SOME CALCULATIONS ON THE PRODUCTION AND DETECTION OF $(\pi\mu)_{atom}$ FROM K_L^0 DECAY*

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(*Received December 28, 1978*)

The branching ratio $R = \Gamma(K_L^0 \to (\pi\mu)_{atom}\nu)/\Gamma(K_L^0 \to \pi\mu\nu)$ is calculated as a function of $\xi = f_-/f_+$, where $f_\pm$ are dimensionless form factors for the K_L^0. The world average values $\xi = -0.2\,(-0.9)$ from $K_{\mu3}/K_{e3}$ (muon polarization) measurements give $R = 3.8\,(2.7)\times 10^{-7}$. Bethe's theory of inelastic collisions is adapted to the calculation of the ionization cross-section σ_{ion} for a relativistic $(\pi\mu)_{atom}$ in the $1S$ state due to its interaction with the screened Coulomb field of a target (foil) atom. In particular, for a $(\pi\mu)_{atom}$ with an energy of $10\,(m_\pi+m_\mu)\,c^2$ incident on an aluminum target (foil) atom, $\sigma_{ion} = 7.4\times 10^{-22}\,\text{cm}^2$. These calculations are relevant to the experiments being currently performed by M. Schwartz and collaborators at Brookhaven and FNAL.

There has recently appeared a letter reporting the detection of hydrogen-like atoms consisting of a pion ($\pi^\mp$) and a muon ($\mu^\pm$) in a Coulomb bound state. These pion-muon atoms $(\pi\mu)_{atom}$ are formed when the π and μ from the decay $K_L^0 \to \pi\mu\nu$ have sufficiently small relative momentum to bind. The $(\pi\mu)_{atom}$, being a hydrogen-like atom, is expected to have a reduced mass m_r of 60.2 MeV/c^2, a Bohr radius a_r of 4.5×10^{-11} cm, and an ionization potential I of 1.6 keV. The total mass $m_{\pi\mu}$ of the $(\pi\mu)_{atom}$, neglecting its binding energy, is 245.2 MeV/c^2. Our purpose here is to report on some calculations that have direct bearing on the experiment.

The prime motivations for the experiment were twofold. Firstly, the value of the branching ratio $R = \dfrac{\Gamma(K_L^0 \to (\pi\mu)\nu)}{\Gamma(K_L^0 \to \pi\mu\nu)}$ is proportional to the square of the π-μ wave function at very small-distances and so an anomaly in its value may be indicative of an anomaly in the π-μ interaction. Secondly, by passage of the atoms through a magnetic field at high velocity ($\gamma \sim 10$) the $2S$ states should be depopulated through Stark mixing with the $2P$ states and consequently decay to the $1S$ states. The extent of this depopulation is sensitive to the Lamb shift of the $2S$ states relative to the $2P$ states and may, if measured

* Research supported in part by National Science Foundation grant PHY 75-18444.

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with some accuracy, lead to a determination of the pion charge radius. The crucial detection of the $(\pi\mu)_{\text{atom}}$ is made possible by its dissociation on passing through a thin aluminum foil.

With regard to the production of the $(\pi\mu)_{\text{atom}}$, we calculate the branching ratio R in the context of conventional current × current weak interaction theory, assuming that only the vector current contributes to the decays. The ratio R is then essentially a function of the parameter $\xi = f_-/f_+$ where f_- and f_+ are dimensionless form factors that parametrize the hadronic weak transition current $\langle\pi|J_{\text{weak}}|K_L^0\rangle$. Unfortunately, there is at the present time no general consensus on the experimental value of ξ and therefore we have no unique prediction for R. The world average values of $\xi \approx -0.2$ and $\xi \approx -0.9$ from $K_{\mu 3}/K_{e3}$ and muon polarization measurements give $R \approx 3.8 \times 10^{-7}$ and $R \approx 2.7 \times 10^{-7}$, respectively. We thereby confirm previous estimates [2] for R. The details of this calculation can be found in Sections 1 and 2.

With regard to the detection of the $(\pi\mu)_{\text{atom}}$, we consider a collision between a fast particle and a $(\pi\mu)_{\text{atom}}$, accompanied by excitation or ionization of the atom. Such inelastic collisions were first discussed nonrelativistically by Bethe and then later generalized to the relativistic case by Bethe and Moller. Both Bethe and Moller studied in detail the case when the target atom was hydrogen-like and the charge distribution of the fast incident particle was a bare Coulomb field. In our case the target atom is indeed hydrogen-like, namely a $(\pi\mu)_{\text{atom}}$; however departures from the bare Coulomb field approximation come at large momentum transfers (i.e., large angles, small impact parameters) and small momentum transfers (i.e., small angles, large impact parameters).

At small momentum transfers screening effects must be taken into account. This is because the fast incident particle in our case (in the $(\pi\mu)_{\text{atom}}$ rest frame) is actually an aluminum atom whose charge distribution is therefore a screened Coulomb field rather than a bare Coulomb field. In Section 3 we derive the relativistic formulas for total inelastic, excitation, and ionization cross-sections appropriate for a screened Coulomb interaction. The essential effect of the screening is to make the various cross-sections finite and constant at ultra-relativistic velocities which would otherwise diverge. A $(\pi\mu)_{\text{atom}}$ in its $1S$ state with an energy of $10m_{\pi\mu}c^2$ on colliding with an aluminum atom is found to have an ionization cross-section of 7.4×10^{-22} cm^2. As expected, the cross-section is of the order of the size of the $(\pi\mu)_{\text{atom}}$. We are thereby justified in neglecting "finite nuclear size" effects which are important at large momentum transfers and expected to be the order of 10^{-26} cm^2 (i.e., square of the nuclear interaction range). The thickness of foil required to break-up a high velocity ($\gamma \sim 10$) beam of $(\pi\mu)_{\text{atom}}$ in the $1S$ state is therefore 2.2×10^{-2} cm ($= 8.8 \times 10^{-3}$ in) of aluminum, in agreement with the Monte Carlo calculations of Ref. [1].

1. *Calculation of* $\Gamma(K_L^0 \to \pi\mu\nu)$

(In this and the following Section we follow the conventions of Ref. [6].)

The differential decay width for the process $K_L^0 \to \pi\mu\nu$ is given by:

$$d\Gamma(K_L^0 \to \pi\mu\nu) = |\mathcal{M}_3^A|^2 \frac{d^3p_\pi d^3p_\mu d^3p_\nu}{(2\pi)^5 2E_K 2E_\pi 2E_\mu 2E_\nu} \delta_4(p_K - p_\pi - p_\mu - p_\nu), \tag{1.1}$$

where the decay amplitude $\mathscr{M}_3^A$ is the product of the hadronic and leptonic weak transition currents:

$$\mathscr{M}_3^A \equiv \mathscr{M}_3^A(p_K; p_\pi, p_\mu, p_\nu)$$

$$= \left[\frac{G}{\sqrt{2}}\sin\theta_c\right][f_+(q^2)(p_K+p_\pi)_\lambda + f_-(q^2)(p_K-p_\pi)_\lambda]\,[\bar{u}_\nu(p_\nu)\gamma_\lambda(1+\gamma_5)v_\mu(p_\mu)]. \quad (1.2)$$

The dimensionless form factors $f_\pm(q^2)$ depend only on $q^2 = (p_K-p_\pi)^2$, the square of the momentum transfer to the leptons, and we assume time reversal invariance so that they are real. The standard parametrization for $f_\pm(q^2)$ is:

$$f_\pm(q^2) = f_\pm(0)\,[1+\lambda_\pm(q/m_\pi)^2], \qquad \xi(q^2) = f_-(q^2)/f_+(q^2). \quad (1.3)$$

Summing over final lepton spin states and performing straightforward integrations we find for the total decay width:

$$\Gamma(K_L^0 \to \pi\mu\nu) = \left[\frac{G}{\sqrt{2}}\sin\theta_c\right]^2 \frac{m_K^5}{(2\pi)^3}\int_{\delta^2}^{\beta^2} |f_+(q^2)|^2[(y-\varepsilon)(y-\beta^2)]^{1/2}$$

$$\times\frac{(y-\delta^2)^2}{2}\left\{\left[\frac{(y+2\delta^2)(y+\varepsilon\beta)^2}{12y^2} - \frac{(2y+\delta^2)}{6y}\right] + \frac{\delta^2}{4y}(y+\varepsilon\beta)(\xi(q^2)-1)\right.$$

$$\left. + \frac{\delta^2}{8}(\xi(q^2)-1)^2\right\}dy, \quad (1.4)$$

where

$$\varepsilon \equiv \frac{(m_\pi+m_K)}{m_K}, \quad \beta \equiv \frac{m_K-m_\pi}{m_K}, \quad \delta \equiv \frac{m_\mu}{m_K}, \quad y \equiv \left(\frac{q}{m_K}\right)^2.$$

The limits on q^2 are $m_\mu^2 \leqslant q^2 \leqslant (m_K-m_\pi)^2$ and from the data of various measurements it is known $\lambda_+ < 0.03$ and $\lambda_- \approx 0$. Thus with an error of a few percent we may safely neglect $\lambda_\pm$ so that $f_\pm(q^2) \approx f_\pm(0) \equiv f_\pm$. For this case ($\lambda_\pm = 0$) we can easily do the integral occurring in (1.4) analytically.

Having obtained the decay width ($K_L^0 \to \pi\mu\nu$) and assuming eμ universality we obtain:

$$\tilde{R}(\lambda_\pm = 0) \equiv \tilde{R} \equiv \frac{\Gamma(K_L^0 \to \pi\mu\nu)}{\Gamma(K_L^0 \to \pi e\nu)} = 0.64512 + 0.12456\xi + 0.018654\xi^2. \quad (1.5)$$

Direct experimental measurement of $\tilde{R}$ suggests $\xi = -0.2$ whereas an independent measurement of the final μ^+ polarization seems to favor $\xi \approx -0.9$. This well-known discrepancy is as yet unresolved.

2. *Calculation of* $(K_L^0 \to (\pi\mu)_{atom}\nu_\mu)$

The differential decay width for the process $K_L^0 \to (\pi\mu)_{atom}\nu$ is given by:

$$d\Gamma(K_L^0 \to (\pi\mu)\nu) = |\mathcal{M}_2^B|^2 \frac{d^3P_{\pi\mu}d^3p_\nu}{(2\pi)^2 2E_K 2E_{\pi\mu} 2E_\nu} \delta_4(p_K - P_{\pi\mu} - p_\nu),$$

$$p_\pi = \frac{m_\pi}{m_{\pi\mu}} P_{\pi\mu} + p_{\pi\mu}, \qquad p_\mu = \frac{m_\mu}{m_{\pi\mu}} P_{\pi\mu} - p_{\pi\mu}, \tag{2.1}$$

where one can think of $P_{\pi\mu}$ and $p_{\pi\mu}$ as the 4 momentum for center of mass and relative motion of the $(\pi\mu)_{atom}$. The decay amplitude is:

$$\mathcal{M}_2^B = \int \frac{d^4p_{\pi\mu}}{(2\pi)^4} \Psi_{BS}(P_{\pi\mu}; p_{\pi\mu}) \mathcal{M}_3^A(p_K; p_\pi; p_\mu; p_\nu), \tag{2.2}$$

where Ψ_{BS} is the Bethe–Salpeter wave function and $\mathcal{M}_3^A$ is defined in (1.2). In principle, the Ψ_{BS} satisfies the full Bethe–Salpeter equation [8] for a relativistic bound state of a π and a μ. Adopting the usual perturbative treatment — due to Salpeter [9] — we replace Ψ_{BS} by Ψ_{BS}^C — the solution of the Bethe–Salpeter equation with an instantaneous Coulomb kernel. In the frame $\vec{P}_{\pi\mu} = 0$ and in the nonrelativistic limit ($p_{\pi\mu} \ll m_\pi, m_\mu$), becomes identical to the nonrelativistic Schrödinger wave function $\Psi(\vec{x})$ ($\vec{x} \equiv \vec{x}_\pi - \vec{x}_\mu$) in momentum space. If we further assume that the weak interaction is local (i.e., effective only at threshold $\vec{p}_{\pi\mu} = \vec{x}_{\pi\mu} = 0$) we find:

$$\Psi_{BS}^C(P_{\pi\mu}^0, \vec{P}_{\pi\mu} = 0; p_{\pi\mu}) = \frac{(2\pi)^4}{i} \delta_4(p_{\pi\mu}) \Psi(\vec{x} = 0); \quad |\Psi(\vec{x} = 0)|^2 = \frac{1}{\pi}\left(\frac{\alpha m_r}{n}\right)^3$$

$$(n = 1, 2, 3, \ldots), \tag{2.3}$$

where $e^2/(\hbar c) \equiv \alpha = (137.036)^{-1}$ in the fine structure constant. Boosting the atom to $\vec{P}_{\pi\mu} \neq 0$ we obtain by Lorentz dilatation:

$$\Psi_{BS}^C(P_{\pi\mu}; p_{\pi\mu}) = \left[\frac{E_{\pi\mu}}{m_{\pi\mu}}\right]^{1/2} \frac{(2\pi)^4}{i} \delta_4(p_{\pi\mu}) \Psi(\vec{x} = 0). \tag{2.4}$$

Since the probability density $|\Psi(\vec{x})|^2 d^3x$ is by definition the same in all inertial frames and length contracts under a Lorentz transformation. This last remark is essential to the Lorentz invariance of the decay width. Thus, in the context of local weak interaction theory we find:

$$\mathcal{M}_2^B = \mathcal{M}_3^A\left(p_K; \frac{m_\pi}{m_{\pi\mu}} P_{\pi\mu}, \frac{m_\mu}{m_{\pi\mu}} P_{\pi\mu}, p_\nu\right)\left[\frac{E_{\pi\mu}}{m_{\pi\mu}}\right]^{1/2} \Psi(\vec{x} = 0). \tag{2.5}$$

Inserting (2.5) into (3.1), integrating, and summing over all principal quantum numbers of the final $(\pi\mu)_{\text{atom}}$ we find for the total decay width:

$$\Gamma(K_L^0 \to (\pi\mu)\nu) = 1.202\left(\frac{G}{\sqrt{2}}\sin\theta_c\right)^2 \frac{|f_+(\bar{q}^2)|^2(m_K^2-m_{\pi\mu}^2)^2}{8\pi^2 m_K^3 m_\pi} m_{\pi\mu}^2(\alpha m_r)^3$$

$$\times\left\{\xi(\bar{q}^2)\left(1-\frac{m_\pi}{m_{\pi\mu}}\right)^2+2\xi(\bar{q}^2)\left(1-\frac{m_\pi^2}{m_{\pi\mu}^2}\right)+\left(1+\frac{m_\pi}{m_{\pi\mu}}\right)^2\right\},$$

$$\bar{q}^2 \equiv (p_K-p_\pi)^2 = \left(p_K-\frac{m_\pi}{m_{\pi\mu}}P_{\pi\mu}\right)^2 = 4.72m_\pi^2, \qquad 1.202 = \sum_{n=1}^{\infty}\frac{1}{n^3}. \tag{2.6}$$

It is amusing to note that $\Gamma(K_L^0 \to (\pi\mu)\nu)$ vanishes for $\xi(\bar{q}^2) = -(2M_\pi+M_\mu)/M_\mu = -3.64$; such a value for ξ does not seem to be favored by $\tilde{R}$ and muon polarization experiments. In Fig. 1 we have plotted $\tilde{R}$ and R over the range $-2 \leqslant \xi \leqslant 2$. Inserting $\xi(\bar{q}^2) \approx \xi(0) \equiv \xi = 0.2$ (0.7) we find $R = 3.8\ (2.7)\times 10^{-7}$. In Fig. 2 we have plotted $\tilde{R}$

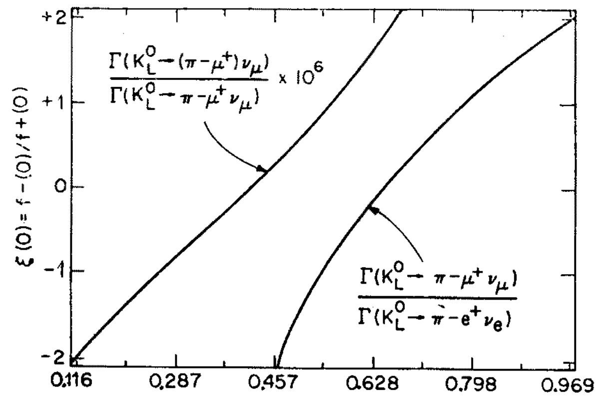

Fig. 1. Plot of the branching ratios for $-2 \leqslant \xi \leqslant 2$

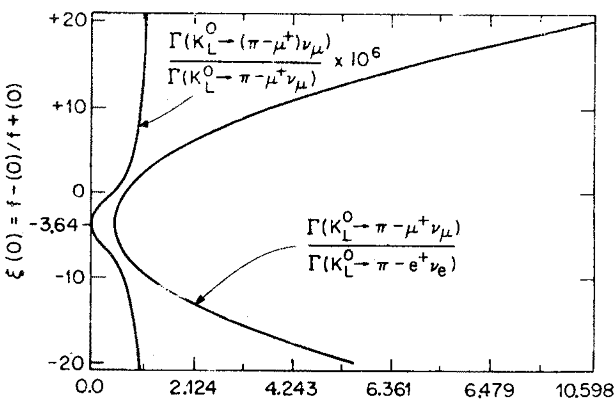

Fig. 2. Plot of the branching ratios for $-20 \leqslant \xi \leqslant 20$

and R over the exaggerated range $-20 \leqslant \xi \leqslant 20$ to emphasize the general behavior, in particular the vanishing of R at $\xi = -3.64$.

The preliminary data [10] on R indicates that a suppression of a factor five to eight is not, at present, inconsistent with experiment. If we interpret this as an anomaly in the π-μ interaction, we are led to postulate the existence of a short range repulsive interaction between the π and μ. A model calculation based on the following potential

$$V_{\pi-\mu}(r) = \begin{cases} 2\text{MeV} & r < r_0 \\ -\dfrac{e^2}{r} & r > r_0 \end{cases} \qquad (r_0 = 2 \times 10^{-12}\,\text{cm}) \tag{2.7}$$

indeed reduces R by a factor of eight without at the same time appreciably altering (by final state interaction) the pion and muon energy spectrum in the predominant decay mode $K_L^0 \to \pi\mu\nu$. In Fig. 3, using the integrand of (1.4), we have plotted, for $\xi(0) = -0.6$,

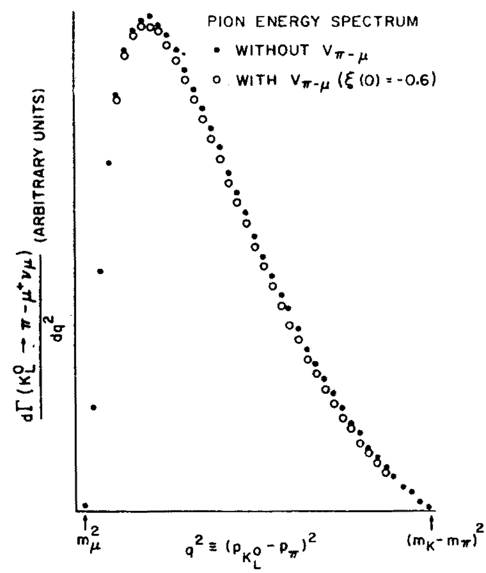

Fig. 3. Plot of the pion energy spectrum for $K_L^0 \to \pi^-\mu^+\nu_\mu$

the pion energy distribution with and without the anomalous interaction $V_{\pi-\mu}$. Note that because of the repulsion the distribution with $V_{\pi-\mu}$ has shifted slightly below that of without $V_{\pi-\mu}$. The anomalous interaction, however, increases the energy levels of the $(\pi\mu)_{\text{atom}}$ by 10% from their conventional (hydrogen-like) values. It seems implausible that such large departures of energy levels could be consistently explained in terms of the known forces; further arguments supporting this view can be found in a recent preprint [11]. We must, of course, keep in mind that no one has actually measured the energy levels of the $(\pi\mu)_{\text{atom}}$. Furthermore, as there exists no published data on R it would be premature for us here to labor the point any further.

3. *Calcualtion of inelastic cross sections*

To fix ideas and notations we begin by summarizing Bethe's [3] original nonrelativistic treatment of inelastic collisions between a fast particle and an atom, in a manner following the lucid presentation of Landau and Lifshits [12]. The relativistic version of the theory as advocated by Bethe [4] and Moller [5], will then be systematically outlined.

Inelastic collisions between a fast particle of nuclear charge Ze and a $(\pi\mu)_{\text{atom}}$ can be considered in the first Born approximation. For the validity of the Born approximation the speed v of the incident particle should be large compared to the orbital speeds in the $(\pi\mu)_{\text{atom}}$. Let $\vec{p}$ and $\vec{p}'$ be the momenta of the fast incident particle before and after the collision and E_0, E_n the corresponding energies of the $(\pi\mu)_{\text{atom}}$. The interaction energy U between the incident particle and the atomic particles is assumed to be a screened Coulomb potential:

$$U(r) = \frac{Ze^2}{r} e^{-r/a_Z}; \quad U(q) = \int e^{i\vec{q}\cdot\vec{r}} U(r) d^3r = \frac{4\pi Ze^2}{q^2 + a_Z^{-2}},$$

(nonrelativistic)

$$a_Z = 1.4 a_0 Z^{-1/3}, \tag{3.1}$$

where screening parameter (size of the incident particle) a_Z is evaluated in the context of the Fermi–Thomas model and a_0 ($= 5.3 \times 10^{-9}$ cm) is the Bohr radius of the electron. The differential inelastic cross-section for a given energy loss (i.e., fixed $|\vec{p}|$ and $|\vec{p}'|$) of the fast incident particle (it is assumed here, as is the case with us, that the mass of the fast incident particle is much greater than M_r) is found to be:

$$d\sigma_n = |U(q)|^2 |\langle \psi_n | e^{-i\vec{q}\cdot\vec{r}} | \psi_0 \rangle|^2 \left[\frac{1}{2\pi} \frac{q dq}{(\hbar v)^2} \right] \tag{3.2a}$$

(nonrelativistic)

$$= 8\pi \left(\frac{Ze^2}{\hbar v} \right)^2 |\langle \psi_n | e^{-i\vec{q}\cdot\vec{r}} | \psi_0 \rangle|^2 \frac{q dq}{(q^2 + a_Z^{-2})^2}, \tag{3.2b}$$

where $-\hbar\vec{q} = (p - p')$ is the momentum transfer to the atom, and ψ_0 (ψ_n) are the initial (final) wave functions of the atom. (In this section we assume the initial state ψ_0 of the $(\pi\mu)_{\text{atom}}$ to be its $1S$ ground state.) Since the collision is inelastic $|\vec{p}| \neq |\vec{p}'|$ and $n \neq 0$. A notable property of (3.2) is that it makes no reference to the mass of the incident particle. The final state (n) of the atom could be either in the discrete spectrum with an energy eigenvalue $E_n = -M_r\alpha^2/(2\hbar^2 n^2)$ ($n = 2, 3, 4, \ldots$) or in the continuous spectrum with energy $E_n = \hbar^2 k^2/(2M_r)$ ($0 < k < \infty$) corresponding to excitation and ionization of the $(\pi\mu)_{\text{atom}}$ respectively. The kinematics of the collision give the following limits in the momentum transfer:

$$\text{(nonrelativistic)} \quad q_{\min} = \frac{(E_n - E_0)}{\hbar v}, \quad q_{\max} = \frac{2m_r v}{\hbar}. \tag{3.3}$$

For hydrogen-like atoms one can calculate the inelastic form factor $|\langle\psi_n|e^{-i\vec{q}\cdot\vec{r}}|\psi_0\rangle|^2 \equiv |(e^{-i\vec{q}\cdot\vec{r}})_{0n}|^2$ analytically and find:

$$|(e^{-i\vec{q}\cdot\vec{r}})_{0n}|^2 = \begin{cases} \begin{bmatrix}\text{discrete} \\ n = 2, 3, 4 \\ E_n = -\dfrac{m_r\alpha^2}{2n^2\hbar^2}\end{bmatrix}: & (qa_r)^2 2^8 n^7 [\tfrac{1}{3}(n^2-1)+(qna_r)^2] \times \dfrac{[(n-1)^2+(qna_r)^2]^{n-3}}{[(n+1)^2+(qna_r)^2]^{n+3}} \quad (3.4a) \\ \begin{bmatrix}\text{continuous} \\ E_n = \dfrac{\hbar^2k^2}{2m_r} \\ 0<k<\infty\end{bmatrix}: & \dfrac{q^2k2^8a_r^{-6}[q^2+\frac{1}{3}(a_r^{-2}+k^2)]}{[(q+k)^2+a_r^{-2}]^3[(q-k)^2+a_r^{-2}]^3} \times \dfrac{\exp[-2(ka_r)^{-1}\tan^{-1}(2ka_r^{-1}/(q^2-k^2+a_r^{-2}))]}{1-\exp[-2\pi/(ka_r)]} \quad (3.4b) \end{cases}$$

If q is small one can make the "dipole" approximation $e^{-i\vec{q}\cdot\vec{r}} \approx 1 - i\vec{q}\cdot\vec{r}$ and obtain the following matrix elements:

$$|(x)_{0n}|^2 = \begin{cases} \text{discrete: } \dfrac{2^8}{3} n^7 \dfrac{(n-1)^{2n-5}}{(n+1)^{2n+5}} a_r^2 & (3.5a) \\ \text{continuous: } \dfrac{2^8}{3} \dfrac{ka_r^{-6}}{(k^2+a_r^{-2})^5} \dfrac{e^{-\frac{4}{ka_r}\tan^{-1}(ka_r)}}{[1-\exp(-2\pi/ka_r)]} & (3.5b) \end{cases}$$

Two important numbers to keep in mind are the following:

$$\sum_{n=2}^{\infty} |(x)_{0n}|^2 = 0.715a_r^2, \qquad \int_0^{\infty} |(x)_{0n}|^2 dk = 0.285a_r^2, \tag{3.6}$$

the sum of which is a_r^2 in accordance with the Compton sum rule.

Let us now consider the necessary modifications to formula (3.2a) for the inelastic cross-section when relativistic effects are to be taken into account. We follow here the treatment of the Bethe and Moller. There are basically three essential changes one must make which we now describe:

(1) In the Fourier transform $U(q)$ of the interaction potential $U(r)$, the momentum transfer square q^2 is replaced by the "four momentum" transfer square $q^2 - \left(\dfrac{E_n-E_0}{\hbar c}\right)^2$. This replacement takes into account the "retardation" effect characteristic of any relativistic interaction. Thus our first change is:

$$\text{(relativistic)} \quad U(q) = \frac{4\pi Ze^2}{q^2 - \left(\dfrac{E_n-E_0}{\hbar c}\right)^2 + a_Z^{-2}}. \tag{3.7}$$

(2) The nonrelativistic inelastic form factor $|\langle\psi_n|e^{-i\vec{q}\cdot\vec{r}}|\psi_0\rangle|^2$ must be replaced by $\int d^3re^{-i\vec{q}\cdot\vec{r}}j_\mu(r)J_\mu(r)$ where j_μ and J_μ are the transition currents for the fast incident particle and the $(\pi\mu)_{\text{atom}}$ respectively. Normalizing all wave functions to one particle per unit volume, the current $j_\mu = (1, \vec{v}/c)$ and $J_\mu = \left(\psi_n^*\psi_0, \psi_n^*\left(\frac{-i\hbar}{m_r c}\vec{\Delta}\right)\psi_0\right)$ where in J_μ we have replaced $v_{\pi\mu}$ by the operator $\frac{-i\hbar}{m_r c}\vec{\Delta}$. Thus our second change is:

$$\text{(relativistic)}\qquad |\langle\psi_n|e^{-i\vec{q}\cdot\vec{r}}|\psi_0\rangle|^2 \to |\langle\psi_n|e^{-i\vec{q}\cdot\vec{r}}\left(1+\frac{i\hbar}{m_r c}\vec{v}\cdot\vec{\nabla}\right)|\psi_0\rangle|^2. \tag{3.8}$$

(3) The nonrelativistic limits $q_{\min}$ and $q_{\max}$ in (3.3) must be replaced by their relativistic analogues

$$\text{(relativistic)}\qquad q_{\min} = \frac{(E_n-E_0)}{\hbar v}, \qquad q_{\max} = \frac{2m_r v\gamma}{\hbar}, \qquad \gamma = (1-v^2/c^2)^{-1/2}, \tag{3.9}$$

the kinematical phase space factor $\left(\frac{1}{2\pi}\frac{qdq}{(\hbar v)^2}\right)$ as it is written in (3.2a) is the same for both non-relativistic and relativistic cases. Collecting these modifications together we arrive at the relativistic expression for the differential inelastic cross-section:

$$d\sigma_n = |U(q)|^2\,|\langle\psi_n|e^{-i\vec{q}\cdot\vec{r}}\left(1+\frac{i\hbar}{m_r c^2}\vec{v}\cdot\vec{\nabla}\right)|\psi_0\rangle|^2\left[\frac{1}{2\pi}\frac{qdq}{(\hbar v)^2}\right] \tag{3.10a}$$

(relativistic)

$$= 8\pi\left(\frac{Ze^2}{\hbar v}\right)^2 |\langle\psi_n|e^{-i\vec{q}\cdot\vec{r}}\left(1+\frac{i\hbar}{m_r c^2}\vec{v}\cdot\vec{\nabla}\right)|\psi_0\rangle|^2$$

$$\times\frac{qdq}{\left[q^2-\left(\frac{E_n-E_0}{\hbar c}\right)^2+a_Z^{-2}\right]^2} \tag{3.10b}$$

Moller has calculated the relativistic inelastic form factor finding

$$\left|\langle\psi_n|e^{-i\vec{q}\cdot\vec{r}}\left(1+\frac{i\hbar}{m_r c^2}\vec{v}\cdot\vec{\nabla}\right)|\psi_0\rangle\right|^2 = \begin{cases} |(x)_{0n}|^2\left[q^2-\left(\frac{E_n-E_0}{\hbar c}\right)^2-\frac{1}{\gamma^2}\left(\frac{E_n-E_0}{\hbar c}\right)^2\right] & \text{for } q \ll \frac{m_r e^2}{\hbar}, \quad (3.11a) \\ |(e^{-i\vec{q}\cdot\vec{r}})_{0n}|^2 & \text{for } q \gg \left(\frac{E_n-E_0}{\hbar c}\right). \quad (3.11b) \end{cases}$$

For $q \gg \frac{(E_n - E_0)}{\hbar c}$ it is convenient to use the following sum rule:

$$\sum_{n \neq 1} |(e^{-i\vec{q}\cdot\vec{r}})_{0n}|^2 = 1 - F^2(q) = 1 - \left[\left(\frac{qa_r}{2}\right)^2 + 1\right]^{-4}, \tag{3.12}$$

where the sum is over both discrete and continuous states, and $F(q)$ is the $1S$ form factor of the $(\pi\mu)_{\text{atom}}$. We are now in a position to calculate the various cross sections of interest.

(A) The total inelastic cross-section σ_{inel} is defined to be:

$$\sigma_{\text{inel}} \equiv \sum_{n \neq 1} \int_{q_{\min}}^{q_{\max}} d\sigma_n. \tag{3.13}$$

We divide the range of integration into two parts, from $q_{\min}$ to q_0 and from q_0 to $q_{\max}$ where q_0 is some value of such that

$$\frac{(E_n - E_0)}{\hbar c} \ll a_Z^{-1} \ll q_0 \ll \frac{m_r e^2}{\hbar^2} = a_r^{-1}.$$

Inserting (3.11a) into (3.10b) the integral from $q_{\max}$ to q_0 is

$$\sigma^{(1)}_{\text{inel}} \equiv \sum_{n \neq 1} \int_{q_{\min}}^{q_0} d\sigma_n$$

$$= 8\pi \left(\frac{Ze^2}{\hbar v}\right)^2 \sum_{n \neq 1} |(x)_{0n}|^2 \int_{\frac{(E_n - E_0)}{\hbar v}}^{q_0} \frac{\left[q^2 - \left(\frac{E_n - E_0}{\hbar c}\right) - \frac{1}{\gamma^2}\left(\frac{E_n - E_0}{\hbar c}\right)^2\right] q\,dq}{\left[q^2 - \left(\frac{E_n - E_0}{\hbar c}\right)^2 + a_Z^{-2}\right]^2}$$

$$\approx \left(\frac{Ze^2}{\hbar v}\right)^2 4\pi \sum_{n \neq 1} |(x)_{0n}|^2 \left\{ - \frac{(v/c)^2 \left[a_Z^{-2} + \frac{1}{\gamma^2}\left(\frac{E_n - E_0}{\hbar c}\right)^2\right]}{\left[\frac{1}{\gamma^2}\left(\frac{E_n - E_0}{\hbar c}\right)^2 + a_Z^{-2}(v/c)^2\right]} \right.$$

$$\left. + \ln \left[\frac{q_0^2 (v/c)^2}{\frac{1}{\gamma^2}\left(\frac{E_n - E_0}{\hbar c}\right)^2 + a_Z^{-2}(v/c)^2}\right] \right\}. \tag{3.14}$$

Similarly, inserting (3.11b) and (3.12) into (3.10b) we obtain the integral from q_0 to $q_{\max}$. Here we can neglect screening since only large momentum transfers are involved and also extend the upper limit to infinity with negligible error, obtaining:

$$\sigma^{(2)}_{\text{inel}} \equiv \sum_{n \neq 1} \int_{q_0}^{q_{\max}} d\sigma_n \approx 8\pi \left(\frac{Ze^2}{\hbar v}\right)^2 \int_{q_0}^{\infty} \left\{1 - \left[\left(\frac{qa_r}{2}\right)^2 + 1\right]^{-4}\right\} \frac{dq}{q^3}$$

$$= \left(\frac{Ze^2}{\hbar v}\right)^2 4\pi a_r^2 \left\{ -\tfrac{13}{12} + \ln \left[\frac{4}{(q_0 a_r)^2}\right] \right\}. \tag{3.15}$$

The sum of (3.14) and (3.15) gives for the total inelastic cross-section:

$$\sigma_{\text{inel}} = \left(\frac{Ze^2}{\hbar v}\right)^2 4\pi a_r^2 \left\{ -\tfrac{13}{12} + \sum_{n\neq 1} \left|\frac{(x)_{0n}}{a_r}\right|^2 \left[-\frac{(v\gamma)^2\left[a_Z^{-2} + \frac{1}{\gamma^2}\left(\frac{E_n-E_0}{\hbar c}\right)^2\right]}{\left[\left(\frac{E_n-E_0}{\hbar}\right)^2 + (v\gamma)^2 a_Z^{-2}\right]} + \ln\left[\frac{4(v\gamma)^2}{a_r^2\left[\left(\frac{E_n-E_0}{\hbar}\right)^2 + (v\gamma)^2 a_Z^{-2}\right]}\right]\right]\right\} \quad (3.16)$$

which is independent of q_0 as it should be. The remaining sum (which includes sum over discrete states and integration over continuous states) must be done numerically on a computer, using the formulas (3.5).

(B) The excitation cross-section σ_{exc} is defined to be:

$$\sigma_{\text{exc}} \equiv \sum_{n=2}^{\infty} \int_{q_{\min}}^{q_{\max}} d\sigma_n. \quad (3.17)$$

We again divide the range of integration into two parts. The integral from $q_{\min}$ to q_0 is identical to (3.14) except the sum now goes only over the discrete states ($n = 2, 3, 4, \ldots$). The integral from q_0 to $q_{\max}$, however, is much more difficult now as we cannot use (3.12) any more but must use (3.4a) instead. Fortunately, Bethe has already evaluated the integral finding:

$$\sigma_{\text{exc}}^{(2)} \equiv \sum_{n=2}^{\infty} \int_{q_0}^{q_{\max}} d\sigma_n \approx \sum_{n=2}^{\infty} \frac{2^{10}\pi\hbar^2 n^7}{3m_r^2 v^2} A_n,$$

$$A_n = \int_{x_0}^{\infty} [(n^2-1)+3x] \frac{[(n-1)^2+x]^{n-3}}{[(n+1)^2+x]^{n+3}} \frac{dx}{x} \qquad (x = (qna_r)^2)$$

$$= \frac{(n-1)^{2n-5}}{(n+1)^{2n+5}} \left\{ -\sum_{\tau=1}^{n+2} \frac{C_{n\tau}}{\tau} + \ln\left[\frac{(n+1)^2}{(nq_0 a_r)^2}\right]\right\},$$

$$C_{n\tau} = 1 - \left(\frac{n+1}{n-1}\right)^{2n-6} \left\{ \sum_{k=0}^{\tau-6} \binom{n-3}{k} \left[\frac{-4n}{(n+1)^2}\right]^k + 3\binom{n-3}{\tau-5}\left[\frac{-4n}{(n+1)^2}\right]^{\tau-5} \left(\frac{n+1}{n-1}\right)\right\}. \quad (3.18)$$

Adding (3.18) to (3.14), restricting the sum to over discrete states only, and making use of some numerical values quoted in Ref. [3] we obtain for the excitation cross-section:

$$\sigma_{\text{exc}} = \left(\frac{Ze^2}{\hbar v}\right)^2 4\pi a_r^2 \left\{ \sum_{n=2}^{\infty} \left[-\frac{(v\gamma)^2 \left[a_Z^{-2} + \frac{1}{\gamma^2}\left(\frac{E_n - E_0}{\hbar c}\right)^2 \right]}{\left[\left(\frac{E_n - E_0}{\hbar}\right)^2 + (v\gamma)^2 a_Z^{-2} \right]} \right.\right.$$

$$\left.\left. + \ln \left[\frac{(v\gamma)^2 (n^2-1)^2}{1.79 n^4 \left[\left(\frac{E_n - E_0}{\hbar}\right)^2 + \left(\frac{v\gamma}{a_Z}\right)^2 \right] a_r^2} \right] \right] \left| \frac{(x)_{0n}}{a_r} \right|^2 \right\} \tag{3.19}$$

and again the dependence on q_0 has disappeared as it should. The ionization cross-section is then simple:

$$\sigma_{\text{ion}} \equiv \sigma_{\text{inel}} - \sigma_{\text{exc}}. \tag{3.20}$$

For the particular case of the bare Coulomb field ($A_Z^{-2} = 0$) we regain the well known Bethe–Moller formulas for the various cross-sections.

$$\sigma_{\text{inel}} = 4\pi a_r^2 \left(\frac{Ze^2}{\hbar v}\right)^2 \left\{ -(v/c)^2 + \ln \left[\frac{m_r(\hbar v\gamma/\alpha)^2}{0.16} \right] \right\},$$

($a_Z^{-2} \equiv 0$)

$$\sigma_{\text{exc}} = 4\pi a_r^2 \left(\frac{Ze^2}{\hbar v}\right)^2 (0.715) \left\{ -(v/c)^2 + \ln \left[\frac{m_r(\hbar v\gamma/\alpha)^2}{0.45} \right] \right\},$$

$$\sigma_{\text{ion}} = 4\pi a_r^2 \left(\frac{Ze^2}{\hbar v}\right)^2 (0.285) \left\{ -(v/c)^2 + \ln \left[\frac{m_r(\hbar v\gamma/\alpha)^2}{0.012} \right] \right\}. \tag{3.21}$$

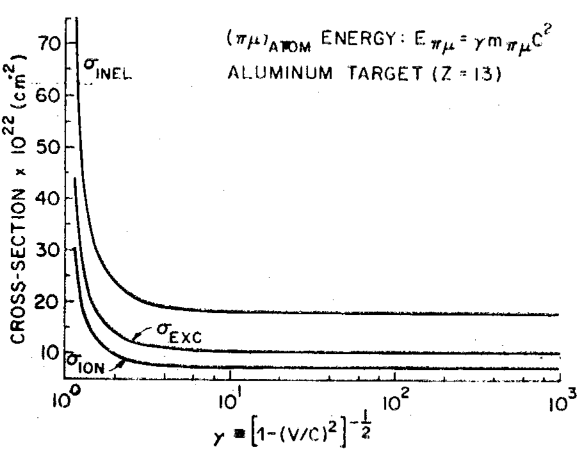

Fig. 4. Plot of the inelastic cross-sections

Computer calculation for σ_{inel}, σ_{exc}, and σ_{ion} for a $(\pi\mu)_{\text{atom}}$ with an energy $E_{\pi\mu} = \gamma m_{\pi\mu} c^2$ $(1.01 \leqslant \gamma \leqslant 10^3)$ colliding on an aluminum atom ($Z = 13$) are shown in Fig. 4. In particular, for $\gamma = 10$ we find $\sigma_{\text{ion}} = 7.4 \times 10^{-22}$ cm^2. Of a large number of $(\pi\mu)_{\text{atom}}$ incident on a thin aluminum foil of thickness t the fraction ionized is given by:

$$\frac{N_{\text{ion}}}{N_{\text{inc}}} = 1 - e^{-\sigma_{\text{ion}} nt},$$

where n is the total number of scattering centers per unit volume. The value of n may be computed from Avogadro's number N_A ($= 6.02 \times 10^{23}$ atoms/g mole), the density p, and the atomic mass M_a of the scattering foil from $n = N_A p / M_a$. In the case of aluminum, for which $Z = 13$, $M_a = 27$ and $p = 2.70$ g/cm^3, a beam of $(\pi\mu)_{\text{atom}}$ with energy $E_{\pi\mu} = 10\, M_{\pi\mu} c^2$ will be completely ionized (i.e., $N_{\text{ion}}/N_{\text{inc}} = 1$) by a foil of thickness $t = 2.2 \times 10^{-2}$ cm ($= 8.8 \times 10^{-3}$ in), in agreement with the Monte-Carlo calculation of Ref. [1]. We should emphasize that the above numbers refer only to direct ionization and not to sequential ionization via excitation.

I am indebted to Professor J. Dirk Walecka for suggesting this investigation and for offering encouragement and advice up to its completion. I am also grateful to Professor M. Schwartz, Dr. R. D. Peccei, and Dr R. D. Viollier for useful discussions during the course of this work.

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