THE PARADOX OF A LORENTZ INVARIANT CURRENT AND CHARGE QUANTIZATION

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It is argued that the electric charge $e' = 4\pi/e$, where e is the charge of the electron, plays a distinguished role, similar to the role of Dirac's magnetic charge $g = 1/2e$.

1. The paradox of a Lorentz invariant current

Classical electrodynamics leads to a paradox which apparently has not received the attention it deserves. The paradox consists in the following. The electric current can be a Lorentz invariant vector field. On the other hand, the electromagnetic field cannot be a Lorentz invariant tensor field. Therefore, the field produced by a Lorentz invariant current must have a deviation from the perfect Lorentz symmetry, which is not implicit in the current.

In Ref. [1] we proposed a solution of this paradox. We have shown that it is possible to choose the electromagnetic field produced by a Lorentz invariant current in such a way that a classical particle scattered by this field emerges with unchanged momentum and angular momentum, which means that the deviation from the perfect Lorentz symmetry cannot be detected.

In this paper another solution is given. It is shown that a Lorentz invariant current can be transformed away by means of a certain (inadmissible) gauge transformation. This, however, cannot be done in Maxwellian electrodynamics but in a generalized electrodynamics introduced a long time ago by Dirac, Fock and Podolsky [2]. The transformation in question is possible if the charge e of a test particle and the charge e' of a Lorentz invariant current are connected by the relation

$$ee' = 4\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

Thus we obtain a principle of charge quantization without Dirac's magnetic charge.

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2. *The electrodynamics of Dirac, Fock and Podolsky*

The equations. As is well known, one of the Maxwell equations of motion (div $\boldsymbol{E} = 0$) does not contain the time derivative; this causes a considerable trouble in the quantum version of the theory. Dirac, Fock and Podolsky introduced a modified theory of a tensor field $F_{\mu\nu} = -F_{\nu\mu}$ and a scalar field F, which fulfil the equations

$$\partial_\lambda F_{\mu\nu}+\partial_\mu F_{\nu\lambda}+\partial_\nu F_{\lambda\mu} = 0, \tag{1}$$

$$\partial^\mu F_{\mu\nu}+\partial_\nu F = 0. \tag{2}$$

These equations are always used in quantum electrodynamics but the field F is considered a nuisance to be argued away, e. g. by means of the Gupta–Bleuler method. In this paper we accept the Dirac–Fock–Podolsky equations (1) and (2) as they stand and derive a consequence which, to our knowledge, has not been noted.

The energy momentum tensor. Multiplying Eq. (2) by $F_{\lambda\cdot}^{\cdot\nu}$ we obtain

$$\begin{aligned} 0 &= F_{\lambda\nu}(\partial_\mu F^{\mu\nu}+\partial^\nu F) \\ &= \partial_\mu(F_{\lambda\nu}F^{\mu\nu})-F^{\mu\nu}\partial_\mu F_{\lambda\nu}+\partial^\nu(F_{\lambda\nu}F)-F\partial^\nu F_{\lambda\nu} \\ &= \partial_\mu(F_{\lambda\nu}F^{\mu\nu}+F_{\lambda\cdot}^{\cdot\mu}F)-\tfrac{1}{2}\,F^{\mu\nu}\partial_\lambda F_{\mu\nu}-F\partial_\lambda F \\ &= \partial_\mu(F_{\lambda\nu}F^{\mu\nu}+FF_{\lambda\cdot}^{\cdot\mu}-\tfrac{1}{4}\,\delta^\mu_\lambda F^{\alpha\beta}F_{\alpha\beta}-\tfrac{1}{2}\,F^2\delta^\mu_\lambda) \end{aligned}$$

which means that the tensor

$$T^\mu_\lambda = \frac{1}{4\pi}\,(\tfrac{1}{4}\,F^{\alpha\beta}F_{\alpha\beta}\delta^\mu_\lambda-F_{\lambda\nu}F^{\mu\nu}+\tfrac{1}{2}\,F^2\delta^\mu_\lambda-FF_{\lambda\cdot}^{\cdot\mu})$$

is conserved. This tensor is a generalization of the Minkowski energy momentum tensor. It is positive definite but not symmetric.

The potential and the principle of least action. Eq. (1) implies that there is a vector field A_μ such that

$$F_{\mu\nu} = \partial_\mu A_\nu-\partial_\nu A_\mu.$$

Let us put

$$F = \partial^\mu A_\mu$$

and consider the integral

$$S = -\frac{1}{16\pi}\int d^4x(F_{\mu\nu}F^{\mu\nu}+2F^2).$$

Varying S with respect to A_μ one finds Eq. (2) as the Euler–Lagrange equation.

The vectors A_μ and $A'_\mu = A_\mu+\partial_\mu f$, where $\Box f = 0$, give the same fields $F_{\mu\nu}$ and F. The transformation

$$A_\mu \to A'_\mu = A_\mu+\partial_\mu f, \quad \Box f = 0,$$

is called a gauge transformation of the second kind.

3. Inadmissible but unobservable gauge transformations of the first kind

Interaction of the Dirac field ψ with the Dirac–Fock–Podolsky field A_μ can be introduced by means of the same well known recipe as in the Maxwellian case.

Let us consider the following gauge transformation of the first kind

$$\psi(x) \to \psi'(x) = e^{2\pi i n(x)}\psi(x),$$

where $n(x)$ is a discontinuous function equal to 0, ± 1, ± 2, ... in domains of continuity. This is obviously an identity transformation

$$\psi \to \psi' \equiv \psi$$

but it generates formally a gauge transformation of the second kind

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \frac{2\pi}{e} n$$

which, in general, is not admissible because, in general,

$$\Box n \neq 0$$

and therefore

$$F_{\mu\nu} \to F'_{\mu\nu} = F_{\mu\nu}$$

but

$$F \to F' = F + \frac{2\pi}{e} \Box n \neq F.$$

Let us interpret the last transformation as a creation of an external (with respect to the Dirac–Fock–Podolsky field) current

$$j_\mu^{\text{ext}} = -\frac{1}{4\pi} \partial_\mu (F' - F) = -\frac{1}{2e} \partial_\mu \Box n.$$

In general nothing can be said about this current; in particular, it does not have to be conserved i. e. it does not have to have a definite charge. (In the Dirac–Fock–Podolsky electrodynamics the current does not have to be conserved; only the Maxwellian part $j_\mu^{\text{ext}} - (1/4\pi)\,\partial_\mu F$ is conserved.) Let us suppose, however, that j_μ^{ext} does have a definite, conserved charge; this will be the case if $\Box\Box\, n = 0$.

Piecewise integer solutions of this equation are investigated in the Appendix. It turns out that each solution of this kind either fulfils the wave equation $\Box\, n = 0$ or is a superposition of solutions which can be obtained from the fundamental solution

$$n(x) = \text{sign}\,(x^0)\theta(xx) = \text{sign}\,(x^0) \begin{cases} 1 \text{ for } xx > 0, \\ 0 \text{ for } xx < 0, \end{cases}$$

by space-time translation and multiplication by an integer. Therefore the total charge of the created current j_μ^{ext} must be always a multiple of the total charge of

$$j_\mu(x) = \frac{1}{2e}\,\partial_\mu\Box\{\text{sign}\,(x^0)\theta(xx)\}$$

i. e. of

$$e' = \frac{4\pi}{e}\,.$$

4. Comparison with the Dirac theory of magnetic charge

There is an obvious analogy between the theory from the preceding section and the Dirac theory of magnetic charge. In the Dirac theory one performs a gauge transformation of the first kind

$$\psi(x) \to \psi'(x) = e^{i\,\text{arctg}\,x^2/x^1}\psi(x).$$

This is an inadmissible gauge transformation because the electromagnetic field is changed:

$$F_{12} \to F'_{12} = F_{12} + \frac{2\pi}{e}\,\delta(x^1)\delta(x^2).$$

There arises an infinitely thin tube of magnetic flux at the end of which there is a magnetic charge

$$g = \frac{1}{2e}\,.$$

Both theories are based on the idea that a discontinuity of phase should be (somehow) admissible if its magnitude is a multiple of 2π. However, the surface of discontinuity is timelike in the Dirac theory and null in our theory, which seems more reasonable in a relativistic theory. Moreover, the existence of a magnetic charge leads to experimental consequences which have never been observed while a Lorentz invariant current is unobservable — if one accepts the argument of Ref. [1]; it does quantize the ordinary charge but does not interfere with the rest of physics.

APPENDIX

On piecewise integer solutions of the equation $\Box\Box\, n = 0$

We shall investigate first local consistency conditions. To this end we put

$$n(x) = \theta(\varphi(x)) = \begin{cases} 1 \text{ for } \varphi(x) > 0, \\ 0 \text{ for } \varphi(x) < 0. \end{cases}$$

$\varphi(x)$ is a continuous function with as many continuous derivatives as is needed in the subsequent calculation.

$$\partial_\mu n = \delta(\varphi)\partial_\mu\varphi, \qquad \Box n = \delta'(\varphi)\partial^\mu\varphi\partial_\mu\varphi + \delta(\varphi)\Box\varphi.$$

$n(x)$ can be discontinuous only across a null surface; therefore

$$\partial^{\mu}\varphi\partial_{\mu}\varphi = 0, \qquad \Box n = \delta(\varphi)\Box\varphi.$$

In the same way

$$\Box\Box n = \delta'(\varphi)\left[(\Box\varphi)^2+2\partial^{\mu}\varphi\partial_{\mu}\Box\varphi\right]+\delta(\varphi)\Box\Box\varphi.$$

Coefficients of $\delta'(\varphi)$ and $\delta(\varphi)$ have to vanish separately. Thus

$$(\Box\varphi)^2+2\partial^{\mu}\varphi\partial_{\mu}\Box\varphi = 0,$$

$$\Box\Box\,\varphi = 0.$$

One can always assume that

$$\varphi(x^0, x^1, x^2, x^3) \equiv x^0-f(x^1, x^2, x^3);$$

then the conditions above take the form

$$(\mathrm{grad}\, f)^2 = 1,$$

$$(\Delta f)^2+2\,\mathrm{grad}\, f\cdot \mathrm{grad}\,\Delta f = 0,$$

$$\Delta\Delta f = 0.$$

The only surfaces $f =$ const for which these conditions hold are planes and spheres. One can prove this as follows: one solves the above conditions in the Gaussian normal coordinates based on the surface $f =$ const and subsequently imposes integrability conditions $R_{ij} = 0$, where R_{ij} is the Ricci tensor of the three-dimensional space $x^0 =$ const. This consideration is local but the result holds globally as well because of the well known rigidity of a sphere [3]; in the case of a plane the solution has the form $f = \boldsymbol{n}\cdot\boldsymbol{x}$, where $\boldsymbol{n}$ is a constant unit vector, and also holds globally.

In the case of plane $\Box\, n(x) = 0$ which means that the gauge transformation $A_{\mu}\to A'_{\mu} = A_{\mu}+\partial_{\mu}(2\pi n/e)$ is admissible. Thus the case of a sphere remains to be considered.

It is clear that in this case the surface $\varphi = 0$ is a light cone of some event. One can assume that this event is at the origin of the coordinate system. One can assume, moreover, that $n(x)$ vanishes outside the light cone since this can be always achieved by subtraction of a constant. Thus

$$n(x) = \begin{cases} n_+ & \text{for } xx > 0,\ x^0 > 0,\\ 0 & \text{for } xx < 0,\\ n_- & \text{for } xx > 0,\ x^0 < 0.\end{cases}$$

The equation $\Box\Box\, n(x) = 0$ is fulfilled if and only if $n_+ = -n_- = n$. In this case

$$n(x) = n\,\mathrm{sign}\,(x^0)\theta(xx),$$

$$\Box\, n(x) = 4n\,\mathrm{sign}\,(x^0)\delta(xx),$$

$$\Box\,\Box\, n(x) = 0.$$

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