REGULAR GENERAL RELATIVISTIC CHARGED FLUID SPHERES

By H. A. Buchdahl

Department of Theoretical Physics, Faculty of Science, Australian National University, Canberra*

(Received November 30, 1978)

Regular (static) fluid spheres are those which satisfy certain conditions of differentiability, of positivity of various quantities, and the like. After a review of general conclusions which can be drawn from an appeal to boundary conditions alone the main body of the paper is concerned with a simple regular charged analogue S of the Schwarzschild interior solution S^*. Any such analogue is here required to share with S^* the property of being conformally flat. Guided by a further property of S^* a second specific assumption then leads to a model with a particularly simple form of g_{44}. This in turn leads to transparent, explicit expressions for various physical quantities of interest. It thus becomes possible to ensure explicitly, without recourse to numerical calculations or to power series, that all conditions of regularity and other conditions arising from physical considerations are satisfied.

1. Introduction

Certain solutions to Einstein's field equations representing a charged static fluid sphere were recently considered in this journal by Singh and Yadav (1978) who also gave a number of references to earlier analogous papers. The equations are not solved by imposing relations between certain field variables which directly reflect desired physical situations — equations of state and relations between mass density and charge density spring to mind — but rather the explicit forms of certain functions which appear in the field equations are arbitrarily prescribed in a way which will lead to a tractable problem. In the case of an uncharged sphere *one* such function may be prescribed, whereas when the sphere can carry an electric charge *two* such functions may be prescribed. This wide freedom of choice entails that the mere derivation of "exact solutions" becomes a somewhat pointless exercise. The least one might demand is that such solutions be, in some ill-defined sense, "simple". One would, however, scarcely regard the solutions of Singh and Yadav referred to above as being particularly simple; nor are the positivity of pressure p and material density ϱ explicitly taken into account — it is not obvious, for instance, that the condition $p \geqslant 0$ can necessarily be accommodated at all (cf. Section 7c).

* Address: Dept. of Theoretical Physics, Australian National University, Canberra, 2600 Australia. Present address: Churchill College, Cambridge, England.

In the light of these remarks it seems that one might proceed along two possible avenues: (1) to investigate what information might be gained about static, charged fluid spheres when only conditions of a general kind are imposed; and (2) to investigate specific (classes of) solutions which are "simple" and reduce to well-known solutions when the sphere happens to be uncharged. With regard to (1) it seems natural to think of establishing general inequalities after the fashion of Buchdahl (1959) (hereafter referred to as B). Not surprisingly, this does not appear to be possible unless one imposes restrictions on the charge density so stringent as to make the problem uninteresting. It remains under this heading to review the general conclusions which may be drawn by merely appealing to boundary conditions; and this is done in Section 4. Under the second heading it is natural to look for possible counterparts to the best-known solution of all, i. e. the Schwarzschild interior solution S^*. The case examined by Kyle and Martin (1967) is of this kind but it is hardly simple in as far as their expression for g_{44} is so involved as to make the consequent explicit examination of their solution quite difficult. (This last remark also applies to the solution of Nduka (1976) for here too one has to have recourse to purely numerical work; but in any event it does not reduce to S^* when there is no charge. Again, Wilson (1969) merely exhibits g_{44} as a power series.) At any rate, one still has endless possibilities and to narrow down the range of available choices I require in the first place that any sphere admitted for examination shall share with the S^* the property of being conformally flat. A particular model is then adopted in Section 5b partly on the basis of an appeal to another property possessed by S^*. It turns out that g_{44} is particularly simple in form and as a result the examination of the physical properties of the model can be carried out explicitly in terms of simple, elementary functions. In particular it is a straightforward task to accommodate the various regularity conditions (see Section 2) and to deal explicitly with questions surrounding the equation of state, the speed of sound, and the like.

2. *Regular spheres*

Canonical coordinates in which the metric has the generic form

$$ds^2 = -e^{\lambda(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + e^{\nu(r)}dt^2 \tag{2.1}$$

are used throughout; and units are so chosen that Newton's constant and the speed of light both take the value unity. Then all (static) spheres which are admitted for consideration must satisfy the following (not necessarily mutually independent) conditions:

(*i*) the material energy momentum tensor has the form diag $(-p, -p, -p, \varrho)$, where ϱ is the material density and p the hydrostatic pressure;

(*ii*) ϱ, p, λ, ν and the (invariant) charge density σ are continuous, sufficiently often differentiable functions of r within the entire sphere whose (coordinate) radius is R;

(*iii*) $\varrho \geqslant 0$, $p \geqslant 0$, $e^\lambda > 0$, $e^\nu > 0$ in the range $0 \leqslant r \leqslant R$;

(*iv*) $d\varrho/dr \leqslant 0$, $dp/dr \leqslant 0$ in the range $0 \leqslant r < R$;

(*v*) $dp/d\varrho > 0$ in the range $0 \leqslant r < R$;

(*vi*) the boundary of S (i. e. the value of R) is defined by the first zero of p as one proceeds outwards from the origin unless $p = 0$ everywhere;

(*vii*) at the boundary λ, ν and $d\nu/dr$ are continuous.

3. *The field equations*

Consider first quantities related to the electric field. The only components of the electromagnetic potential ϕ_i and of the current J^i which do not necessarily vanish are $\phi_4(=:\phi)$ and $J^4 = \sigma e^{-\nu/2}$. Then, with

$$u := \frac{1}{8\pi} e^{-\lambda-\nu}\phi'^2, \tag{3.1}$$

the electromagnetic energy tensor is diag $(u, -u, -u, u)$; see, for example, Bonnor (1960). (Primes denote differentiation with respect to r.) Now, for the most part following the notation of B, set

$$P := 4\pi p, \quad \hat{\varrho} = 4\pi\varrho, \quad U := 4\pi u, \quad \varrho^* := \hat{\varrho} + U,$$

$$x := r^2, \quad \zeta := e^{\nu/2}, \quad w := r^{-3}\int_0^r r^2\varrho^* dr =: r^{-3}m(r),$$

$$y := (1-2xw)^{1/2}, \quad q := 4\pi\int_0^r r^2\sigma e^{\lambda/2}dr. \tag{3.2a–i}$$

The Einstein–Maxwell equations then become here

$$2r^2(P-U) = e^{-\lambda}(r\nu'+1)-1, \tag{3.3}$$

$$2r^2\varrho^* = e^{-\lambda}(r\lambda'-1)+1, \tag{3.4}$$

$$(P-U)' = -\tfrac{1}{2}\nu'(P+\hat{\varrho})+4r^{-1}U, \tag{3.5}$$

$$(r^2e^{-(\lambda+\nu)/2}\phi')' = -4\pi\sigma r^2 e^{\lambda/2}. \tag{3.6}$$

Bearing the regularity of S in mind, (3.4) is equivalent to

$$e^{-\lambda} = y^2(= 1-2m(r)/r), \tag{3.7}$$

whilst (3.6) leads directly to

$$U = \tfrac{1}{2}x^{-2}q^2. \tag{3.8}$$

Indicating derivatives with respect to some variable z by a subscript z following a comma, the two remaining equations may be written as

$$P = U-w+2y^2\zeta_{,x}/\zeta, \tag{3.9}$$

$$P_{,x} = U_{,x}-(2xw_{,x}+3w+P-U)\zeta_{,x}/\zeta+2U/x, \tag{3.10}$$

and upon eliminating P between these one gets an equation analogous to B (2.12), viz.

$$(1-2xw)\zeta_{,xx}-(xw_{,x}+w)\zeta_{,x}-(\tfrac{1}{2}w_{,x}+U/x)\zeta = 0. \tag{3.11}$$

Two remarks should be made at this stage. First the kind of argument which led in B to the powerful inequality $\Delta \geqslant \frac{1}{9}$ (where Δ is the boundary value of ζ^2) is not now available — in fact, on account of the positivity of U one no longer has a general inequality of this kind. Second, if one's objective is merely to construct explicit solutions of the field equations one need only choose the functions w and U in any way which leads to elementary solutions of the linear equation (3.11); and to this extent one is faced with an essentially trivial problem.

4. Consequences of boundary conditions

Bearing (3.2g) in mind one has for $r \geqslant R$

$$m(r) = m(R)+ \int_R^r Ur^2 dr. \tag{4.1}$$

Here

$$m(R) = m_\varrho + m_\sigma, \tag{4.2}$$

where

$$m_\varrho = \int_0^R r^2 \hat{\varrho} dr, \qquad m_\sigma = \int_0^R r^2 U dr. \tag{4.3}$$

According to (3.3) and (3.4) $\lambda + \nu = 0$ when $r \geqslant R$, so that with $q(R) = : Q$ one has from (3.6) and (3.8)

$$-r^2\phi' = Q, \qquad U = Q^2/2r^4. \tag{4.4}$$

Q is evidently the total charge on S. Still for $r \geqslant R$, one now finds from (4.1) and (4.4) that

$$m(r) = m(R)+Q^2/2R-Q^2/2r,$$

whence in view of (3.7) and (4.2)

$$e^{-\lambda} = 1-(2m_\varrho+2m_\sigma+Q^2/R)r^{-1}+Q^2r^{-2}. \tag{4.5}$$

By contemplating the motion of a distant test particle one concludes that the factor multiplying r^{-1} here is twice the gravitational field producing mass M of S:

$$M = m_\varrho+m_\sigma+Q^2/2R. \tag{4.6}$$

Now, using (3.8) in (4.3), an integration by parts leads to the equation

$$m_\sigma = -\tfrac{1}{2}Q^2/R+\int_0^R (qq'/r)dr$$

so that

$$M = m_\varrho+ \int_{r=0}^R r^{-1} q dq. \tag{4.7}$$

The mass of S is thus the sum of two parts: in the chosen coordinate system the first is *formally* the "Newtonian mass of the material" of the sphere and the second is the "Newtonian self-energy of the charge distribution". (Note that Eq. (3.13) of Singh and

Yadav (1978) is incorrect.) That the simple result (4.7) holds for any regular sphere independently of its detailed structure does not appear to be always explicitly recognized.

Henceforth subscripts b and c will indicate boundary and central values respectively. In particular

$$\Delta := y_b^2 = \zeta_b^2 = 1-2M/R+Q^2/R^2, \tag{4.8}$$

whilst it follows from the continuity of v' that

$$\zeta_b\zeta_b' = M/R^2-Q^2/R^3. \tag{4.9}$$

Eq. (3.9) is of course now satisfied at the boundary. For future reference

$$w_b = M/R^3-Q^2/2R^4, \quad U_b = Q^2/2R^4. \tag{4.10}$$

5. *On charged analogues of the Schwarzschild interior solution*

(a) Generic remarks: Conformal flatness

The Schwarzschild interior solution S^* may be characterized by the constancy of ϱ or the constancy of w. Furthermore, it is well known that S^* is conformally flat. One may take the view that this is the most interesting property of S^*; and I shall therefore require any regular charged analogue of S^* to be conformally flat.

In the present notation the condition that (2.1) be conformally flat reduces to the simple equation

$$(1-2xw)\zeta_{,xx}-(xw_{,x}+w)\zeta_{,x}+\tfrac{1}{2}\, w_{,x}\zeta = 0. \tag{5.1}$$

In view of (3.11) this means that the relation

$$xw_{,x}+U = 0 \tag{5.2}$$

must be satisfied.

To arrive at a definite model a second condition needs to be imposed (cf. the remarks at the end of Section 3). Before considering this problem it is worth deriving some relations which follow as a consequence of the adoption of (5.2) alone. First,

$$\hat{\varrho} = 2xw_x+3w-U = 3(w-U) \tag{5.3}$$

so that by (4.10)

$$\hat{\varrho}_b = 3R^{-3}(M-Q^2/R). \tag{5.4}$$

I shall call a sphere gaseous if $\varrho_b = 0$ and (5.4) then entails that $M = Q^2/R$. Moreover for any regular sphere subject to (5.2) one must certainly have

$$M \geqslant Q^2/R. \tag{5.5}$$

It is useful to introduce the parameter

$$\chi := Q/M. \tag{5.6}$$

Then, for instance, when $\varrho_b = 0$

$$\Delta = 1-\chi^{-2} \tag{5.7}$$

so that one certainly cannot have a gaseous sphere of this kind unless $\chi > 1$; cf. the conclusion stated after Eq. (5.12).

From (3.5) one has at the boundary

$$P'_b = U'_b + 4U_b/R - \hat{\varrho}_b \zeta'_b/\zeta_b.$$

Using (3.8), (4.8–10) and (5.4) this gives

$$P'_b = R^{-5}[RQq'_b - 3\Delta^{-1}(M-Q^2/R)^2]. \tag{5.8}$$

If one is to have $P'_b \leqslant 0$ this relation constitutes a restriction on the possible values of σ_b.

(b) Choice of a particular model

In deciding upon a particular model one or other of the choices $w' = 0$, $\varrho' = 0$ naturally springs to mind. However, since U does not vanish everywhere the first of these is in immediate conflict with (5.2). The constancy of ϱ on the other hand would imply, in view of (5.2, 3) that the sphere is not regular. Accordingly, some other condition has to be imposed and it will have to suffice to be guided by considerations of simplicity.

For S^* one has

$$M = w_c R^3 \tag{5.9}$$

and this relation will now be assumed to continue to hold. Therefore, on account of (4.10), $w_b + U_b = w_c$ and the simplest way to ensure that this relation will hold is to require that $w+U = w_c$ throughout the range $0 \leqslant r \leqslant R$. Taking (5.2) into account one is therefore led to taking the relations

$$w = a-bx, \qquad U = bx \tag{5.10a, b}$$

as characterizing (the interior of) a simple analogue S of S^*, where a and b are positive constants:

$$a = M/R^3, \qquad b = Q^2/2R^6. \tag{5.11}$$

An important conclusion may be drawn straight away. From (5.10b) and (3.8) it follows that $q'_b = 3Q/R$. Inserting this value of q'_b in (5.8) one has

$$P'_b = -3(M^2-Q^2)/R^5\Delta. \tag{5.12}$$

Accordingly P will become negative as one goes inwards from the boundary unless $M \geqslant Q$, i. e. S must certainly have $\chi \leqslant 1$.

Combining (5.10b) with (3.8) and (5.11) one finds that

$$q = Q(r/R)^3. \tag{5.13}$$

If $\hat{\sigma} := 4\pi\sigma$ it then follows from (3.2i) that

$$\hat{\sigma} = (3Q/R^3)y. \tag{5.14}$$

In particular $\hat{\sigma}_c = 3Q/R^3$, a harmonious counterpart to the relation (5.9), i. e. $\hat{\varrho}_c = 3M/R^3$, which obtains here. Note that

$$(\sigma/\varrho)_c = \chi; \tag{5.15}$$

and, like the material density, the charge density falls off outwards; see also Section 8.

6. *Solution of the equation for ζ when $Q < M$*

It has already been remarked just after Eq. (5.12) that the case $\chi > 1$ is of no interest here. The sphere with $\chi = 1$ will be left aside for the time being (see Section 8) and it remains to consider the case $\chi < 1$ in detail. Given (5.10), the equation (3.11) for ζ is

$$(1-2ax+2bx^2)\zeta_{,xx}-(a-2bx)\zeta_{,x}-\tfrac{1}{2}\,b\zeta = 0.$$

The change of variable

$$t := k^{-1}(1-\chi^2 ax), \tag{6.1}$$

with $k := (1-\chi^2)^{\frac{1}{2}}$, transforms the equation into

$$(t^2-1)\zeta_{,tt}+t\zeta_{,t}-\tfrac{1}{4}\,\zeta = 0. \tag{6.2}$$

This has a very simple solution, viz.

$$\zeta = A(t+1)^{1/2}+B(t-1)^{1/2}, \tag{6.3}$$

where A and B are constants of integration. It may be noted in passing that in terms of the variable t

$$y^2 = (k/\chi)^2(t^2-1), \qquad \varrho = \varrho_c kt, \tag{6.4}$$

and that the boundary value of t is

$$t_b = (M-Q^2/R)\,(M^2-Q^2)^{-1/2}. \tag{6.5}$$

7. *Physical features of the solution when $Q < M$*

(a) The constants A and B

From (6.4) $\zeta_b = (k/\chi)\,(t_b^2-1)^{\frac{1}{2}}$, whilst $(\zeta_{,t})_b$ may be found as follows. From (4.9), using (5.11) and (6.1) in turn, there comes $\zeta_b\zeta_b' = Rakt_b$; and $\zeta_b' = -(2\chi^2 aR/k)\,(\zeta_{,t})_b$, because of (6.1) and (3.2e). Combining these results one finds that $(\zeta_{,t})_b = -(k/2\chi)t_b(t_b^2-1)^{-\frac{1}{2}}$. Equating the expressions just found for ζ_b and $(\zeta_{,t})_b$ to those given by (6.3) one has two equations for A and B whose solution is

$$A = (k/2\chi)\,(2t_b+1)\,(t_b-1)^{1/2}, \qquad B = -(k/2\chi)\,(2t_b-1)\,(t_b+1)^{1/2}. \tag{7.1}$$

(b) First limitation on Δ

Since $kt_c = 1$ one has at once

$$\zeta_c = k^{-1/2}[A(1+k)^{1/2}+B(1-k)^{1/2}]. \tag{7.2}$$

The regularity condition $\zeta_c > 0$ now becomes after a little manipulation

$$4t_b^3-3t_b-1/k > 0, \tag{7.3}$$

or explicitly

$$t_b > [(1+\chi)/8k]^{1/3}+[(1-\chi)/8k]^{1/3}. \tag{7.4}$$

This implies that

$$\Delta > \frac{1-\chi^2}{4\chi^2}\left\{\left(\frac{1+\chi}{1-\chi}\right)^{1/3}+\left(\frac{1-\chi}{1+\chi}\right)^{1/3}-2\right\}, \tag{7.5}$$

which is an instructive result; for when $\chi \to 0$ it shows that

$$\Delta > \tfrac{1}{9}-\tfrac{7}{243}\chi^2+O(\chi^4), \tag{7.6}$$

the first term representing the correct limit for S^*, whereas when $\chi \to 1$ one has

$$\Delta > [\tfrac{1}{2}(1-\chi)]^{2/3}+O(1-\chi). \tag{7.7}$$

This shows that by a suitable choice of χ in the range $0 \leqslant \chi < 1$ one can always have a sphere S for which Δ has any arbitrarily small desired value; bearing in mind that the right hand member of (7.5) is the actual value of Δ when $\zeta_c = 0$.

(c) The pressure

The calculation of P from (3.9) is straightforward. One finds that

$$P/ak = -\frac{A(2t-1)(t+1)^{1/2}+B(2t+1)(t-1)^{1/2}}{A(t+1)^{1/2}+B(t-1)^{1/2}}. \tag{7.8}$$

It is sometimes useful to use in place of t the auxiliary variable

$$z := \operatorname{ar\,cosh} t. \tag{7.9}$$

Then, for instance,

$$\zeta = k\chi^{-1}\sinh\tfrac{1}{2}(3z_b-z). \tag{7.10}$$

Eq. (7.8) becomes

$$P = ak\frac{\sinh\frac{3}{2}(z-z_b)}{\sinh\frac{1}{2}(3z_b-z)}. \tag{7.11}$$

Evidently, by inspection, $P \geqslant 0$, $P' \leqslant 0$ everywhere in S, as required.

(d) The equation of state

The density is given by

$$\hat{\varrho} = 3akt \tag{7.12}$$

so that $\varrho > 0$ $(0 \leqslant r \leqslant R)$ and $\varrho' < 0$ $(0 < r \leqslant R)$, as required. With (7.12), (7.8) is in effect the explicit equation of state of the fluid constituting the sphere. Beyond its relative simplicity it does not seem to have any features of special interest; however, see also Sections 7e, f.

(e) The speed of sound

The sphere being charged, it is not possible on the present naive phenomenological level to say what the speed of propagation of a sound wave of arbitrary frequency will be. However, if the frequency is sufficiently large the adiabatic speed of propagation by the fluid is presumably given by

$$v^2 = dP/d\hat{\varrho} = (1/3ak)P_{,t}, \tag{7.13}$$

with P given by (7.8). Thus

$$v^2 = -\tfrac{2}{3}\left(1+\frac{AB(t^2-1)^{-1/2}}{[A(t+1)^{1/2}+B(t-1)^{1/2}]^2}\right) \equiv \tfrac{2}{3}\left(\frac{\sinh 3z_b}{4\sinh z \sinh^2(\tfrac{3}{2}z_b-\tfrac{1}{2}z)}-1\right). \tag{7.14}$$

As one approaches the boundary

$$v_b^2 = \tfrac{1}{2}(t_b^2-1)^{-1}, \tag{7.15}$$

and elsewhere v^2 is certainly positive, bearing in mind that $z < 3z_b$ (cf. (7.10)). The stability condition $dp/d\varrho > 0$ is therefore satisfied everywhere, as required.

The inequality (7.4) may be used in (7.15). Then as $\chi \to 0$ (7.15) reduces to

$$v_b^2 < 9/2\chi^2, \tag{7.16}$$

whereas when $\chi \to 1$ it reduces to

$$v_b^2 < [4(1-\chi)]^{1/3}. \tag{7.17}$$

Again, $v_b = 1$ when $t_b = (3/2)^{\frac{1}{2}}$. On the other hand, bearing in mind that v reaches its maximum at the centre, the condition $v_c = 1$ leads to a quadratic equation for cosh (3 arcosh t_b). The point of interest here is that it leads to acceptable values of t_b only if $0 < \chi < \frac{1}{4}$. Evidently when $\chi \geqslant \frac{1}{4}$ the material of the sphere is too soft to be capable of transmitting sound (of sufficiently high frequency) with unit speed anywhere in S.

(f) $p_c = \frac{1}{3}\varrho_c$: Second limitation on Δ

It is not unusual to take it for granted that the trace of the energy momentum tensor must be non-negative. Here this condition requires $p \leqslant \frac{1}{3}\varrho$. Some straight-forward work based upon (7.8) and (7.12) leads to the conclusion that p/ϱ takes its greatest value at the centre so that the condition above is accommodated everywhere if $p_c \leqslant \frac{1}{3}\varrho_c$. Since $t_c = k^{-1}$ this inequality requires that

$$4t_b^3 - 3t_b - (9-5k^2)/[k(k^2+3)] \geqslant 0, \tag{7.18}$$

again from (7.8) and (7.12). Therefore t_b must be not less than the positive zero of the left hand member of (7.18). Although the zero is an elementary algebraic function of k it is simpler to write

$$t_b \geqslant \cosh\left(\tfrac{1}{3}\operatorname{arcosh}\frac{9-5k^2}{k(3+k^2)}\right). \tag{7.19}$$

Since $\Delta = (k/\chi)^2(t_b^2-1)$ there follows the inequality

$$\Delta \geqslant \frac{1-\chi^2}{\chi^2}\sinh^2\left(\tfrac{1}{3}\operatorname{arcosh}\frac{4+5\chi^2}{(1-\chi^2)^{1/2}(4-\chi^2)}\right). \tag{7.20}$$

When $\chi \to 0$ the right hand member of this tends correctly to $\frac{4}{9}$; cf. B(3.10) with $\delta = 1$. On the other hand when $1-\chi$ is sufficiently small (7.20) reduces to

$$\Delta \geqslant [\tfrac{3}{4}(1-\chi)]^{2/3}. \tag{7.21}$$

(7.18) entails incidentally that v_c is finite as long as χ is non-zero.

(g) The potential energy

Kyle and Martin (1967) and, following them, Wilson (1969) consider a quantity δM which they call the "mass defect". It is defined as the difference between M and the integral M_0 of the material density ϱ taken over the elements of proper volume of the sphere. In the present notation

$$\delta M := M - M_0 := M - \int_0^R \hat{\varrho} y^{-1} r^2 dr. \tag{7.22}$$

It is difficult to attach a physical significance to this quantity which does not explicitly include the electrostatic energy density. The local proper energy density is T_4^4, not ϱ, and it would seem more appropriate to define a quantity

$$\Omega := M - M_0^* =: M - \int_0^\infty \varrho^* y^{-1} r^2 dr, \tag{7.23}$$

so that M_0^* is the total bare mass of the system. Ω occurs in Section 6 of B (with sign reversed) where it is called the "gravitational potential energy".

The integrals on the right of (7.22) and (7.23) may be evaluated in terms of elliptic integrals of the first and second kind, together with elementary functions. It does not seem to be worthwhile reproducing the explicit results here for they are not very enlightening. Moreover, one has to exercise care when attempting physical interpretations — as distinct from writing down merely formal results — because the equations being considered involve R; and this is a coordinate dependent quantity.

(h) The physical radius

As has just been remarked, R, unlike Q and M, is a coordinate-dependent quantity. In its place one should use the physical radius

$$\mathscr{R} := \int_0^R dr/y. \tag{7.24}$$

Let

$$F(x; n) := \int_0^x [(1-t^2)(1-nt^2)]^{-1/2} dt \tag{7.25}$$

denote the incomplete elliptic integral of the first kind. Then (7.24) becomes here

$$\mathscr{R} = R\xi^{-1}F(\xi; n) \tag{7.26}$$

with

$$\xi^2 := (1+k)M/R, \qquad n = (1-k)/(1+k). \tag{7.27}$$

In the limit of sufficiently small k (7.26) becomes

$$\mathscr{R} = R\eta^{-1} \operatorname{artanh} \eta + O(k^2), \tag{7.28}$$

where $\eta^2 = M/R$. It may be noted that $\xi^2 < 1$ always since

$$\Delta = (1-\xi^2)(1-n\xi^2). \tag{7.29}$$

One therefore has here the result that

$$1 < \mathscr{R}/R < K(n), \tag{7.30}$$

where $K(n)$ is the complete elliptic integral of the first kind.

If, as remarked previously, expressions like those for δM or Ω are to be given invariant form one has to eliminate R in favour of $\mathscr{R}$. This requires one to solve (7.26) for R. Evidently even Δ already becomes a very complicated function of M, Q and $\mathscr{R}$.

8. *The case* $Q = M$

It was already concluded in Section 5b that one must have $\chi \leqslant 1$. It therefore remains to examine the case with $\chi = 1$. Accordingly, $a^2 = 2b$ now and, with $t := 1-ax$, the equation for ζ becomes

$$t^2\zeta_{,tt} + t\zeta_{,t} - \tfrac{1}{4}\zeta = 0,$$

whence

$$\zeta = At^{1/2} + Bt^{-1/2}, \tag{8.1}$$

where A and B are constants of integration. For P one obtains the expression

$$P = -2At^2/(B+At). \tag{8.2}$$

When $A \neq 0$ the vanishing of P requires that t_b be zero. However, if P is to be non-negative B cannot vanish and the vanishing of t_b would entail ζ becoming singular at the boundary. It follows that A must vanish so that

$$P = 0 \text{ everywhere}; \tag{8.3}$$

cf. Bonnor (1960). The material density is given by

$$\hat{\varrho} = 3at. \tag{8.4}$$

t_b may be taken arbitrarily within the range $0 < t_b < 1$. Clearly,

$$t_b = \varrho_b/\varrho_c. \tag{8.5}$$

Also,

$$y = t, \tag{8.6}$$

and therefore

$$y_b = \zeta_b = 1 - M/R = t_b = Bt_b^{-1/2}, \tag{8.7}$$

whence

$$B = t_b^{3/2}. \tag{8.8}$$

Again, from (5.14),

$$\hat{\sigma} = 3at, \tag{8.9}$$

so that

$$\sigma = \varrho \text{ everywhere}. \tag{8.10}$$

Finally,

$$\mathscr{R} = R\eta^{-1} \operatorname{artanh} \eta, \tag{8.11}$$

cf. (7.28). To find R as a function of $\mathscr{R}$ one therefore has to invert the relation

$$\mathscr{R}/M = \eta^{-3} \operatorname{artanh} \eta. \tag{8.12}$$

In particular, when $1-\eta$ is sufficiently small

$$R \sim M(1+4e^{-2\mathscr{R}/M}). \tag{8.13}$$

Evidently as R approaches the value M the physical radius becomes indefinitely large.

9. Concluding remark

The case $\chi > 1$ may of course be dealt with in detail after the fashion of Sections 6 and 7 above. Consistently with (5.12) it then emerges that the pressure is negative throughout the interior of the sphere and this is unacceptable. One might conceivably argue that the sphere is under these circumstances to be regarded as solid and so capable of sustaining tensions; but it is difficult to see why the pressure should then be taken to be isotropic in the first place.

REFERENCES

Bonnor, W. B., *Z. Phys.* **160**, 59 (1960).
Buchdahl, H. A., *Phys. Rev.* **116**, 1027 (1959).
Kyle, C. F., Martin, W. A., *Nuovo Cimento* **50A**, 583 (1967).
Nduka, A., *Gen. Rel. Grav.* **7**, 493 (1976).
Singh, T., Yadav, R. B. S., *Acta Phys. Pol.* **B9**,475 (1978).
Wilson, S. J., *Can. J. Phys.* **47**, 2401 (1969).