TWO-FERMION EQUATION WITH FORM-FACTORS AND ELECTROMAGNETIC HIGH MASS RESONANCES*

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Using a relativistic two-body one-time formalism we derive static potentials which include form-factors for both particles. Another new feature is the derivation of fourth order coupled radial eigenvalue problems taking into account full relativistic angular momentum analysis and spins for both particles. For a particular choice of magnetic form-factors and particular total quantum numbers we demonstrate numerically the existence of high mass electromagnetic resonances.

1. Introduction

The main purpose of this paper is to investigate some consequences of the effective potential between two fermions taking into account form-factors and full relativistic spinorial kinematics. The motivation stems from the existence of high-mass resonances in pure electrodynamics in simpler models (called super-positronium) and the aim is to understand better the existence and to characterize the quantum numbers of such possible resonances using these potentials.

The effective static potential between two fermions in the lowest order in quantum electrodynamics (QED) is well known and leads to the Breit equation. This approximation is adequate for weak binding. We are mainly interested in strong binding and therefore we need a highly off-mass extrapolation of such potentials. There are no satisfactory and complete treatments of strong binding [1]. A method often used in particle physics is to introduce form-factors at the vertices to take into account the self-energy and radiative corrections in an already renormalized form. This approximation would be adequate if

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we have a satisfactory way of determining the form-factors without introducing phenomenological functions for them. We shall show that this can be done under certain conditions.

If one of the constituent particles is heavy, it has been shown that the limiting Dirac equation with an anomalous magnetic moment coupling (including a magnetic form-factor) leads to high energy narrow resonances [2]. Physically this is due to the fact that the magnetic interactions become very strong at small distances and lead to almost binding in the continuum, i.e. positive mass states, hence to resonances. From the point of view of QED it is entirely a non-perturbative self-energy effect and cannot be seen in treatments using interparticle potentials only. To understand this problem in the equal mass case it is natural to consider form-factors for both of the particles. It should be emphasized that form-factors in principle are of course contained in QED. But their inclusion into the bound state problem, even to the lowest order, has not been, to our knowledge, carried out yet. Usually in the two-body problem, one considers the exchange interaction terms between the particles and omits the self-energy terms. The form-factors provide a way to put back these self-energy terms, presumably already summed to all orders, if we knew the correct functional form of the form-factors.

In this paper we shall describe a possible self-consistent determination of resonances due to multipole and in particular magnetic interactions. There are of course a great many other forms of two-body equations in the literature. Most of them are equivalent to the lowest order. In Section 2 we discuss the properties of a relativistic two-fermion equation with form-factors, using a one-time formalism. In this framework under the assumption that the form-factors are given functions of q^2, we derive the explicit form of effective potential in the static approximation. Next we show that the Breit equation, reduced one-body Dirac equation and Barut–Kraus equtiaon follow from the general dynamical equation as very special cases.

Next in Section 3 we derive the explicit form of general dynamical equation in radial coordinates. It turns out that in the case where the total angular momentum J of the two-fermion system equals zero, the general dynamical equation reduces to a Schrödinger-like radial equation with an energy dependent potential. However, in the case $J > 0$ the dynamical two-fermion equation reduces to fourth order radial equation. The occurrence of these fourth order equations is a new phenomenon, we believe, in Quantum Theory. Thus the eigenvalue problem changes from the Sturm–Liouville type to the fourth order radial equation type. The spectral analysis of the fourth order radial equations is just beginning to develop and might provide a number of new aspects of physical importance.

The explicit form of the magnetic form-factor is suggested in Section 4. The calculation is based on the assumption that near the resonance energy the lowest order QED radiative corrections to the point vertex when the mass m of the electron is replaced by the M of the bound state provide a reasonable approximation of the physical form-factor. In other words, we look for a self-consistent form-factor for a highly localized resonance state.

In Section 5 we carry out a numerical analysis of the two-fermion radial equation under the assumption that masses $m^{(1)}$ and $m^{(2)}$ of the constituents satisfy $m^{(2)} \gg m^{(1)}$. The resulting effective potential is energy and angular momentum dependent. We first show that effective potential for a fixed energy is "resonating", i. e. it is of the form of the potential well capable of producing resonances. It is interesting that it is more resonating for the higher total angular momenta. Next we show the dependence of the potential on energy at a fixed J. It turns out that in this case the potential is more resonating up to a certain energy after which it becomes less resonating. Next we show that for certain energies the radial wave function has the characteristic resonance peak which disappears if we go away from resonance energy. We have also calculated the phase shifts as a function of energy. The numerical analysis shows that for every resonance energy for which we have a resonance peak in the wave function there exists a change of the phase shift by π. This seems to confirm that the present approach provides a description of high mass resonances of two-fermion system with purely electromagnetic interactions.

Thus for a given form-factor with parameter M we can determine the resonance mass M_r. At the end of the paper we comment on a possible self-consistent determination of M, by relating M to M_r.

2. *Relativistic one-time two-fermion equation with form-factors*

Covariant one-time two-body equations have been considered by several authors [3–6]. If in the two-body wave function $\Psi(x_1, x_2)$, the coordinates x_1 and x_2 are on the same space-like σ, then Ψ has a direct probability amplitude interpretation at fixed σ (or t). To the lowest order in the coupling constant one has then the basic equations:

$$\begin{aligned}(\not\partial^{(1)}+m^{(1)})\Psi(x_1, x_2) &= -ie^{(1)}e^{(2)}\int dxS^{(2)\,\mathrm{ret}}(x_2-x)\gamma_\mu^{(1)}\gamma^{(2)\mu}D_F(x-x_1)\Psi(x_1, x),\\ (\not\partial^{(2)}+m^{(2)})\Psi(x_1, x_2) &= -ie^{(1)}e^{(2)}\int dxS^{(1)\,\mathrm{ret}}(x_1-x)\gamma_\mu^{(1)}\gamma^{(2)\mu}D_F(x-x_2)\Psi(x, x_2),\\ x_1 \in \sigma, &\quad x_2 \in \sigma. \end{aligned} \tag{2.1}$$

These equations correctly reduce to the Bethe–Salpeter equation in the ladder approximation. In this form these equations may be derived from various general settings (Appendix I).

According to the motivation and discussion we gave in Section 1 we shall use the generalization of Eqs. (2.1) in which $\gamma_\mu^{(1)}$ in the first equation, and $\gamma_\mu^{(2)}$ in the second equation are replaced respectively by

$$\gamma_\mu \to \Gamma_\mu(x) = \frac{1}{(2\pi)^4}\int e^{iqx}d^4{}_q[\gamma_\mu F_1(q^2)+i\sigma_{\mu\nu}q^\nu F_2(q^2)],$$
$$\sigma_{\mu\nu} = \tfrac{1}{2}(\gamma_\nu\gamma_\mu-\gamma_\mu\gamma_\nu). \tag{2.2}$$

More generally, both $\gamma_\mu^{(1)}$ and $\gamma_\mu^{(2)}$ simultaneously may be replaced by $\Gamma_\mu^{(1)}$ and $\Gamma_\mu^{(2)}$. But this leads to nonlocal potentials involving a time integration as we shall point out later. Thus we shall consider from now on the system of equations

$$\begin{aligned}(\not\partial^{(1)}+m^{(1)})\Psi(x_1, x_2) &= -ie^{(1)}e^{(2)}\iint dxd\xi S^{(2)\,\mathrm{ret}}(x_2-x)\Gamma_\mu^{(1)}(x-\xi)\gamma_\mu^{(2)}D_F(\xi-x_1)\Psi(x_1, x),\\ (\not\partial^{(2)}+m^{(2)})\Psi(x_1, x_2) &= -ie^{(1)}e^{(2)}\iint dxd\xi S^{(1)\,\mathrm{ret}}(x_1-x)\gamma_\mu^{(1)}\Gamma^{\mu(2)}(x-\xi)D_F(\xi-x_2)\Psi(x, x_2).\end{aligned} \tag{2.3}$$

We may write Eqs. (2.3) in the general form as (cf. Eq. (A.2))

$$(\partial^{(1)}+m^{(1)})\Psi(x_1, x_2) = -ie^{(1)}e^{(2)} \iint_{\sigma_1=\sigma_2=t} A^{(1)}_{[\sigma_1,\sigma_2]}(x_1, x_2; x_1', x_2')\Psi(x_1', x_2')d\sigma_1 d\sigma_2. \quad (2.4)$$

Similarly for particle (2).

Multiplying the equation for particle (1) with $\gamma_4^{(1)}$ and that for particle (2) with $\gamma_4^{(2)}$ and adding, we have the Hamiltonian form

$$\left(2i\frac{\partial}{\partial t} - H_0^{(1)} - H_0^{(2)}\right)\Psi(x_1, x_2) = -ie^{(1)}e^{(2)} \iint_{\sigma_1=\sigma_2=t} d\sigma_1 d\sigma_2$$

$$\times[\gamma_4^{(1)}A^{(1)}_{[\sigma_1,\sigma_2]}(x_1, x_2; x_1', x_2')+\gamma_4^{(2)}A^{(2)}_{[\sigma_1,\sigma_2]}(x_1, x_2; x_1', x_2')]\Psi(x_1', x_2')] \quad (2.5)$$

which can be further written as

$$\left(2i\frac{\partial}{\partial t} - H_0^{(1)} - H_0^{(2)}\right)\Psi(x_1, x_2) = \int_\sigma V_\sigma(x_1, x_2, x_1', x_2')\Psi(x_1'x_2')dx_1'dx_2'. \quad (2.6)$$

We now introduce the new variables

$$X = bx^{(1)}+(1-b)x^{(2)} \equiv (\boldsymbol{R}, T), \quad b = m^{(1)}/(m^{(1)}+m^{(2)}),$$
$$x = x^{(1)}-x^{(2)} \equiv (\boldsymbol{r}, t). \quad (2.7)$$

In these variables we obtain

$$H_0^{(1)} = \gamma_4^{(1)}\left[\gamma^{(1)}\cdot\left(b\frac{\partial}{\partial \boldsymbol{R}}+\frac{\partial}{\partial \boldsymbol{r}}\right)+m^{(1)}\right],$$

$$H_0^{(2)} = \gamma_4^{(2)}\left[\gamma^{(2)}\cdot\left((1-b)\frac{\partial}{\partial \boldsymbol{R}}-\frac{\partial}{\partial \boldsymbol{r}}\right)+m^{(2)}\right]. \quad (2.8)$$

Furthermore, on the space-like surface $\sigma_1 = \sigma_2 = t$ we have $d\sigma_1 d\sigma_2 \to d^3\boldsymbol{R}'d^3\boldsymbol{r}'$. Taking the Fourier transform with respect to t, and R' and passing to the center-of-mass variables we can write the equation with $2i\partial/\partial t \to E$,

$$\left[E+(\gamma_4^{(1)}\gamma^{(2)}-\gamma_4^{(1)}\gamma^{(1)})\frac{\partial}{\partial \boldsymbol{r}}-(\gamma_4^{(2)}m^{(2)}+\gamma_4^{(1)}m^{(1)})\right]\Psi_E(\boldsymbol{r}) = \int V_E(\boldsymbol{r}, \boldsymbol{r}')\Psi_E(\boldsymbol{r}')d\boldsymbol{r}'. \quad (2.9)$$

The direct calculation shows that the resulting potential is nonlocal. To obtain a well defined effective potential we make a static approximation in the expression for $V_E(\boldsymbol{r}, \boldsymbol{r}')$. This consists in setting to zero the fourth components of momentum vectors appearing in the integrands of kernels (2.3). Performing the elementary calculations one obtains

$$\int V_E(\boldsymbol{r}, \boldsymbol{r}')\Psi(\boldsymbol{r}')d\boldsymbol{r}'$$
$$= (16\pi^3)^{-1}\gamma_4^{(1)}\gamma_4^{(2)}\int d\boldsymbol{q}e^{i\boldsymbol{q}\boldsymbol{r}}(\boldsymbol{q}^2)^{-1}(e^{(2)}\Gamma_\mu^{(1)}(\boldsymbol{q}^2)\gamma^{(2)\mu}+e^{(1)}\Gamma_\mu^{(2)}(\boldsymbol{q}^2)\gamma^{(1)\mu}\Psi(\boldsymbol{r}) = V_E(\boldsymbol{r})\Psi(\boldsymbol{r})$$
$$(2.10)$$

Assume now that Dirac and Pauli form-factors are entire functions of q^2 i.e.

$$F_1^{(k)}(q^2) = \sum_{n=0}^{\infty} \varepsilon_n^{(k)}(-q^2)^n, \quad F_2^{(k)}(q^2) = \sum_{n=0}^{\infty} \mu_n^{(k)}(-q^2)^n. \tag{2.11}$$

Here the coefficients ε_n and μ_n are constants characterizing the interactions. For $n = 0$ the coefficients $\varepsilon_0^{(k)}$ and $\mu_0^{(k)}$ are the charge ($= e^{(k)}$) and the static anomalous magnetic moment ($= a^{(k)}e^{(k)}/2m^{(k)}$) of the k^{th} fermion.

Inserting (2.11) into (2.10), one obtains the explicit expression for the potential $V_E(r)$ oin terms of Fourier integrals of powers of q. Such integrals must be treated with proper care since they represent in fact the Gelfand–Shilov generalized functions. Using the technique of regularization of divergent integrals presented in Appendix B, we obtain the following formula for the effective potential:

$$V(r) = \frac{\Gamma}{2}\left\{e^{(2)}\left[\frac{\varepsilon_0^{(1)}}{r} - \mu_0^{(1)}\frac{\gamma^{(1)}r}{r^3}\right] + e^{(1)}\left[\frac{\varepsilon_0^{(2)}}{r} + \mu_0^{(2)}\frac{\gamma^{(2)}r}{r^3}\right]\right.$$
$$\left. + e^{(2)}(E^{(1)} + M^{(1)}) + e^{(1)}(E^{(2)} - M^{(2)})\right\}, \tag{2.12}$$

where

$$\Gamma = 1 - \alpha^{(1)}\alpha^{(2)},$$

$$E^{(k)} = -4\pi \sum_{n=1}^{\infty} \varepsilon_n^{(k)} \frac{\prod_{m=1}^{2(n-1)}(-1-m)}{[2(n-1)]!}\,\delta^{[2(n-1)]}(r), \quad k = 1, 2$$

$$M^{(k)} = -4\pi\gamma^{(k)}\partial/\partial r \sum_{n=1}^{\infty} \mu_n^{(k)} \frac{\prod_{m=1}^{2(n-1)}(-1-m)}{[2(n-1)]!}\,\delta^{[2(n-1)]}(r). \tag{2.13}$$

The dynamical equation (2.9) with the potential given by (2.12) describes a general system of two-fermion with form-factors. In order to provide a better insight into the meaning of Eq. (2.9) consider several special cases.

Limiting cases

(*i*) $F_1^{(k)}q^2) = \varepsilon_0^{(k)} = e^{(k)}, \quad F_2^{(k)} = 0$. Then

$$V(r) = \frac{e^{(1)}e^{(2)}}{r}(1 - \alpha^{(1)}\alpha^{(2)}). \tag{2.14}$$

Thus Eq. (2.9) represents in this approximation the Breit equation with part of the retardation effect taken into account (cf. Eq. (14) of Ref. [5] where the retardation effect is absent in the same approximation though the original equations derived from field theory are one-time equations).

(*ii*) Take the approximation (*i*) and in addition assume $m^{(2)} \gg m^{(1)}$. To obtain one-body equation we perform first the Fourier transform with respect to the $\boldsymbol{r}$-variables and obtain

$$[E+H_0^{(1)}(\boldsymbol{p})+H_0^{(2)}(-\boldsymbol{p})]\Psi_E(\boldsymbol{p}) = \int V_E(\boldsymbol{p}, \boldsymbol{p}')\Psi_E(\boldsymbol{p}')d\boldsymbol{p}', \tag{2.15}$$

where $H_0^{(i)}$ is the free hamiltonian of i^{th} particle. Performing now a Foldy–Wouthausen transformation of the second particle given by the formula (cf. [7] Ch. 4, § VI)

$$U^{(2)} = \exp[iS^{(2)}], \quad S^{(2)} = -\frac{i}{2m^{(2)}}\gamma\boldsymbol{p}\omega\left(\frac{|\boldsymbol{p}|}{m^{(2)}}\right) \tag{2.16}$$

with

$$\omega\left(\frac{|\boldsymbol{p}|}{m^{(2)}}\right) = \frac{m^{(2)}}{|\boldsymbol{p}|}\operatorname{arc\,tg}\left(\frac{|\boldsymbol{p}|}{m^{(2)}}\right),$$

we obtain

$$H_0^{(2)\prime} = U^{(2)}H_0^{(2)}U^{(2)^{-1}} = \gamma_4^{(2)}\sqrt{\boldsymbol{p}^2+m^{(2)2}}. \tag{2.17}$$

Hence in the case when $m^{(2)}$ becomes very large we have

$$H_0^{(2)\prime} \cong \gamma_4^{(2)}\left(m^{(2)}+\frac{\boldsymbol{p}^2}{2m^{(2)}}\right) \cong \gamma_4^{(2)}m^{(2)}. \tag{2.17'}$$

Consequently, since $U^{(2)}$ commutes with $H_0^{(1)}$, Eq. (2.15) takes the form

$$[E+H_0^{(1)}(\boldsymbol{p})+\gamma_4^{(2)}m^{(2)}]\Psi_E'(\boldsymbol{p}) = \int V_E'(\boldsymbol{p}, \boldsymbol{p}')\Psi(\boldsymbol{p}')d\boldsymbol{p}', \tag{2.18}$$

where $\Psi' = U^{(2)}\Psi$ and $V_E' = U^{(2)}V_E U^{(2)^{-1}}$. Using now the assumptions (*i*) we obtain that the potential V_E has the form (2.14) in the Dirac representation. In this representation the operator $\boldsymbol{\alpha}^{(2)} = [H_0^{(2)}, \boldsymbol{r}]$ represents a velocity operator. In the Foldy–Wouthausen representation the velocity operator takes the form

$$\boldsymbol{\alpha}^{(2)} \to U^{(2)}[H_0^{(2)}, \boldsymbol{r}]U^{(2)^{-1}} = [H_0^{(2)\prime}, \boldsymbol{r}'] = \frac{\boldsymbol{p}}{E(\boldsymbol{p})}\,\frac{\gamma_4^{(2)}m^{(2)}+\gamma_4^{(2)}\gamma^{(2)}\boldsymbol{p}}{E(\boldsymbol{p})}, \tag{2.19}$$

with $E(\boldsymbol{p}) = \sqrt{\boldsymbol{p}^2+m^{(2)2}}$. Hence in the case when $m^{(2)}$ becomes very large we obtain from (2.14)

$$V_E' = \frac{e^{(1)}e^{(2)}}{r}. \tag{2.20}$$

Performing now the Fourier transformation with respect to the time variable in the form

$$\Psi_E(\boldsymbol{p}, \tau) = \Psi_E(\boldsymbol{p})e^{-\frac{i}{2}(E+\frac{1}{2}\gamma_4{}^{(2)}m^{(4)}]\tau}$$

one reduces Eq. (2.9) to the following equation:

$$[E-\gamma_4^{(1)}(\gamma^{(1)}\partial+m^{(1)})]\Psi_E(\boldsymbol{r}) = \frac{e^{(1)}e^{(2)}}{r}\Psi_E(\boldsymbol{r}), \tag{2.21}$$

which is the Dirac equation for the charge fermion in the Coulomb field. Let us stress that the Bethe–Salpeter equation does not lead directly to the Dirac equation in an external field when the mass of one of the particles tends to infinity. Thus in contrast to the rather complicated procedures of other methods we obtained the above limits (i.e. the Breit and the Dirac equations) in a rather natural and straightforward manner.

(*iii*) Coulomb and anomalous magnetic moment interactions. Take

$$F_1^{(k)}(\boldsymbol{q}^2) = \varepsilon_0^{(k)} = e^{(k)}, \quad F_2^{(k)} = \mu_0^{(k)} = \frac{a^{(k)}e^{(k)}}{2m^{(k)}}.$$

Inserting this into (2.12) we obtain

$$V(r) = e^{(1)}e^{(2)}\Gamma\left[\frac{1}{r} - \frac{a^{(1)}}{4m^{(1)}}\frac{\gamma^{(1)}\cdot \boldsymbol{r}}{r^3} + \frac{a^{(2)}}{4m^{(2)}}\frac{\gamma^{(2)}\cdot \boldsymbol{r}}{r^3}\right]. \tag{2.22}$$

The first term in (2.22) corresponds to the Breit potential with part of the retardation effects included. The second and the third terms in the potential arise due to the inclusion of anomalous magnetic moments of both particles. In the limit $m^{(2)} \gg m^{(1)}$ using the same analysis as in (*ii*) one obtains from (2.9)

$$[E-\gamma_4^{(1)}(\gamma^{(1)}\partial+m^{(1)})]\Psi_E(\boldsymbol{r}) = e^{(1)}e^{(2)}\left(\frac{1}{r} - \frac{a^{(1)}}{4m^{(1)}}\frac{\gamma^{(1)}\cdot \boldsymbol{r}}{r^3}\right)\Psi_E(\boldsymbol{r}). \tag{2.23}$$

This is the Barut–Kraus equation for a single Dirac particle with anomalous magnetic moment in the Coulomb field of the other heavy particle if $m \to m_{\text{red}}$ [2]. This approximation might be very good in the case when one of the particles is much heavier than the other. In the case of two-fermions with equal masses equation (2.9) with potential (2.22) should be used in which both particles are treated on an equal footing.

Let us note that taking into consideration more terms in the form-factors provides, on the basis of formula (2.12), additional terms in the potential, which are proportional to derivatives of $\delta(\boldsymbol{r})$. Thus any finite number of multipoles leads to terms in the potential strongly singular at origin, and does not result in a well defined eigenvalue problem. Consequently, it seems that it is more reasonable to calculate, even approximately, the functional form of the form-factors and deduce from them by means of formula (2.10) the explicit form of effective potentials. This is presented in Section 4.

3. *Derivation of the radial equation*

It is evident from Eqs. (2.9) and (2.12) that dynamical equations in the center of mass system with potential (2.12) are invariant with respect to the ordinary rotations. This implies that the energy, the total angular momentum J and its third component J_3 form a system of commuting operators for the system (2.9). We have

$$J_k = -i(L_k+S_k), \quad L_k = x_i\partial_j - x_j\partial_i, \quad S_k = 1/2(\gamma_i^{(1)}\gamma_j^{(1)}+\gamma_i^{(2)}\gamma_j^{(2)}), \tag{3.1}$$

$k, i, j,$ is a cyclic permutation of 1, 2, 3.

It was observed in [4] (see also [6]) that the unitary transformation $U = \exp[S_2\vartheta]$ $\exp[S_3\varphi]$ diagonalizes J^2 and J_3 and we have

$$U\gamma^{(k)}\boldsymbol{r}U^{-1} = r\gamma_3^{(k)},$$

$$U\gamma^{(1)}\partial U^{-1}Z_0^\beta(\vartheta,\varphi)\Psi_E^J(r) = \left[\gamma_3^{(1)}\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)+\frac{1}{2r}(\gamma_1^{(1)}\gamma_1^{(2)}+\gamma_2^{(1)}\gamma_2^{(2)})\gamma_3^{(2)}\right.$$

$$\left.+r^{-1}\sqrt{J(J+1)}\,\gamma_2^{(1)}\gamma_1^{(2)}\gamma_2^{(2)}\right)Z_0^J(\vartheta,\varphi)\Psi_E^J(r), \tag{3.2}$$

$$U\gamma^{(2)}\partial U^{-1}Z_0^J(\vartheta,\varphi)\Psi_E^J(r) = \left[\gamma_3^{(2)}\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)+\frac{1}{2r}(\gamma_1^{(1)}\gamma_1^{(2)}+\gamma_2^{(1)}\gamma_2^{(2)})\gamma_3^{(1)}\right.$$

$$\left.+r^{-1}\sqrt{J(J+1)}\,\gamma_2^{(2)}\gamma_1^{(1)}\gamma_2^{(1)}\right)Z_0^J(\vartheta,\varphi)\Psi_E^J(r),$$

$$[U,\Gamma] = 0.$$

The functions (3.2) are eigenfunctions of J^2 and J_3. Using (2.9), (2.12) and (3.2) we obtain the following radial equation for the function $\Psi_E^J(r)$

$$\left[E+(\gamma_4^{(2)}\gamma_3^{(2)}-\gamma_4^{(1)}\gamma_3^{(1)})\left(\frac{d}{dr}+\frac{1}{r}\right)+\frac{1}{2r}(\gamma_3^{(2)}\gamma_4^{(1)}-\gamma_3^{(1)}\gamma_4^{(2)})(\gamma_1^{(1)}\gamma_1^{(2)}+\gamma_2^{(1)}\gamma_2^{(2)})\right.$$

$$\left.-\frac{1}{r}(\gamma_4^{(1)}\gamma_1^{(2)}-\gamma_4^{(2)}\gamma_1^{(1)})\gamma_2^{(1)}\gamma_2^{(2)}\sqrt{J(J+1)}-\gamma_4^{(1)}m^{(1)}+\gamma_4^{(2)}m^{(2)}\right]\Psi_E^J(r) = V(r)\Psi_E^J(r) \tag{3.3}$$

with

$$V(r) = \frac{\Gamma}{2}\left\{e^{(2)}\left[\frac{\varepsilon_0^{(1)}}{r}-\mu_0^{(1)}\frac{\gamma_3^{(1)}}{r^2}\right]+e^{(1)}\left[\frac{\varepsilon_0^{(2)}}{r}+\frac{\mu_0^{(2)}\gamma_3^{(2)}}{r^2}\right]+e^{(2)}E^{(1)}+2e^{(2)}\right.$$

$$\times\left[\gamma_3^{(1)}\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)+\frac{1}{2r}(\gamma_1^{(1)}\gamma_1^{(2)}+\gamma_2^{(1)}\gamma_2^{(2)})\gamma_3^{(2)}+\left(\frac{\sqrt{J(J+1)}}{r}\right)\gamma_2^{(1)}\gamma_1^{(1)}\gamma_2^{(2)}\right]M^{(1)}+e^{(1)}E^{(2)}$$

$$\left.-2e^{(1)}\left[\gamma_3^{(1)}\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)+\frac{1}{2r}(\gamma_1^{(1)}\gamma_1^{(2)}+\gamma_2^{(1)}\gamma_2^{(2)})\gamma_3^{(1)}+\frac{\sqrt{J(J+1)}}{r}\gamma_2^{(2)}\gamma_1^{(1)}\gamma_2^{(1)}\right]M^{(2)}\right\}. \tag{3.4}$$

Since neither $\gamma_2^{(1)}$ nor $\gamma_2^{(2)}$ appear alone in Eq. (3.3), but only in the combination $\gamma_2^{(1)}\gamma_2^{(2)}$, there exists a normal divisor

$$N = \gamma_1^{(1)}\gamma_3^{(1)}\gamma_4^{(1)}\gamma_1^{(2)}\gamma_3^{(2)}\gamma_4^{(2)} \tag{3.5}$$

which commutes with all terms in Eq. (3.3) and has eigenvalues ± 1. This implies that the original 16-dimensional spinor space can be split into two independent 8-dimensional subspaces in which N has the eigenvalue $+1$ or -1 respectively. In the case $J = 0$ there exists still another normal divisor of the form:

$$M = \gamma_2^{(1)}\gamma_3^{(1)}\gamma_4^{(1)}\gamma_2^{(2)}\gamma_3^{(2)}\gamma_4^{(2)}. \tag{3.6}$$

M has also eigenvalue ± 1 and therefore together with N splits the original 16-dimensional spinor space into four four-dimensional subspaces. In these subspaces the $\gamma^{(k)}$-matrices have the following representation

$$\gamma_3^{(1)} = \sigma_2 \otimes 1, \quad \gamma_4^{(1)} = \sigma_1 \otimes 1, \quad \gamma_3^{(2)} = 1 \otimes \sigma_2, \quad \gamma_4^{(2)} = 1 \otimes \sigma_1,$$

$$\gamma_1^{(1)}\gamma_2^{(1)} = \begin{cases} -\sigma_3 \otimes \sigma_3 & \text{for } N = 1 \\ \sigma_3 \otimes \sigma_3 & \text{for } N = -1 \end{cases}, \qquad \gamma_2^{(1)}\gamma_2^{(2)} = \begin{cases} -\sigma_3 \otimes \sigma_3 & \text{for } M = 1 \\ \sigma_3 \otimes \sigma_3 & \text{for } M = -1 \end{cases},$$

$$\gamma_1^{(1)}\gamma_1^{(2)} + \gamma_2^{(1)}\gamma_2^{(2)} = \begin{cases} 2\sigma_3 \otimes \sigma_3 & N = M = -1 \\ -2\sigma_3 \otimes \sigma_3 & \text{if } N = M = 1 \\ 0 & MN = -1 \end{cases}. \tag{3.7}$$

Using formula (3.7) for $J = 0$ and the corresponding representations of $\gamma^{(1)}$ and $\gamma^{(2}$ for $J \neq 0$ (given for instance in [6], Eq. (2.3)) one may write the reduced 4×4 and 8×8 equation for the general potential (3.4). However, for the sake of simplicity we shall present the explicit form of the dynamical equation for the most important case (*ii*) of Section 2 only. By virtue of (3.4) the effective potential has in this case the form:

$$V(r) = e^{(1)}e^{(2)}\Gamma\left(\frac{1}{r} - \frac{a^{(1)}\gamma_3^{(1}}{4m^{(1)}r^2} + \frac{a^{(2)}\gamma_3^{(2)}}{4m^{(2)}r^2}\right). \tag{3.8}$$

Using now (3.7), (3.4) and (3.8) one obtains for $J = 0$, $N = M = -1$:

$$\begin{bmatrix} E-2A, & -m^{(1)}+2i(B^{(1)}+B^{(2)}), & -m^{(2)}-2i(B^{(1)}+B^{(2)}), & -2A \\ -m^{(1)}-2iB^{(1)}, & E+2i\partial, & 2i/r+2A, & -m^{(2)}-2iB^{(2)} \\ -m^{(2)}+2iB^{(2)}, & -2i/r+2A, & E-2i\partial & -m^{(1)}+2iB^{(1)} \\ -2A, & -m^{(2)}+2i(B^{(1)}+B^{(2)}), & -m^{(1)}-2i(B^{(1)}+B^{(2)}), & E-2A \end{bmatrix}$$

$$\times \begin{bmatrix} \chi_{ab}^1 \\ \chi_{ab}^2 \\ \chi_{ab}^3 \\ \chi_{ab}^4 \end{bmatrix} = 0, \tag{3.9}$$

where

$$A = \frac{e^{(1)}e^{(2)}}{r}, \quad B^{(k)} = \frac{e^{(1)}e^{(2)}a^{(k)}}{4m^{(k)}r^2}, \quad a, b = +, - \quad \text{and} \quad \partial = d/dr, \quad \chi(r) = r\Psi.$$

The inspection of Eq. (3.9) shows that in this case it represents a system of two algebraic and two first order ordinary differential equations. Hence one can get a Schrödinger-like eigenvalue equation for a single component $\chi^3_{ab} \equiv \varphi$ of the form

$$\left[\frac{d^2}{dr^2} + V(E, r)\right] \varphi(r) = 0. \tag{3.10}$$

Let us note that in the case $J > 0$ using (3.5) one reduces the dynamical equation to a system of 8×8 matrix differential equations from which 4 are algebraic and the remaining 4 are first order ordinary differential equations. This system can be reduced to a single fourth order ordinary radial differential equation. Hence the spectral properties of two fermion system with $J > 0$ are entirely different than in the case $J = 0$. The detailed analysis of the exact eigenvalue problem for $J > 0$ will be given elsewhere. However, in order to obtain approximate information on the behaviour of two-fermion system for $J > 0$ we shall analyze in Section 5 the case $m^{(2)} \gg m^{(1)}$. It is discovered there by numerical analysis that the two-fermion system becomes more resonance-like if the total angular momentum J increases. The form of the resulting potential $V_0(r, E)$ depends strongly on the eigenvalue of N and M.

4. *Self-consistent determination of the form-factors*

It is well known that the anomalous magnetic moment coupling is associated with a form-factor. In the original Dirac–Pauli potential model this form-factor is neglected, one assumes a constant static anomalous magnetic moment throughout. However, at short distances (or high energies) the effect of the form-factor will be important. For the study of magnetic resonances we discuss here the magnetic form-factor because in this case the magnetic forces will dominate over the Coulomb parts.

It has been shown recently by calculating Liènard–Wiechert potentials for localized charge distribution that for a localized state (localized at a distance of r_0) the form-factor will have appreciable effect only for $r \leqslant r_0$. Here we proceed differently. If there exists a fairly sharp resonance of mass M, corresponding to a localized state of size r_0, then each constituent is strongly bound. The properties of each constituent in a localized region of interaction are described in electromagnetic theory by form-factors resulting from the summation of all radiative correction. Clearly an explicit form of form-factors is at present unknown and presumably it will be very difficult to calculate it to all orders in perturbation theory. Therefore we are justified to introduce a reasonable approximate form-factor reflecting the fact that we have the electromagnetic interactions of localized constituents. We postulate that the true form-factor may be approximated by the lowest order radiative corrections to a point vertex $e\gamma_\mu$ with the electron mass m_e replaced by a certain effective mass $M_{\text{eff}} \sim r_0^{-1}$. Our postulate takes into account the fact that the lowest radiative corrections reflect in the simplest possible manner the localization property of each constituent. With this potential, we search for a resonance numerically. It is remarkable and significant, as the numerical analysis shows, that the static magnetic potential alone without the form-factor has not quite the right shape to lead to the formation of resonances.

The magnetic form-factor has been evaluated in standard quantum electrodynamics (see Ref. [7], Ch. 15) and in dispersion theory [8]. In momentum space the magnetic vertex is given by the well known function to the lowest order

$$\Gamma_{\mu}^{\text{magn}} = \sigma_{\mu\nu} q^{\nu} \frac{2\theta}{\sin 2\theta},$$

with

$$\sin^2 \theta = -q^2/4M_{\text{eff}}^2.$$

In the coordinate space the vertex correction leads to a change in potential [9]

$$\delta V(r) \sim \beta(\boldsymbol{\alpha}\nabla) \int d^3\boldsymbol{q} \frac{e^{i\boldsymbol{q}\boldsymbol{r}}}{|\boldsymbol{q}|^2} \frac{2\theta}{\sin 2\theta} = \beta(\boldsymbol{\alpha}\nabla) I(r).$$

Using the integral representation

$$\frac{2\theta}{\sin 2\theta} = \int_0^1 \frac{dy}{1 + \dfrac{\boldsymbol{q}^2}{M_{\text{eff}}^2} y(1-y)}$$

the integral $I(r)$ can be evaluated to yield

$$I(r) = \frac{2\pi^2}{r} \left[1 - \int_0^1 dx e^{-M_{\text{eff}} r/\sqrt{x(1-x)}} \right].$$

The potential $\delta V(r)$ is proportional to $\dfrac{d}{dr} I(r)$. The derivative can be exactly evaluated [10] and gives

$$\frac{dI(r)}{dr} = -\frac{2\pi}{r^2} (1 - 2M_{\text{eff}} K_1(2M_{\text{eff}} r)).$$

The expression

$$g(r) = 1 - 2M_{\text{eff}} K_1(2M_{\text{eff}} r) \qquad (4.1)$$

is just the factor with which we have to multiply the potential calculated for constant $F_2(0)$. It has the asymptotic limits

$$g(r) = 1 - \sqrt{\frac{\pi}{2}} \exp[-2M_{\text{eff}} r], \qquad \text{for } M_{\text{eff}} r \gg 1$$

$$= -2M_{\text{eff}}^2 r^2 [\ln M_{\text{eff}} r + \gamma - 1], \qquad \text{for } M_{\text{eff}} r \ll 1$$

and the form shown in Fig. 1. As we expect, only for $r \leqslant r_0$ will the effect of the form--factor be important, and this in a significant way.

The electric form-factor has no effect in the region of magnetic resonances and we have omitted it for the time being.

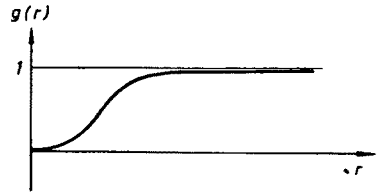

Fig. 1. Dependence of the magnetic form-factor $g(r)$ on r

5. *Numerical analysis of dynamical equation*

The general dynamical equations are too complicated for spectral analysis and the problem of explicit localization of eigenvalues of energy is rather difficult at this time. Therefore, in order to verify whether there exist bound or resonance states of the dynamical equations one has to utilize numerical analysis. We performed such an analysis under the following assumptions

$$(i) \quad m^{(2)} \gg m^{(1)}, \qquad (ii) \quad \varepsilon_n^{(i)} = \delta_{0n} e^{(i)}, \qquad \mu_n^{(i)} = \delta_{0n} \frac{a^{(i)}}{4m^{(i)}} .$$

This leads to a one-body dynamical equation with the effective potential given by the formula:

$$V(r) = e^{(1)} e^{(2)} \left[\frac{1}{r} - \frac{a^{(1)}}{4m^{(1)}} g(r) \frac{\gamma^{(1)} \cdot r}{r^3} \right], \tag{5.1}$$

where the form-factor $g(r)$ is given by formula (4.1).

Passing to the radial coordinates one obtains a system of four equations, which can be reduced to Schrödinger-like equation

$$\left[\frac{d^2}{dr^2} + \lambda(E) - V(r, E, J) \right] \Psi_E^J(r) = 0, \tag{5.2}$$

with the energy and angular momentum dependent potential.

We first analysed the dependence of V on J at a fixed E. The numerical analysis reveals an interesting fact that the potential becomes more resonance-like at small distances if the total angular momentum increases (see Fig. 2). This phenomenon persists in a large interval of energy.

Next we analysed the dependence of V on energy at fixed J. Again it turns out that the potential becomes more resonance-like if energy increases. However, this phenomenon persists up to a certain energy only after which the potential becomes less resonance-like (see Fig. 3).

Finally we calculated resonance energy E_r and the shape of the wave function corresponding to the resonance energy. As seen from (5.3), the potential is strongly singular at origin, a fact which causes a serious numerical problem. We have used a variant of

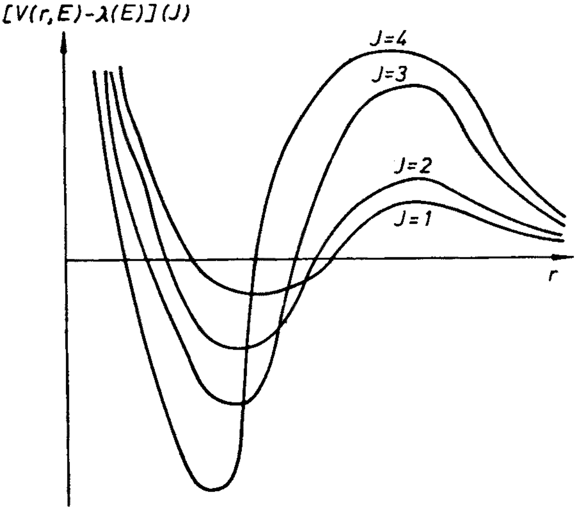

Fig. 2

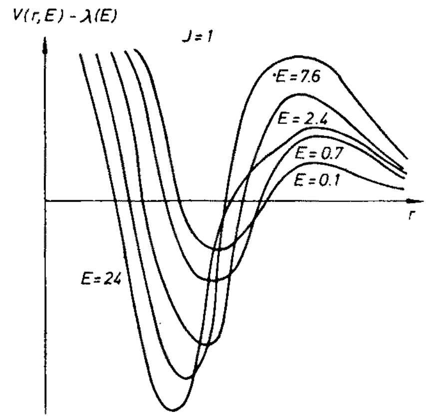

Fig. 3

Fig. 2. Dependence of $V(r, E, J)$ on total angular momentum J at a fixed E
Fig. 3. Dependence of $V(r, E, J)$ on energy for $J = 1$

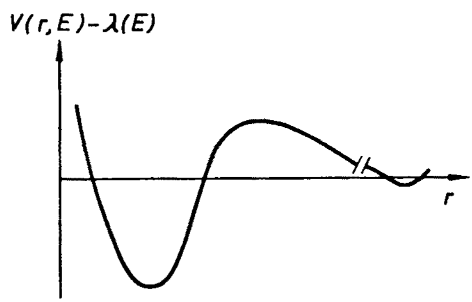

Fig. 4

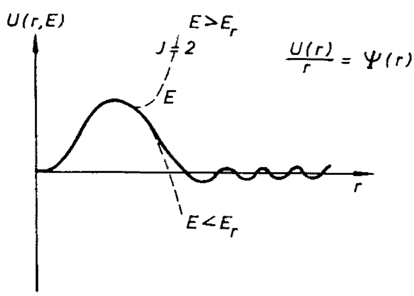

Fig. 5

Fig. 4. Dependence of $V(r, E, J)$ on r for fixed E and J
Fig. 5. Dependence of the radial wave function on energy E at fixed $J = 2$ in the region of resonance energy E_r

W.K.B. method adapted for singular potentials by Skorupski. The description of this method will be published elsewhere [11]. On Fig. 5 the typical shape of the wave function at resonance energy, corresponding to $J = 2$, is depicted. It is shown that for a certain energy E_r the wave function has a single resonance peak. If we take $E > E_r$, or $E < E_r$ even by relatively small amount then the resonance peak disappears. For the purpose of illustration we depicted on Fig. 4 the shape of the potential in the same units. It is evident from Figs. 4 and 5 that the resonance peak of wave function occurs at distances corresponding to the minimum of the potential wall, as should be expected from the classical analysis. This phenomenon persists for various J in a large interval of angular momenta.

The value of resonance energy E_r depends on the localization parameter $r_0(\sim M_{eff}^{-1})$. The dependence of E_r on r_0 for various J was calculated numerically. It turns out that for higher angular momenta the resonances are more massive hence more localized.

In order to identify more unambiguously the resonances another numerical program for calculation of phase shifts for singular potentials was elaborated by Skorupski. Fig. 6 shows the dependence of the phase shift as a function of energy for various values of E_r

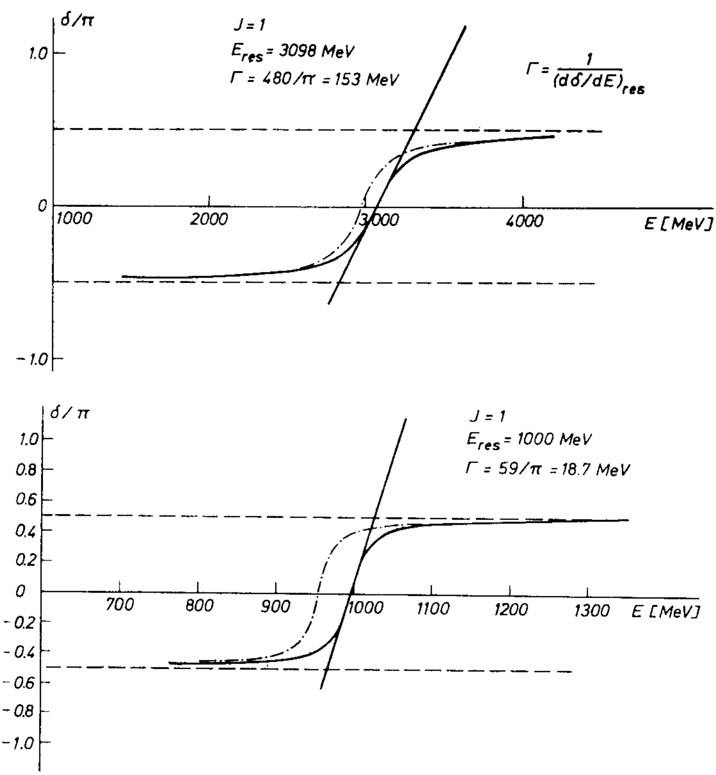

Fig. 6. Dependence of the phase shifts on energy for $J = 1$ and $E_r = 3098$ and 1000 MeV respectively

obtained from the determination of the resonance peak of wave function. It is seen from Fig. 6 that phase shift changes sharply by π when energy passes the value of E_r in the neighbourhood of the resonance. This is clearly the most characteristic feature of narrow resonances. The region around E_r where a change of phase shift by π occurs, gives also the possibility of a precise determination of half-width Γ of resonances. The corresponding values of Γ for various E_r and $J = 1$ are given in Fig. 6.

Thus the above analysis strongly confirms that there might indeed exist high mass resonances in a two-fermion system with purely electromagnetic interactions. Numerical analysis was also performed by Anders [10] with similar conclusions.

In further work we shall determine the parameter r_0 in the form-factor in a self-consistent way. In the present numerical calculations the resonance energy E_r does not completely match $M_{eff} = 1/r_0$. The reason lies in the fact that our form-factor, being the lowest radiative correction, does not reflect the form-factor correctly at still shorter distances, and E_r is very sensitive to the form-factor at these distances. We propose to modify the effect of this inner region in a self-consistent manner.

APPENDIX A

A relativistic two-fermion equation was introduced by Günther in the following form [3]

$$(\partial^{(i)}+m^{(i)})\Psi(x_1, x_2) = \iint G^{(i)}(x_1, x_2 : x_1', x_2')\Psi(x_1', x_2')dx_1'dx_2', \quad i = 1, 2, \tag{A.1}$$

where $G^{(i)}$ are general kernel functions. It was observed by Królikowski and Rzewuski [4] that one obtains a probabilistic interpretation of the wave function $\Psi(x_1, x_2)$ if, using an equivalence theorem, one replaces the system (A.1) with the following system

$$(\partial^{(i)}+m^{(i)})\Psi(x_1, x_2) = \int_{\sigma_1}\int_{\sigma_2} A^{(i)}_{[\sigma_1,\sigma_2]}(x_1, x_2; x_1', x_2')\gamma_\mu^{(1)}\gamma_\nu^{(2)}\Psi(x_1', x_2')d\sigma_1^\mu d\sigma_2^\nu. \tag{A.2}$$

Here σ_1, σ_2 are spacelike surfaces and the kernels $A^{(i)}_{[\sigma_1,\sigma_2]}$ which are functionals of σ_1, σ_2 are determined by the original kernels $G^{(i)}$ by means of a linear integral equation

$$A^{(i)}_{[\sigma_1,\sigma_2]}(x_1, x_2; x_1', x_2') = \{[G^{(i)}S]+[G^{(i)}RS]\}(x_1, x_2; x_1', x_2')$$
$$-\int_{\sigma_1}\int_{\sigma_2} A^{(i)}_{[\sigma_1,\sigma_2]}(x_1, x_2; x_1'', x_2'')\gamma_\mu^{(1)}\gamma_\nu^{(2)}[RS](x_1'', x_2''; x_1', x_2')d\sigma_1''^\mu d\sigma_2''^\nu, \tag{A.3}$$

where

$$S(x_1, x_2; x_1', x_2') = S^{(1)}(x_1-x_1')S^{(2)}(x_2-x_2'), \qquad S^{(i)}(x) = (\partial^{(i)}+m^{(i)})\Delta^{(i)}(x),$$

and R is the resolvent kernel defined by the equation

$$\Psi(x_1, x_2) = \Psi_0(x_1, x_2)+ \iint R(x_1, x_2; x_1', x_2')\Psi_0(x_1', x_2')dx_1'dx_2',$$

and the symbol $[A\,B...C]$ is given by the formula

$$[AB \dots C](x_1, x_2; x_1', x_2') = \int \dots \int A(x_1, x_2; \xi_1, \xi_2)d\xi_1 d\xi_2 B(\xi_1, \xi_2; \dots) \dots C(\dots; x_1'x_2').$$

Since the functional derivative with respect to σ_1 and σ_2 of r.h.s. of (A.2) vanishes, (A.2) does not depend on σ_1 or σ_2. Consequently, setting $\sigma_1 = \sigma_2 = \sigma$ and choosing σ to be a hyperplane τ = const, and also assuming $x^{(1)}, x^{(2)}$ to be situated on σ we obtain from (A.2) a one-time equation.

APPENDIX B

During the process of calculation of explicit form of potential from form-factors (cf. Eq. (2.10)) one meets the calculation of the following typical integrals

$$\int \frac{(q^2)^n}{q^2} e^{iqr} dq = \Delta^n \left(\frac{1}{r}\right).$$

The calculation of the results of an action of Δ^n on the "function" r^{-1} requires a special care since this function has a locally nonintegrable singularity at the origin. In order to assure the proper treatment we shall treat the "function" r^{-1} as a distribution in the Schwartz test function space $S(R^3)$ and we shall use Gelfand–Shilov technique of analytic regulation of distributions with power singularities. In the case of r^{-1} distribution this technique uses r^λ distribution with λ any complex number. Gelfand and Shilov proved the following results:

(*i*) The distribution r^λ is well defined on $S(R^3)$ for all complex λ except $\lambda = -3, -5, -7, \ldots$.

(*ii*) At exceptional points $\lambda = -3, -5, -7, \ldots$, the distribution has single poles.

(*iii*) The action of the Laplace operator Δ on r^λ is given by the formula

$$\Delta r^\lambda = \lambda(\lambda+1)r^{\lambda-2}. \tag{B.1}$$

(*iv*) The Laurent's series expansion of r^λ around the singular points $\lambda = -3-2k$ is given by the formula

$$r^\lambda = \frac{4\pi}{(\lambda+2k+3)\,(2k)!}\,\delta^{(2k)}(r)+4\pi r^{-2k-3}+4\pi(\lambda+2k+3)r^{-2k-3}\ln r+\ \ldots, \tag{B.2}$$

where on the r.h.s. the symbol r^{-2k-3} represents the main part of the Laurent's series of r^λ (and not a distribution r^{-2k-3} which would have a pole).

We shall now calculate the distribution $\Delta^n \frac{1}{r}$ as the limit

$$\Delta^n \frac{1}{r} = \lim_{\lambda\to-1} \Delta^n r^\lambda.$$

Using (B.1) for $n = 1$ we have

$$\Delta \frac{1}{r} = \lim_{\lambda\to-1} \lambda(\lambda+1)r^{\lambda-2}.$$

By virtue of equation (B.2) for $\lambda' = \lambda-2 = -3$ we have $k = 0$ and

$$\Delta \frac{1}{r} = 4\pi \lim_{\lambda\to-1} \lambda(\lambda+1)\left[\frac{\delta(r)}{\lambda-2+3}+r^{-3}+(\lambda-2+3)r^{-3}\ln r+\ \ldots\right] = -4\pi\delta(r).$$

For $\Delta^2 \frac{1}{r}$ using the same steps we obtain

$$\Delta^2 \frac{1}{r} = \lim_{\lambda \to -1} \lambda(\lambda+1)(\lambda-2)(\lambda-1)r^{\lambda-4}.$$

Now for $\lambda \to -1$, $r^{\lambda-4} \to r^{-5}$ and therefore $k = 1$. Hence we have to use expansion (B.2) around the point $\lambda' = \lambda - 4 = -5$. Hence

$$\Delta^2 \frac{1}{r} = 4\pi \lim_{\lambda \to -1} \lambda(\lambda+1)(\lambda-2)(\lambda-1)\left[\frac{\delta^{(2)}(r)}{2(\lambda+1)} + r^{-5} + (\lambda+1)r^{-5}\ln r + \ldots\right]$$

$$= -4\pi \frac{3!}{4!}\delta^{(2)}(r).$$

Similarly we calculate

$$\Delta^n \frac{1}{r} = 4\pi \lim_{\lambda \to -1} \prod_{m=-1}^{2(n-1)} (\lambda - n)\left[\frac{\delta^{[2(n-1)]}(r)}{(\lambda+1)[2(n-1)]!} + r^{-(2n+1)} + (\lambda+1)r^{-(2n+1)}\ln r + \ldots\right],$$

or

$$\Delta^n \frac{1}{r} = \frac{-4\pi \prod_{m=1}^{2(n-1)} (-1-m)}{[2(n-1)]!}\delta^{[2(n-1)]}(r).$$

Using these results one obtains the formula (2.10) for the effective potential.

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