear-free	E_4 is a 2-fold D–P vector.

TABLE X

Type $[2N_1-2N]_{(1-2)}$ (complex)

1	$C_{33}=1$ $C_{34}=N$ $C_{12}=-N$ (other components vanish)	[3–1] [4] ⊗ [A] [–]	E_4 is a multiple left D–P vector	$\Gamma_{424}=\Gamma_{422}=0$; $N_{,4}=(2\Gamma_{412}-\Gamma_{421})N$.
		[2–1–1] [2–2] ⊗ [A]	E_4 is a multiple left D–P vector	$C^{(3)}\neq\frac{2}{3}N\Rightarrow\Gamma_{424}=0$; $C^{(3)}\neq-\frac{2}{3}N\Rightarrow\Gamma_{422}=0$.
2	$C_{33}=1$ $C_{34}=N$ $C_{12}=-N$	[3–1] [4] ⊗ [A] [–]	E_4 is a multiple left D–P vector, $R_{,4}=0$	$\Gamma_{424}=\Gamma_{422}=\Gamma_{421}=0$; $N_{,4}=0\Leftrightarrow\Gamma_{412}=0$.
			E_4 is a multiple left D–P vector, $R_{,4}=R_{,2}=0$	$\Gamma_{424}=\Gamma_{422}=\Gamma_{421}=\Gamma_{423}=0$; $N_{,4}=0\Leftrightarrow\Gamma_{412}=0$; $R=0$; $N_{,2}=0\Leftrightarrow\Gamma_{324}=0$.
3	$C_{33}=1$ $C_{34}=N$ $C_{12}=-N$	[4] [–] ⊗ [4] [–]	E_4 is a multiple left and right D–P vector	$\Gamma_{424}=\Gamma_{422}=\Gamma_{414}=\Gamma_{411}$ $=\Gamma_{421}-\Gamma_{412}=\Gamma_{423}-\Gamma_{324}$ $=\Gamma_{413}-\Gamma_{314}=\Gamma_{321}-\Gamma_{312}=0$.
			E_4 is a multiple left and right D–P vector, $R_{,4}=R_{,2}=R_{,1}=0$	$\Gamma_{424}=\Gamma_{422}=\Gamma_{414}=\Gamma_{411}$ $=\Gamma_{321}-\Gamma_{312}=\Gamma_{421}=\Gamma_{412}$ $=\Gamma_{423}=\Gamma_{413}=\Gamma_{324}=\Gamma_{314}$ $=0$; $N_{,4}=N_{,2}=N_{,1}=0$; $R=0$.
4	$C_{33}=1$ $C_{34}=N$ $C_{12}=-N$	[4] [–] ⊗ [–]	E_4 is a multiple left D–P vector	$\Gamma_{424}=\Gamma_{422}=\Gamma_{414}=\Gamma_{411}$ $=\Gamma_{421}-\Gamma_{412}=\Gamma_{423}-\Gamma_{324}$ $=\Gamma_{413}-\Gamma_{314}=\Gamma_{321}-\Gamma_{312}$ $=\Gamma_{322}=0$.
			E_4 is a multiple left D–P vector, $R_{,4}=R_{,2}-R_{,1}=0$	$\Gamma_{424}=\Gamma_{422}=\Gamma_{414}=\Gamma_{411}$ $=\Gamma_{321}-\Gamma_{312}=\Gamma_{421}=\Gamma_{412}$ $=\Gamma_{423}=\Gamma_{413}=\Gamma_{324}$ $=\Gamma_{314}=0$; $\Gamma_{323}=0\Leftrightarrow\Gamma_{342}=0$; $R=0$; $N_{,4}=N_{,2}=N_{,1}=0$; $N_{,3}=0\Leftrightarrow\Gamma_{321}=0$.
5	$C_{33}=1$ $C_{34}=N$ $C_{12}=-N$	[A] ⊗ [B]	$\Gamma_{424}=\Gamma_{422}=0$	E_4 is a multiple left D–P vector.

Other propositions are obtained by interchanging: $2\leftrightarrow 1$, left ↔ right, [A] ⊗ [B] ↔ [B] ⊗ [A], $C^{(a)}\to\bar{C}^{(a)}$.

TABLE XI

Type $[2S_1-2N]_{(1-2)}$ (real)

1	$C_{33} = \pm 1$ $C_{34} = N$ $C_{12} = -N$ (other components vanish)	[3–1] [4] [–]	E_4 is a multiple D–P vector	$\Gamma_{424} = \Gamma_{422} = 0$; $\Gamma_{421}-\Gamma_{412} = 0$.
		[2–1–1] [2–2]	E_4 is a multiple D–P vector	$C^{(3)} \neq \frac{2}{3}N \Rightarrow \Gamma_{424} = 0$; $C^{(3)} \neq -\frac{2}{3}N \Rightarrow \Gamma_{422} = 0$.
2	$C_{33} = \pm 1$ $C_{34} = N$ $C_{12} = -N$	[3–1] [4] [–]	E_4 is a multiple D–P vector, $R_{,4} = 0$	$\Gamma_{424} = \Gamma_{422} = \Gamma_{421} = 0$; $N_{,4} = 0$; $R_{,2} = 0 \Rightarrow (\Gamma_{423} = 0,\ R = 0,$ $N_{,2} = 0 \Leftrightarrow \Gamma_{324} = 0)$.
3	$C_{33} = \pm 1$ $C_{34} = N$ $C_{12} = -N$	[4] [–]	E_4 is a multiple D–P vector	$\Gamma_{424} = \Gamma_{422} = \Gamma_{412}-\Gamma_{421}$ $= \Gamma_{423}-\Gamma_{324} = \Gamma_{321}-\Gamma_{312} = 0$.
			E_4 is a multiple D–P vector, $R_{,4} = R_{,2} = 0$	$\Gamma_{424} = \Gamma_{422} = \Gamma_{321}-\Gamma_{312} = \Gamma_{421}$ $= \Gamma_{423} = \Gamma_{324} = 0$; $R = N_{,4} = N_{,2} = 0$.
4	$C_{33} = \pm 1$ $C_{34} = N$ $C_{12} = -N$	[–]	—	$\Gamma_{424} = \Gamma_{422} = \Gamma_{421}-\Gamma_{412}$ $= \Gamma_{423}-\Gamma_{324} = \Gamma_{322} = \Gamma_{321}$ $-\Gamma_{312} = 0$.
			$R_{,4} = R_{,2} = 0$	$\Gamma_{424} = \Gamma_{422} = \Gamma_{421} = \Gamma_{423}$ $= \Gamma_{324} = \Gamma_{322} = \Gamma_{321}$ $-\Gamma_{312} = 0$; $\Gamma_{323} = 0 \Leftrightarrow \Gamma_{342} = 0$; $R = 0;\ N_{,4} = N_{,2} = 0$; $N_{,3} = 0 \Leftrightarrow \Gamma_{321} = 0$.
5	$C_{33} = \pm 1$ $C_{34} = N$ $C_{12} = -N$	[A]	$\Gamma_{424} = \Gamma_{422} = 0$	E_4 is a multiple D–P vector.

TABLE XII

Type $[C_1-3N]_3$ (complex)

1	$C_{11}=-2N$ $C_{22}=-2N$ $C_{33}=1$ $C_{12}=-N$ $C_{34}=N$ (other compo-nents vanish)	$[A]\otimes[B]$	E_4 is a multiple left and right D–P vector, $\frac{C^{(3)}+\overline{C}^{(3)}}{2}\neq\frac{3C^{(3)}\cdot\overline{C}^{(3)}}{4N}$	$\Gamma_{424}=\Gamma_{414}=0.$
2	$C_{11}=-2N$ $C_{22}=-2N$ $C_{33}=1$ $C_{12}=-N$ $C_{34}=N$	$\begin{matrix}[3-1]\\ [4]\\ [-]\end{matrix}\otimes[A]$ or $\left([A]\otimes\begin{matrix}[3-1]\\ [4]\\ [-]\end{matrix}\right)$	E_4 is a multiple left or (right) D–P vector	$\Gamma_{414}+\Gamma_{424}=0.$
3	$C_{11}=-2N$ $C_{22}=-2N$ $C_{33}=1$ $C_{12}=-N$ $C_{34}=N$	$\begin{matrix}[3-1]\\ [4]\\ [-]\end{matrix}\otimes[3-1]$ or $[3-1]\otimes\begin{matrix}[3-1]\\ [4]\\ [-]\end{matrix}$	E_4 is a multiple right and left D–P vector, $C^{(2)}-\overline{C}^{(2)}\neq 0$	$\Gamma_{424}=\Gamma_{414}=0;$ $R_{,4}=0\Leftrightarrow N_{,4}=0.$
			E_4 is a multiple right and left D P vector, $C^{(2)}-\overline{C}^{(2)}\neq 0,$ $R_{,4}=0$	$\Gamma_{424}=\Gamma_{414}=0;\ N_{,4}=0;$ $\Gamma_{412}=0\Rightarrow\Gamma_{422}=\Gamma_{124}=0;$ $\Gamma_{422}=0\Rightarrow\Gamma_{412}=\Gamma_{124}=0;$ $\Gamma_{421}=0\Rightarrow\Gamma_{411}=\Gamma_{124}=0;$ $\Gamma_{411}=0\Rightarrow\Gamma_{421}=\Gamma_{124}=0$

TABLE XIII

Type $[S_1-3N]_3$ (real)

1	$C_{11}=-2N$ $C_{22}=-2N$ $C_{33}=\pm1$ $C_{12}=-N$ $C_{34}=N$ (other compo-nents vanish)	[2–1–1] [2–2]	E_4 is a multiple D–P vector, $\operatorname{Re}C^{(3)}\neq\frac{3\lvert C^{(3)}\rvert^2}{4N}$	$\Gamma_{424}=0.$
2	$C_{11}=-2N$ $C_{22}=-2N$ $C_{33}=\pm1$ $C_{12}=-N$ $C_{34}=N$	[3–1] [4] [–]	E_4 is a multiple D–P vector	$\Gamma_{424}+\Gamma_{414}=0.$
3	$C_{11}=-2N$ $C_{22}=-2N$ $C_{33}=\pm1$ $C_{12}=-N$ $C_{34}=N$	[3–1]	E_4 is a multiple D–P vector, $\operatorname{Im}C^{(2)}\neq 0$	$\Gamma_{424}=0;$ $R_{,4}=0\Leftrightarrow N_{,4}=0.$
			E_4 is a multiple D–P vector, $\operatorname{Im}C^{(2)}\neq 0,\ R_{,4}=0$	$\Gamma_{424}=0;$ $\Gamma_{421}=0\Rightarrow\Gamma_{422}=\Gamma_{124}=0;$ $\Gamma_{422}=0\Rightarrow\Gamma_{421}=\Gamma_{124}=0.$

TABLE XIV

Type $[C_1-C_2-2N]_4$ (complex)

1	$C_{11}=\frac{1}{2}(C_1-C_2)$ $C_{22}=\frac{1}{2}(C_1-C_2)$ $C_{33}=1$ $C_{12}=-N$ $C_{34}=N$ (other components vanish)	$[A]\otimes[B]$	E_4 is a multiple left and right D–P vector, $N\frac{C^{(3)}+\bar{C}^{(3)}}{2}\neq\frac{3C^{(3)}\bar{C}^{(3)}}{4}+\frac{3N^2+C_1C_2}{12}$	$\Gamma_{424}=\Gamma_{414}=0.$
2	$C_{11}=\frac{1}{2}(C_1-C_2)$ $C_{22}=\frac{1}{2}(C_1-C_2)$ $C_{33}=1$ $C_{12}=-N$ $C_{34}=N$	[3–1] [3–1] [4] ⊗ [4] [–] [–]	E_4 is a multiple left and right D–P vector	$\Gamma_{424}=\Gamma_{414}=0$; $R,_4=0\Leftrightarrow N,_4=0.$
			E_4 is a multiple left and right D–P vector, $R,_4=0$	$\Gamma_{424}=\Gamma_{414}=0$; $N,_4=0$; $N\cdot\Gamma_{421}=0\Leftrightarrow\Gamma_{411}=0$; $N\cdot\Gamma_{412}=0\Leftrightarrow\Gamma_{422}=0.$
3	$C_{11}=\frac{1}{2}(C_1-C_2)$ $C_{22}=\frac{1}{2}(C_1-C_2)$ $C_{33}=1$ $C_{12}=-N$ $C_{34}=N$	[3–1] [3–1] [4] ⊗ [4] [–] [–]	E_4 is a multiple left and right D–P vector, $N=0$	$\Gamma_{424}=\Gamma_{422}=\Gamma_{414}=\Gamma_{411}=0$; $R,_4=0.$
		[4] ⊗ [4] [–] [–]	E_4 is a multiple left and right D–P vector, $N=0$, $R,_2=R,_1=0$	$\Gamma_{424}=\Gamma_{422}=\Gamma_{414}=\Gamma_{411}=\Gamma_{423}=\Gamma_{413}=\Gamma_{324}=\Gamma_{314}=0$; $R,_4=0.$

TABLE XV

Type $[S_1-S_2-2N]_4$ (real)

1	$C_{11}=\frac{1}{2}(S_1-S_2)$ $C_{22}=\frac{1}{2}(S_1-S_2)$ $C_{33}=\pm1$ $C_{12}=-N$ $C_{34}=N$ (other components vanish)	$[A]$	E_4 is a multiple D–P vector, $N\cdot\mathrm{Re}\,C^{(3)}\neq\frac{3\lvert C^{(3)}\rvert^2}{4}+\frac{3N^2+S_1S_2}{12}$	$\Gamma_{424}=0.$
2	$C_{11}=\frac{1}{2}(S_1-S_2)$ $C_{22}=\frac{1}{2}(S_1-S_2)$ $C_{33}=\pm1$ $C_{12}=-N$ $C_{34}=N$	[3–1] [4] [–]	E_4 is a multiple D–P vector	$\Gamma_{424}=0$; $R,_4=0\Leftrightarrow N,_4=0.$
			E_4 is a multiple D–P vector, $R,_4=0$	$\Gamma_{424}=0$; $N,_4=0$; $N\cdot\Gamma_{421}=0\Leftrightarrow\Gamma_{422}=0.$
3	$C_{11}=\frac{1}{2}(S_1-S_2)$ $C_{22}=\frac{1}{2}(S_1-S_2)$ $C_{33}=\pm1$ $C_{12}=-N$ $C_{34}=N$	[3–1] [4] [–]	E_4 is a multiple D–P vector, $N=0$	$\Gamma_{424}=\Gamma_{422}=0$; $R,_4=0.$
		[4]	E_4 is a multiple D–P vector, $N=0$, $R,_2=0$	$\Gamma_{424}=\Gamma_{422}=\Gamma_{423}=\Gamma_{324}=0$; $R,_4=0.$

Type $[4N]_1$ (complex). See theorems 2.2 (t), 2.3 (t), 2.4 for $C_{ab}=0$. Type $[4T]_1$ (real). See theorems 2.5 (t), 2.6 for $C_{ab}=0$.

TABLE XVI

Type $[2N_1-2N]_2$ (complex)

1	$C_{12}=-N$ $C_{34}=N$ (other components vanish)	$[A]\otimes[B]$	E_4 is a multiple left D–P vector	$C^{(3)}\neq\frac{2}{3}N\Rightarrow\Gamma_{424}=0$; $C^{(3)}\neq-\frac{2}{3}N\Rightarrow\Gamma_{422}=0$.
		$\begin{matrix}[3\text{-}1]\\ [4]\\ [-]\end{matrix}\otimes[A]$	E_4 is a multiple left D–P vector	$\Gamma_{424}=\Gamma_{422}=0$.
2	$C_{12}=-N$ $C_{34}=N$	$\begin{matrix}[3\text{-}1]\\ [4]\\ [-]\end{matrix}\otimes[A]$	E_4 is a multiple left D–P vector	$R_{,4}=0\Leftrightarrow N_{,4}-2N\Gamma_{412}=0$; $R_{,2}=0\Leftrightarrow N_{,2}+2N\Gamma_{324}=0$.
			E_4 is a multiple left D–P vector, $R_{,4}=0$	$\Gamma_{424}=\Gamma_{422}=\Gamma_{421}=0$; $N_{,4}=0\Leftrightarrow\Gamma_{412}=0$.
			E_4 is a multiple left D–P vector, $R_{,4}=0$, $R_{,2}=0$	$\Gamma_{424}=\Gamma_{422}=\Gamma_{421}$ $=\Gamma_{423}=0$; $R=0$; $N_{,4}=0\Leftrightarrow\Gamma_{412}=0$; $N_{,2}=0\Leftrightarrow\Gamma_{324}=0$.
		$\begin{matrix}[3\text{-}1]\\ [4]\\ [-]\end{matrix}\otimes\begin{matrix}[3\text{-}1]\\ [4]\\ [-]\end{matrix}$	E_4 is a multiple left and right D–P vector	$R_{,4}=0\Leftrightarrow N_{,4}=0$.
3	$C_{12}=-N$ $C_{34}=N$	$\begin{matrix}[4]\\ [-]\end{matrix}\otimes[A]$	E_4 is a multiple left D–P vector, $R_{,4}=R_{,2}=0$	$\Gamma_{42}=0$; $R=0$; $\Gamma_{314}=\Gamma_{312}=0$; $N_{,4}+2N\Gamma_{413}=0$; $N_{,3}-2N\Gamma_{321}=0$.
		$\begin{matrix}[4]\\ [-]\end{matrix}\otimes\begin{matrix}[4]\\ [-]\end{matrix}$	E_4 is a multiple left and right D–P vector	$R_{,a}=0\Leftrightarrow N_{,a}=0$. $(a=1,2,3,4)$
4	$C_{12}=-N$ $C_{34}=N$	$\begin{matrix}[3\text{-}1]\\ [4]\\ [-]\end{matrix}\otimes[A]$	E_4 is a multiple left D-P vector, $R_{,2}=0$	$\Gamma_{424}=\Gamma_{422}=\Gamma_{423}=0$.
5	$C_{12}=-N$ $C_{34}=N$	$\begin{matrix}[4]\\ [-]\end{matrix}\otimes[A]$	E_4 is a multiple left D-P vector, $R_{,1}=0$	$\Gamma_{424}=\Gamma_{422}=\Gamma_{314}=0$.
6	$C_{12}=-N$ $C_{34}=N$	$\begin{matrix}[4]\\ [-]\end{matrix}\otimes[A]$	E_4 is a multiple left D-P vector, $R_{,3}=0$	$\Gamma_{424}=\Gamma_{422}=\Gamma_{312}=0$.
7	$C_{12}=-N$ $C_{34}=N$	$[-]\otimes[A]$	—	$\Gamma_{424}=\Gamma_{422}=\Gamma_{313}$ $=\Gamma_{311}=0$.
8	$C_{12}=-N$ $C_{34}=N$	$[A]\otimes[B]$	$\Gamma_{422}=\Gamma_{424}=0$	E_4 is a multiple left D–P vector.

The propositions concerning E_3 as a multiple D–P vector are found by interchanging $4\leftrightarrow3$, $2\leftrightarrow1$ (but $C^{(3)}\to C^{(3)}$). The propositions concerning the right D–P vectors are obtained from 1–8 by interchanging: $2\leftrightarrow1$, left $\leftrightarrow$ right, $C^{(a)}\to\bar{C}^{(a)}$), $[A]\otimes[B]\leftrightarrow[B]\otimes[A]$.

TABLE XVII

Type $[2S_1-2T]_2$ (real)

1	$C_{12}=-T$ $C_{34}=T$ (other components vanish)	$[A]$	E_4 is a multiple D–P vector	$C^{(3)} \neq \frac{2}{3}T \Rightarrow \Gamma_{424}=0$; $C^{(3)} \neq -\frac{2}{3}T \Rightarrow \Gamma_{422}=0$.
		[3–1] [4] [–]	E_4 is a multiple D–P vector	$\Gamma_{424}=\Gamma_{422}=0$.
2	$C_{12}=-T$ $C_{34}=T$	[3–1] [4] [–]	E_4 is a multiple D–P vector, $R,_4=0$	$\Gamma_{424}=\Gamma_{422}=\Gamma_{421}=0$; $T,_4=0$.
			E_4 is a multiple D–P vector, $R,_4=R,_2=0$	$\Gamma_{424}=\Gamma_{422}=\Gamma_{421}=\Gamma_{423}=0$; $T,_4=0$; $R=0$; $T,_2=0 \Leftrightarrow \Gamma_{324}=0$.
			E_4 is a multiple D–P vector	$R,_4=0 \Leftrightarrow T,_4=0$; $R,_2=0 \Leftrightarrow T,_2+2T\Gamma_{324}=0$.
3	$C_{12}=-T$ $C_{34}=T$	[4] [–]	E_4 is a multiple D–P vector, $R,_4=R,_2=0$	$\Gamma_{42}=0$; $\Gamma_{324}=\Gamma_{321}=0$; $R=0$; $T=$ const.
			E_4 is a multiple D–P vector	$R,_a=0 \Leftrightarrow T,_a=0$. $(a=1, 2, 3, 4)$
4	$C_{12}=-T$ $C_{34}=T$	[3–1] [4] [–]	E_4 is a multiple D–P vector, $R,_2=0$	$\Gamma_{424}=\Gamma_{422}=\Gamma_{423}=0$.
5	$C_{12}=-T$ $C_{34}=T$	[4] [–]	E_4 is a multiple D–P vector, $R,_2=0$	$\Gamma_{424}=\Gamma_{422}=\Gamma_{423}=\Gamma_{314}=0$.
6	$C_{12}=-T$ $C_{34}=T$	[4] [–]	E_4 is a multiple D–P vector, $R,_3=0$	$\Gamma_{424}=\Gamma_{422}=\Gamma_{312}=0$.
7	$C_{12}=-T$ $C_{34}=T$	[–]	—	$\Gamma_{424}=\Gamma_{422}=\Gamma_{313}=\Gamma_{311}=0$.
8	$C_{12}=-T$ $C_{34}=T$	$[A]$	$\Gamma_{424}=\Gamma_{422}=0$	E_4 is a multiple D–P vector.

Similar propositions concerning E_3 as a multiple D–P vector are obtained by interchanging: $2 \leftrightarrow 1$, $4 \leftrightarrow 3$ (but $C^{(3)} \to C^{(3)}$).

TABLE XVIII

Type $[C_1-3N]_2$ (complex)

1	$C_{11}=-2N$ $C_{22}=-2N$ $C_{12}=-N$ $C_{34}=N$ (other components vanish)	$[A]\otimes[B]$	E_4 is a multiple left and right D–P vector, $\frac{C^{(3)}+\bar{C}^{(3)}}{2}\neq\frac{3C^{(3)}\bar{C}^{(3)}}{4N}$	$\Gamma_{424}=\Gamma_{414}=0.$
2	$C_{11}=-2N$ $C_{22}=-2N$ $C_{12}=-N$ $C_{34}=N$	[3–1] [4] $\otimes$ $[A]$ [–]	E_4 is a multiple left D–P vector	$\Gamma_{424}+\Gamma_{414}=0.$
		$[A]\otimes$ [3–1] [4] [–]	E_4 is a multiple right D–P vector	$\Gamma_{424}+\Gamma_{414}=0.$
3	$C_{11}=-2N$ $C_{22}=-2N$ $C_{12}=-N$ $C_{34}=N$	[3–1] [4] $\otimes$ [3–1] [–] or [3–1] $\otimes$ [3–1] [4] [–]	E_4 is a multiple left and right D–P vector, $C^{(2)}-\bar{C}^{(2)}\neq 0$	$\Gamma_{424}=\Gamma_{414}=0;$ $R,_4=0\Leftrightarrow N,_4=0.$
			E_4 is a multiple left and right D–P vector, $C^{(2)}-\bar{C}^{(2)}\neq 0,$ $R,_4=0$	$\Gamma_{424}=\Gamma_{414}=0;$ $\Gamma_{412}=0\Rightarrow\Gamma_{422}=\Gamma_{124}=0;$ $\Gamma_{422}=0\Rightarrow\Gamma_{412}=\Gamma_{124}=0;$ $\Gamma_{421}=0\Rightarrow\Gamma_{411}=\Gamma_{124}=0;$ $\Gamma_{411}=0\Rightarrow\Gamma_{421}=\Gamma_{124}=0;$ $N,_4=0.$

Analogous propositions for E_3 are obtained by interchanging: $4\leftrightarrow 3$, $2\leftrightarrow 1$, $C^{(3)}\to C^{(3)}$, $\bar{C}^{(3)}\to\bar{C}^{(3)}$, $C^{(2)}\to -C^{(4)}$, $\bar{C}^{(2)}\to -\bar{C}^{(4)}$.

TABLE XIX

Type $[S_1-3T]_2$ (real)

1	$C_{11}=-2T$ $C_{22}=-2T$ $C_{12}=-T$ $C_{34}=T$ (other components vanish)	$[A]$	E_4 is a multiple D–P vector, $\mathrm{Re}\,C^{(3)}\neq 3\frac{\lvert C^{(3)}\rvert^2}{4T}$	$\Gamma_{424}=0.$
2	$C_{11}=-2T$ $C_{22}=-2T$ $C_{12}=-T$ $C_{34}=T$	[3–1] [4] [–]	E_4 is a multiple D–P vector	$\Gamma_{424}+\Gamma_{414}=0.$
3	$C_{11}=-2T$ $C_{22}=-2T$ $C_{12}=-T$ $C_{34}=T$	[3–1]	E_4 is a multiple D–P vector, $\mathrm{Im}\,C^{(2)}\neq 0$	$\Gamma_{424}=0;$ $R,_4=0\Leftrightarrow T,_4=0.$
			E_4 is a multiple D–P vector, $\mathrm{Im}\,C^{(2)}\neq 0,$ $R,_4=0$	$\Gamma_{424}=0;$ $\Gamma_{421}=0\Rightarrow\Gamma_{422}=\Gamma_{124}=0;$ $\Gamma_{422}=0\Rightarrow\Gamma_{421}=\Gamma_{124}=0;$ $T,_4=0.$

Analogous propositions for E_3 are obtained by interchanging: $4\leftrightarrow 3$, $2\leftrightarrow 1$, $C^{(3)}\to C^{(3)}$, $C^{(2)}\to -C^{(4)}$.

TABLE XX

Type $[C_1-C_2-2N]_3$ (complex)

1	$C_{11} = \frac{1}{2}(C_1-C_2)$ $C_{22} = \frac{1}{2}(C_1-C_2)$ $C_{12} = -N$ $C_{34} = N$ (other components vanish)	$[A] \otimes [B]$	E_4 is a multiple left and right D–P vector, $N\frac{C^{(3)}+\bar{C}^{(3)}}{2} \neq \frac{3C^{(3)}\bar{C}^{(3)}}{4} + \frac{3N^2+C_1C_2}{12}$	$\Gamma_{424} = \Gamma_{414} = 0.$
2	$C_{11} = \frac{1}{2}(C_1-C_2)$ $C_{22} = \frac{1}{2}(C_1-C_2)$ $C_{12} = -N$ $C_{34} = N$	$\begin{matrix}[3-1] \\ [4] \\ [-]\end{matrix} \otimes \begin{matrix}[3-1] \\ [4] \\ [-]\end{matrix}$	E_4 is a multiple left and right D–P vector	$\Gamma_{424} = \Gamma_{414} = 0;$ $R,_4 = 0 \Leftrightarrow N,_4 = 0.$
			E_4 is a multiple left and right D–P vector, $R,_4 = 0$	$\Gamma_{424} = \Gamma_{414} = 0;$ $N\Gamma_{421} = 0 \Leftrightarrow \Gamma_{411} = 0;$ $N\Gamma_{412} = 0 \Leftrightarrow \Gamma_{422} = 0;$ $N,_4 = 0.$
3	$C_{11} = \frac{1}{2}(C_1-C_2)$ $C_{22} = \frac{1}{2}(C_1-C_2)$ $C_{12} = -N$ $C_{34} = N$	$\begin{matrix}[3-1] \\ [4] \\ [-]\end{matrix} \otimes \begin{matrix}[3-1] \\ [4] \\ [-]\end{matrix}$	E_4 is a multiple left and right D–P vector, $N = 0$	$\Gamma_{424} = \Gamma_{422} = \Gamma_{414}$ $= \Gamma_{411} = 0;$ $R,_4 = 0.$
		$\begin{matrix}[4] \\ [-]\end{matrix} \otimes \begin{matrix}[4] \\ [-]\end{matrix}$	E_4 is a multiple left and right D–P vector, $N = 0,$ $R,_2 = R,_1 = 0$	$\Gamma_{424} = \Gamma_{422} = \Gamma_{414}$ $= \Gamma_{411} = \Gamma_{423} = \Gamma_{413}$ $= \Gamma_{314} = \Gamma_{324} = 0;$ $R,_4 = 0.$

Propositions valid for E_3 are obtained by interchanging: $4 \leftrightarrow 3$, $2 \leftrightarrow 1$ but $C^{(3)} \to C^{(3)}$, $\bar{C}^{(3)} \to \bar{C}^{(3)}$

TABLE XXI

Type $[S_1-S_2-2T]_3$ (real)

1	$C_{11} = \frac{1}{2}(S_1-S_2)$ $C_{22} = \frac{1}{2}(S_1-S_2)$ $C_{12} = -T$ $C_{34} = T$ (other components vanish)	$[A]$	E_4 is a multiple D–P vector, $T\,\mathrm{Re}\, C^{(3)} \neq \frac{3\lvert C^{(3)}\rvert^2}{4} + \frac{3T^2+S_1S_2}{12}$	$\Gamma_{424} = 0.$
2	$C_{11} = \frac{1}{2}(S_1-S_2)$ $C_{22} = \frac{1}{2}(S_1-S_2)$ $C_{12} = -T$ $C_{34} = T$	[3–1] [4] [–]	E_4 is a multiple D–P vector	$\Gamma_{424} = 0;$ $R,_4 = 0 \Leftrightarrow T,_4 = 0.$
			E_4 is a multiple D–P vector, $R,_4 = 0$	$\Gamma_{424} = 0;$ $T,_4 = 0;$ $T\Gamma_{421} = 0 \Leftrightarrow \Gamma_{422} = 0.$
3	$C_{11} = \frac{1}{2}(S_1-S_2)$ $C_{22} = \frac{1}{2}(S_1-S_2)$ $C_{12} = -T$ $C_{34} = T$	[3–1] [4] [–]	E_4 is a multiple D–P vector, $T = 0$	$\Gamma_{424} = \Gamma_{422} = 0;$ $R,_4 = 0.$
		[4] [–]	E_4 is a multiple D–P vector, $T = 0,$ $R,_2 = 0$	$\Gamma_{424} = \Gamma_{422} = \Gamma_{423}$ $= \Gamma_{324} = 0;$ $R,_4 = 0.$

Propositions valid for E_3 are obtained by interchanging: $4 \leftrightarrow 3$, $2 \leftrightarrow 1$ but $C^{(3)} \to C^{(3)}$.

APPENDIX

The Bianchi identities [8, 14]

$$dR^a{}_b + \Gamma^a{}_c \wedge R^c{}_b - \Gamma^c{}_b \wedge R^a{}_c = 0, \tag{A.1}$$

where $R^a{}_b$ is the curvature form, $\Gamma^a{}_b$ is the connection form, can be written in the spinor terms as follows [12, 8]

$$\nabla^S{}_{\dot{D}} C_{SABC} + \nabla_{(A}{}^{\dot{S}} C_{BC)\dot{D}\dot{S}} = 0, \tag{A.2a}$$

$$\nabla_D{}^{\dot{S}} \bar{C}_{\dot{S}\dot{A}\dot{B}\dot{C}} + \nabla^S{}_{(\dot{A}} C_{|DS|\dot{B}\dot{C})} = 0, \tag{A.2b}$$

$$\nabla^{P\dot{S}} C_{AP\dot{B}\dot{S}} + \tfrac{1}{8} \nabla_{A\dot{B}} R = 0, \tag{A.2c}$$

here $\nabla_{A\dot{B}} := g^a{}_{A\dot{B}} \nabla_a$.

Identities (A.2a) we call "the left (heavenly) Bianchi identities", (A.2b) we call "the right (hellish) Bianchi identities". By tedious algebraic manipulations one finds the explicit form of identities (A.2a) (in the appropriate pair of spinor bases) [8, 13]:

$$\begin{aligned} &-[\partial_1 - 2(\Gamma_{121} + \Gamma_{341}) + \Gamma_{314}]C^{(5)} + [\partial_4 - (\Gamma_{124} + \Gamma_{344}) - 4\Gamma_{421}]C^{(4)} + 3\Gamma_{424}C^{(3)} \\ &\quad = [\partial_2 - 2\Gamma_{342} + \Gamma_{324}]C_{44} - [\partial_4 - (\Gamma_{124} + \Gamma_{344}) - 2\Gamma_{412})C_{42} + 2\Gamma_{422}C_{41} \\ &\qquad - 2\Gamma_{424}C_{12} - \Gamma_{414}C_{22}, \end{aligned} \tag{A.3a}$$

$$\begin{aligned} &-[\partial_2 + 2(\Gamma_{122} + \Gamma_{342}) + \Gamma_{423}]C^{(1)} - [\partial_3 + (\Gamma_{123} + \Gamma_{343}) - 4\Gamma_{312}]C^{(2)} + 3\Gamma_{313}C^{(3)} \\ &\quad = [\partial_1 + 2\Gamma_{341} + \Gamma_{413}]C_{33} - [\partial_3 + (\Gamma_{123} + \Gamma_{343}) - 2\Gamma_{321}]C_{31} + 2\Gamma_{311}C_{32} \\ &\qquad - 2\Gamma_{313}C_{12} - \Gamma_{323}C_{11}, \end{aligned} \tag{A.3b}$$

$$\begin{aligned} &[\partial_3 - 2(\Gamma_{123} + \Gamma_{343}) - \Gamma_{312}]C^{(5)} + [\partial_2 - (\Gamma_{122} + \Gamma_{342}) + 4\Gamma_{423}]C^{(4)} + 3\Gamma_{422}C^{(3)} \\ &\quad = [\partial_2 - (\Gamma_{122} + \Gamma_{342}) + 2\Gamma_{324}]C_{42} - [\partial_4 - 2\Gamma_{124} - \Gamma_{412}]C_{22} + 2\Gamma_{422}C_{12} \\ &\qquad + 2\Gamma_{424}C_{32} - \Gamma_{322}C_{44}, \end{aligned} \tag{A.3c}$$

$$\begin{aligned} &[\partial_4 + 2(\Gamma_{124} + \Gamma_{344}) - \Gamma_{421}]C^{(1)} - [\partial_1 + (\Gamma_{121} + \Gamma_{341}) + 4\Gamma_{314}]C^{(2)} + 3\Gamma_{311}C^{(3)} \\ &\quad = [\partial_1 + (\Gamma_{121} + \Gamma_{341}) + 2\Gamma_{413}]C_{31} - [\partial_3 + 2\Gamma_{123} - \Gamma_{321}]C_{11} + 2\Gamma_{311}C_{12} \\ &\qquad + 2\Gamma_{313}C_{41} - \Gamma_{411}C_{33}, \end{aligned} \tag{A.3d}$$

$$\begin{aligned} &\Gamma_{311}C^{(5)} - [\partial_1 - (\Gamma_{121} + \Gamma_{341}) + 2\Gamma_{314}]C^{(4)} + [\partial_4 - 3\Gamma_{421}]C^{(3)} + 2\Gamma_{424}C^{(2)} \\ &= -\tfrac{1}{3}[\partial_3 - 2\Gamma_{343} - \Gamma_{321} + 2\Gamma_{312}]C_{44} - \tfrac{1}{3}[\partial_1 + 2\Gamma_{413} - (\Gamma_{121} + \Gamma_{341}) - 2\Gamma_{314}]C_{42} \\ &\quad + \tfrac{2}{3}[\partial_2 - \Gamma_{423} + \Gamma_{324} - (-\Gamma_{122} + \Gamma_{342})]C_{41} - \tfrac{2}{3}[\partial_4 + \Gamma_{421} - 2\Gamma_{412}]C_{12} \\ &\qquad - \tfrac{1}{3}\Gamma_{411}C_{22} + \tfrac{2}{3}\Gamma_{422}C_{11} + \tfrac{2}{3}\Gamma_{424}C_{31} + \tfrac{2}{3}\Gamma_{414}C_{32}, \end{aligned} \tag{A.3e}$$

$$\begin{aligned} &\Gamma_{422}C^{(1)} + [\partial_2 + (\Gamma_{122} + \Gamma_{342}) + 2\Gamma_{423}]C^{(2)} + [\partial_3 - 3\Gamma_{312}]C^{(3)} - 2\Gamma_{313}C^{(4)} \\ &= -\tfrac{1}{3}[\partial_4 + 2\Gamma_{344} - \Gamma_{412} + 2\Gamma_{421}]C_{33} - \tfrac{1}{3}[\partial_2 + (\Gamma_{122} + \Gamma_{342}) + 2\Gamma_{324} - 2\Gamma_{423}]C_{31} \end{aligned}$$

$$+\tfrac{2}{3}[\partial_1-\Gamma_{314}+\Gamma_{413}+(-\Gamma_{121}+\Gamma_{341})]C_{32}-\tfrac{2}{3}[\partial_3+\Gamma_{312}-2\Gamma_{321}]C_{12}$$
$$-\tfrac{1}{3}\Gamma_{322}C_{11}+\tfrac{2}{3}\Gamma_{311}C_{22}+\tfrac{2}{3}\Gamma_{313}C_{42}+\tfrac{2}{3}\Gamma_{323}C_{41}, \tag{A.3f}$$

$$-\Gamma_{313}C^{(5)}+[\partial_3-(\Gamma_{123}+\Gamma_{343})-2\Gamma_{312}]C^{(4)}+[\partial_2+3\Gamma_{423}]C^{(3)}+2\Gamma_{422}C^{(2)}$$
$$=-\tfrac{1}{3}[\partial_3-(\Gamma_{123}+\Gamma_{343})-2\Gamma_{321}+2\Gamma_{312}]C_{42}-\tfrac{1}{3}[\partial_1-2\Gamma_{121}+\Gamma_{413}-2\Gamma_{314}]C_{22}$$
$$+\tfrac{2}{3}[\partial_2-\Gamma_{423}+2\Gamma_{324}]C_{12}+\tfrac{2}{3}[\partial_4+(-\Gamma_{124}+\Gamma_{344})+\Gamma_{421}-\Gamma_{412}]C_{32}$$
$$+\tfrac{1}{3}\Gamma_{323}C_{44}-\tfrac{2}{3}\Gamma_{424}C_{33}-\tfrac{2}{3}\Gamma_{422}C_{31}-\tfrac{2}{3}\Gamma_{322}C_{41}, \tag{A.3g}$$

$$-\Gamma_{424}C^{(1)}-[\partial_4+(\Gamma_{124}+\Gamma_{344})-2\Gamma_{421}]C^{(2)}+[\partial_1+3\Gamma_{314}]C^{(3)}-2\Gamma_{311}C^{(4)}$$
$$=-\tfrac{1}{3}[\partial_4+(\Gamma_{124}+\Gamma_{344})-2\Gamma_{412}+2\Gamma_{421}]C_{31}-\tfrac{1}{3}[\partial_2+2\Gamma_{122}+\Gamma_{324}-2\Gamma_{423}]C_{11}$$
$$+\tfrac{2}{3}[\partial_1-\Gamma_{314}+2\Gamma_{413}]C_{12}+\tfrac{2}{3}[\partial_3-(-\Gamma_{123}+\Gamma_{343})+\Gamma_{312}-\Gamma_{321}]C_{41}$$
$$+\tfrac{1}{3}\Gamma_{414}C_{33}-\tfrac{2}{3}\Gamma_{313}C_{44}-\tfrac{2}{3}\Gamma_{311}C_{42}-\tfrac{2}{3}\Gamma_{411}C_{32}. \tag{A.3h}$$

Interchanges $C^{(a)}\to\bar{C}^{(a)}$, $2\leftrightarrow 1$ lead to the explicit form of the right Bianchi identities (A.2b).

Now, we can write the explicit form of Bianchi identities (A.2c) as follows [8, 13]:

$$\tfrac{1}{4}\partial_1 R=[\partial_1+2\Gamma_{413}+2\Gamma_{314}]C_{12}+[\partial_2+2\Gamma_{122}+\Gamma_{423}+\Gamma_{324}]C_{11}$$
$$+[\partial_3-(-\Gamma_{123}+\Gamma_{343})-\Gamma_{321}-2\Gamma_{312})C_{41}+[\partial_4+(\Gamma_{124}+\Gamma_{344})-\Gamma_{421}-2\Gamma_{412}]C_{31}$$
$$-\Gamma_{311}C_{42}-\Gamma_{411}C_{32}-\Gamma_{313}C_{44}-\Gamma_{414}C_{33}, \tag{A.4a}$$

$$\tfrac{1}{4}\partial_2 R=[\partial_2+2\Gamma_{423}+2\Gamma_{324}]C_{12}+(\partial_1+2\Gamma_{211}+\Gamma_{413}+\Gamma_{314}]C_{22}$$
$$+[\partial_3-(\Gamma_{123}+\Gamma_{343})-\Gamma_{312}-2\Gamma_{321}]C_{42}+[\partial_4+(-\Gamma_{124}+\Gamma_{344})-\Gamma_{412}-2\Gamma_{421}]C_{32}$$
$$-\Gamma_{322}C_{41}-\Gamma_{422}C_{31}-\Gamma_{323}C_{44}-\Gamma_{424}C_{33}, \tag{A.4b}$$

$$\tfrac{1}{4}\partial_3 R=[\partial_1+(-\Gamma_{121}+\Gamma_{341})+\Gamma_{413}+2\Gamma_{314}]C_{32}$$
$$+[\partial_2+(\Gamma_{122}+\Gamma_{342})+\Gamma_{423}+2\Gamma_{324}]C_{31}+[\partial_4+2\Gamma_{344}-\Gamma_{421}-\Gamma_{412}]C_{33}$$
$$-[\partial_3-2\Gamma_{321}-2\Gamma_{312}]C_{12}+\Gamma_{311}C_{22}+\Gamma_{322}C_{11}+\Gamma_{323}C_{41}+\Gamma_{313}C_{42}, \tag{A.4c}$$

$$\tfrac{1}{4}\partial_4 R=[\partial_1-(\Gamma_{121}+\Gamma_{341})+2\Gamma_{413}+\Gamma_{314}]C_{42}$$
$$+[\partial_2-(-\Gamma_{122}+\Gamma_{342})+\Gamma_{324}+2\Gamma_{423}]C_{41}+[\partial_3-2\Gamma_{343}-\Gamma_{312}-\Gamma_{321}]C_{44}$$
$$-[\partial_4-2\Gamma_{421}-2\Gamma_{412}]C_{12}+\Gamma_{411}C_{22}+\Gamma_{422}C_{11}+\Gamma_{424}C_{31}+\Gamma_{414}C_{32}. \tag{A.4d}$$

(Of course $\partial_a := E_a{}^{\mu}\partial/\partial z^{\mu}$).

One can verify that identities (A.2a), (A.2b) are equivalent to the following identities [9, 13]:

$$C^a{}_{bcd;a}=C_{b[c;d]}+\tfrac{1}{3}\,g_{b[c}C^a{}_{d];a} \tag{A.5a}$$

and (A.2c) are equivalent to

$$C^a{}_{b;a}=\tfrac{1}{4}R_{;b}. \tag{A.5b}$$

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