ON THE MASTER EQUATIONS FOR PARTON DISTRIBUTIONS

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The evolution equations derived for parton distributions by Altarelli and Parisi are reformulated so, as to include explicit loss terms. This gives equations closer to master equations from statistical physics and supplies the necessary regularization of divergences without involving arguments foreign to the parton model. Relations with other approaches are also discussed.

There is growing evidence that deviations from scaling observed in deep inelastic lepton hadron scattering experiments are correctly described by quantum chromodynamics [1]. Since, however, the quark-parton model (QPM) language is much simpler than the language of quantum gauge field theories, much effort is devoted to a parton interpretation of the results. It has been found that the results of the standard (leading log) approximation to quantum chromodynamics (LLQCD) can be understood in terms of QPM (cf. e. g. [2–5]). In particular Altarelli and Parisi (AP) in Ref. [2] found that, if the structure functions of quarks ($q(x, t)$) and gluons ($G(x, t)$) are assumed to satisfy the equation (see below for the notation):

$$\frac{dq^i(x, t)}{dt} = \frac{\alpha(t)}{2\pi}\int\limits_x^1 \frac{dy}{y}\left[\sum_{j=1}^{2f} q^j(y, t)P_{q^iq^j}\left(\frac{x}{y}\right) + G(y, t)P_{q^iG}\left(\frac{x}{y}\right)\right], \tag{1}$$

$$\frac{dG(x, t)}{dt} = \frac{\alpha(t)}{2\pi}\int\limits_x^1 \frac{dy}{y}\left[\sum_{j=1}^{2f} q^j(y, t)P_{Gq^j}\left(\frac{x}{y}\right) + G(y, t)P_{GG}\left(\frac{x}{y}\right)\right], \tag{2}$$

then the resulting formulae for moments of the structure functions agree with LLQCD.

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Equations (1) and (2) have been interpreted by AP as master equations with the variable $t = \ln(Q^2/Q_0^2)$ playing the role of time. The coefficients $P_{ab}(z)$ are obtained, by a suitable regularization procedure, from the Born transition probabilities $b \to a$:

$$K_{q_iq_j}(z) = \tfrac{4}{3}\,\frac{1+z^2}{1-z}\,\delta_{ij}, \tag{3}$$

$$K_{Gq_i}(z) = \tfrac{4}{3}\,\frac{1+(1-z)^2}{z}, \tag{4}$$

$$K_{GG}(z) = 6\left[\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)\right], \tag{5}$$

$$K_{q_iG}(z) = \tfrac{1}{2}\,[z^2+(1-z)^2], \tag{6}$$

where everything is calculated in the infinite momentum frame and z is the ratio of the longitudinal momentum of particle a to the longitudinal momentum of particle b. The non-integrable singularities at $z = 1$ must be removed by a regularization procedure, in order to obtain from Eqs (1), (2) finite moments of the structure functions. The usual justification for the regularization prescription, however, uses elements of renormalization theory.

Here we suggest an alternative approach closer both to QPM and to what is done in statistical physics. Instead of $P_{ab}(z)$ we substitute into equation (1) and (2) the QPM transition probabilities $K_{ab}(z)$, but we note that then the equations are incomplete. The right hand sides contain only gains to the structure functions at x due to decays of partons with momentum fractions exceeding x. In order to have a master equation, we must subtract losses due to decays of partons with momentum fraction x. Using also for losses the transition probabilities $K_{ab}(z)$ we find the master equations

$$\frac{dq^i(x,t)}{dt} = \frac{\alpha(t)}{2\pi}\left\{\int_x^1 \frac{dy}{y}\left[\sum_{j=1}^{2f} q^j(y,t)K_{q_iq_j}\left(\frac{x}{y}\right) + G(y,t)K_{q_iG}\left(\frac{x}{y}\right)\right]\right.$$
$$\left. - q^i(x,t)\int_0^x \frac{dy}{x}\sum_{j=1}^{2f} K_{q_jq_i}\left(\frac{y}{x}\right)\right\}, \tag{7}$$

$$\frac{dG(x,t)}{dt} = \frac{\alpha(t)}{2\pi}\left\{\int_x^1 \frac{dy}{y}\left[\sum_{j=1}^{2f} q^j(y,t)K_{Gq_j}\left(\frac{x}{y}\right) + G(y,t)K_{GG}\left(\frac{x}{y}\right)\right]\right.$$
$$\left. - G(x,t)\int_0^x \frac{dy}{x}\left[\sum_{j=1}^{f} K_{q_jG}\left(\frac{y}{x}\right) + \theta(2y-x)K_{GG}\left(\frac{y}{x}\right)\right]\right\}. \tag{8}$$

The θ function in the last term in Eq. (8) is necessary in order to avoid double counting. With $x/2 \leqslant y \leqslant 1$ each possible final gluon pair is counted exactly once. For the same reason there are no K_{Gq} and $K_{\bar{q}G}$ contributions to the loss terms. A simple calculation shows that in the right hand side of Eqs (7), (8) all divergences cancel and that the results for the moments of the structure functions agree with LLQCD. Thus these equations can be considered as a reformulation of the AP equations, which is closer to QPM (no parton loops and "naive" transition probabilities (3)–(6)) and to statistical physics (the master equation is a balance equation between gains and losses, which taken separately do not have to be finite).

In order to make contact with the usual field theory approach [3–7] let us note that (cf. e. g. [5], Eqs (38a, b))

$$-\int_0^x \frac{dy}{x} K_{qq}\left[\frac{y}{x}\right] = \gamma_F, \qquad -\int_0^x \frac{dy}{x}\left[f K_{qG}\left(\frac{y}{x}\right) + \theta(2y-x) K_{GG}\left(\frac{y}{x}\right)\right] = \gamma_G, \tag{9}$$

are just the anomalous dimensions of the quark and gluon fields, respectively. Multiplying both sides of Eqs (7) and (8) respectively by

$$d_i(t) = \exp\left(-\gamma_i \int_{t_0}^{t} \frac{\alpha(\tau)d\tau}{2\pi}\right), \qquad i = F, G \tag{10}$$

and combining the derivatives on the left hand sides with the loss terms to derivatives of qd_F and Gd_G we obtain from (7) and (8) Eq (30) from Ref. [5] and its analogue for gluons (cf. also [7]).

To summarize: the right hand side of the AP equations contains both gains and losses to the structure functions. This is explicit when the form (7), (8) is used. Eliminating the loss terms as described above one obtains the differential form of the DDT equations. It is also possible to eliminate the gain terms and obtain still another form of the equations. Equations (7), (8) are equivalent with and simply connected to equation derived from other approaches. They seem, however, to be particularly suitable for arguments based on the QPM.

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