RELATIVISTIC TWO-BODY EQUATION FOR ONE DIRAC AND ONE KLEIN-GORDON PARTICLE

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(Received February 27, 1979)

A new relativistic two-body wave equation is conjectured for a system of one Dirac and one Klein-Gordon particle. The equation is solved exactly for Coulombic bound states in the case of equal, non-zero masses. In particular, a massless stable Coulombic bound state with $j = 1/2$ is shown to exist, provided the coupling constant α assumes its critical value 2. Such a situation would appear for every unbroken supersymmetric massive pair of one Dirac and one Klein-Gordon particle if only they interacted mutually through a critical Coulombic field.

As is well known, most of our information on elementary systems came out from investigating two-particle systems and forces acting therein. So, two-body wave equations belonged always to the most popular part of the quantum theory. In the non-relativistic approximation, these equations have an universal character of the two-body Schrödinger equation with an appropriate potential. Their relativistic form, however, becomes specific for a given kind of particles, being strongly spin- and force-dependent. In particular, for two Dirac particles in the static approximation with a potential transforming as the time-component of a 4-vector one gets the Breit equation [1]

$$[E-V(\vec{r}_1-\vec{r}_2)-(\vec{\alpha}_1\cdot\vec{p}_1+\beta_1 m_1)-(\vec{\alpha}_2\cdot\vec{p}_2+\beta_2 m_2)]\psi(\vec{r}_1,\vec{r}_2) = 0 \tag{1}$$

if one considers the single-particle theory. In the hole theory, one obtains the Salpeter equation [2], where instead of the potential $V(\vec{r}_1-\vec{r}_2)$ one has the interaction

$$[\Lambda_1^{(+)}(\vec{p}_1)\Lambda_2^{(+)}(\vec{p}_2)-\Lambda_1^{(-)}(\vec{p}_1)\Lambda_2^{(-)}(\vec{p}_2)]V(\vec{r}_1-\vec{r}_2) \tag{2}$$

which decouples the components $\psi^{(+-)}(\vec{r}_1,\vec{r}_2)$ and $\psi^{(-+)}(\vec{r}_1,\vec{r}_2)$.

The wave equation (1) (or (1) with the modified interaction (2)) represents the one-time approach to the relativistic two-body problem which can be derived either from the

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two-time approach provided by the Bethe–Salpeter equation or directly from the quantum field theory [3].

In the present note we consider a system of one Dirac and one Klein–Gordon particle and conjecture for this system the following relativistic two-body wave equation (being an analogue of the Breit equation (1)):

$$\{[E-V(\vec{r}_1-\vec{r}_2)-(\vec{\alpha}\cdot\vec{p}_1+\beta m_1)]^2-(\vec{p}_2^{\,2}+m_2^2)\}\psi(\vec{r}_1,\vec{r}_2)=0. \tag{3}$$

Eq. (3) can be rewritten in the form

$$\left\{\vec{\alpha}\cdot\vec{p}_1+\beta m_1-\frac{1}{2(E-V)}(\vec{p}_1^{\,2}+m_1^2-\vec{p}_2^{\,2}-m_2^2)-\frac{\vec{\alpha}\cdot[\vec{p}_1,V]}{2(E-V)}-\frac{E-V}{2}\right\}\psi(\vec{r}_1,\vec{r}_2)=0 \tag{4}$$

which in the center-of-mass frame, where $\vec{p}_1=-\vec{p}_2\equiv\vec{p}$ *and* $\vec{r}_1-\vec{r}_2\equiv\vec{r}$, *reduces to the Dirac-like equation with an effective potential describing the internal motion,*

$$\left(\vec{\alpha}\cdot\vec{p}+\beta m_1-\tfrac{1}{2}\frac{m_1^2-m_2^2+\alpha\cdot[\vec{p},V]}{E-V}-\frac{E-V}{2}\right)\psi(\vec{r})=0. \tag{5}$$

Notice that the wave equation (3) or (4) implies the following stationary form of the continuity equation for probability:

$$\mathrm{div}_1\left\{\psi^\dagger\left[(E-V)\vec{\alpha}-\frac{\vec{p}_1-\overleftarrow{p}_1}{2}\right]\psi\right\}+\mathrm{div}_2\left(\psi^\dagger\frac{\vec{p}_2-\overleftarrow{p}_2}{2}\psi\right)=0, \tag{6}$$

which reduces in the center-of-mass frame to the equation

$$\mathrm{div}\,[\psi^\dagger(E-V)\vec{\alpha}\psi]=0. \tag{7}$$

In the probability current in Eq. (6) $\vec{p}_i=-i\vec{\partial}_i$ and $\overleftarrow{p}_i=-i\overleftarrow{\partial}_i$ act to the right and to the left, respectively. In the probability current in Eq. (7) $\vec{p}=-i\vec{\partial}$ does not appear.

In the case of a central potential $V(\vec{r})=V(r)$ we obtain from Eq. (5) the following radial equation

$$\left[-i\alpha_r\left(\frac{d}{dr}+\frac{1+\beta k}{r}-\tfrac{1}{2}\frac{\frac{dV}{dr}}{E-V}\right)+\beta m_1-\tfrac{1}{2}\frac{m_1^2-m_2^2}{E-V}-\frac{E-V}{2}\right]\psi(r)=0, \tag{8}$$

where we have $k=\pm(j+1/2)$ in correspondence to the total parity $P=\eta(-1)^{j\mp 1/2}$. Here, $j=1/2, 3/2, 5/2, \ldots$ is the total angular momentum, whereas $\eta=\pm1$ denotes the intrinsic parity of the Klein–Gordon particle, relative to the Dirac particle. In the representation where

$$\alpha_r=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \beta=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \psi=\begin{pmatrix}\psi^+\\ \psi^-\end{pmatrix}, \tag{9}$$

Eq. (8) takes the form

$$\left(\frac{d}{dr}+\frac{k}{r}-\frac{1}{2}\frac{\frac{dV}{dr}}{E-V}\right)r\psi^{-}+\left(\frac{E-V}{2}-m_1+\frac{1}{2}\frac{m_1^2-m_2^2}{E-V}\right)r\psi^{+}=0,$$

$$\left(\frac{d}{dr}-\frac{k}{r}-\frac{1}{2}\frac{\frac{dV}{dr}}{E-V}\right)r\psi^{+}-\left(\frac{E-V}{2}+m_1+\frac{1}{2}\frac{m_1^2-m_2^2}{E-V}\right)r\psi^{-}=0. \tag{10}$$

Here, the superscript $\pm$ of the components $\psi^{\pm}$ refers to the intrinsic parity $\beta = \pm 1$ of the Dirac particle, described by the matrix β. Obviously, $P = \eta\beta(-1)^{l} = \eta(-1)^{j\mp 1/2}$ for both components ψ^{+} and ψ^{-} because $l = j\mp 1/2$ for ψ^{+} and $l = j\pm 1/2$ for ψ^{-} (hence $l = |k|-1$ or $|k|$ for ψ^{+} and $l = |k|$ or $|k|-1$ for ψ^{-} as $|k| = k$ or $-k$).

In the case of a Coulombic potential $V(r) = -\varepsilon\alpha/r$ with $\varepsilon = \pm 1$ and an arbitrary $\alpha > 0$, we can rewrite Eq. (10) as follows

$$\left[\frac{d}{dr}+\frac{k}{r}-\frac{\varepsilon\alpha}{2r(Er+\varepsilon\alpha)}\right]r\psi^{-}-\left[m_1-\frac{E}{2}-\frac{\varepsilon\alpha}{2r}-\frac{(m_1^2-m_2^2)r}{2(Er+\varepsilon\alpha)}\right]r\psi^{+}=0,$$

$$\left[\frac{d}{dr}-\frac{k}{r}-\frac{\varepsilon\alpha}{2r(Er+\varepsilon\alpha)}\right]r\psi^{+}-\left[m_1+\frac{E}{2}+\frac{\varepsilon\alpha}{2r}+\frac{(m_1^2-m_2^2)r}{2(Er+\varepsilon\alpha)}\right]r\psi^{-}=0. \tag{11}$$

Eq. (11) shows that

$$r\psi^{\pm} \underset{r\to 0}{\sim} r^{p+1/2}, \qquad r\psi^{\pm} \underset{r\to\infty}{\sim} e^{-\mu r}, \tag{12}$$

where

$$p = \pm\sqrt{(j+\tfrac{1}{2})^2-\frac{\alpha^2}{4}}, \qquad \mu = \pm\sqrt{m_1^2-\left(\frac{E}{2}+\frac{m_1^2-m_2^2}{2E}\right)^2}. \tag{13}$$

Notice that p is real or imaginary if $\alpha \leqslant 2j+1$ or $\alpha > 2j+1$, respectively, while μ (corresponding to bound states) is real and $\neq 0$ if $0 < |m_1-m_2| < E < m_1+m_2$ when $m_1 \neq m_2$ or $0 \leqslant E < 2m$ when $m_1 = m_2 \equiv m > 0$. Obviously, the lower sign at p is allowed by the regularity condition $r\psi^{\pm} \to 0$ at $r \to 0$ only if $0 \leqslant p^2 < 1/4$ or $p^2 < 0$ i.e. $\alpha^2 > (2j+1)^2-1$, whereas for bound states the lower sign at μ is always excluded by the normalization condition at $r \to \infty$. In particular, for $j = 1/2$ the lower sign at p is allowed if $\alpha > \sqrt{3}$ and p is imaginary if $\alpha > 2$.

After the standard substitution

$$r\psi^{\pm} = r^{p+1/2}e^{-\mu r}w^{\pm} \tag{14}$$

Eq. (11) takes the form

$$\left[\frac{d}{dr}+\frac{p+1/2+k}{r}-\frac{\varepsilon\alpha}{2r(Er+\varepsilon\alpha)}-\mu\right]w^{-}-\left[\mu^{+}-\frac{\varepsilon\alpha}{2r}-\frac{(m_1^2-m_2^2)r}{2(Er+\varepsilon\alpha)}\right]w^{+}=0,$$

$$\left[\frac{d}{dr}+\frac{p+1/2-k}{r}-\frac{\varepsilon\alpha}{2r(Er+\varepsilon\alpha)}-\mu\right]w^{+}-\left[\mu^{-}+\frac{\varepsilon\alpha}{2r}+\frac{(m_1^2-m_2^2)r}{2(Er+\varepsilon\alpha)}\right]w^{-}=0, \tag{15}$$

where

$$\mu^{\pm} = m_1 \mp \frac{E}{2}. \tag{16}$$

In the case of $m_1 = m_2 \equiv m$

$$\mu = \sqrt{m^2 - \left(\frac{E}{2}\right)^2} = \sqrt{\mu^+\mu^-}, \quad \mu^{\pm} = m \mp \frac{E}{2}. \tag{17}$$

Now, making the second substitution

$$w^{\pm} = (Er+\varepsilon\alpha)^{-1/2} v^{\pm} \tag{18}$$

we transform Eq. (15) into

$$\left(\frac{d}{dr} + \frac{p+k}{r} - \mu\right) v^- - \left[\mu^+ - \frac{\varepsilon\alpha}{2r} - \frac{(m_1^2 - m_2^2)r}{2(Er+\varepsilon\alpha)}\right] v^+ = 0,$$

$$\left(\frac{d}{dr} + \frac{p-k}{r} - \mu\right) v^+ - \left[\mu^- + \frac{\varepsilon\alpha}{2r} + \frac{(m_1^2 - m_2^2)r}{2(Er+\varepsilon\alpha)}\right] v^- = 0. \tag{19}$$

In the case of $m_1 = m_2 \equiv m$, Eq. (19) reduces to the form

$$\left(\frac{d}{dr} + \frac{p+k}{r} - \mu\right) v^- - \left(\mu^+ - \frac{\varepsilon\alpha}{2r}\right) v^+ = 0,$$

$$\left(\frac{d}{dr} + \frac{p-k}{r} - \mu\right) v^+ - \left(\mu^- + \frac{\varepsilon\alpha}{2r}\right) v^- = 0, \tag{20}$$

where μ and $\mu^{\pm}$ are given in Eq. (17). It is not difficult to demonstrate that for $m > 0$ Eq. (20) produces the polynomial solutions

$$v^{\pm} = \sum_{\nu=0}^{n_r} c_\nu^{\pm} r^\nu \quad (n_r = 0, 1, 2, \ldots) \tag{21}$$

if and only if

$$p - \frac{\varepsilon\alpha/2}{\sqrt{\left(\frac{2m}{E}\right)^2 - 1}} = -n_r, \tag{22}$$

the bound state solutions for $\varepsilon = -1$ being excluded by the normalization condition at $r \to \alpha/E$, unless $E = 0$ (cf. Eq. (18)). Here we have $k = \pm(j+1/2)$ and $P = \eta(-1)^{j \mp 1/2}$, except for $n_r = 0$ when $k = \varepsilon(j+1/2)$ and $P = \eta(-1)^{j-\varepsilon/2}$ only [4]. In the case of $p > 0$ the bound state condition (22) can be satisfied only for $\varepsilon = +1$, resulting in the Sommerfeld formula (with $2m$ and $\alpha/2$ substituted for m and α),

$$E = \frac{2m}{\sqrt{1 + \left[\frac{\alpha/2}{n_r + \sqrt{(j+1/2)^2 - \alpha^2/4}}\right]^2}} \quad (n_r = 0, 1, 2, \ldots \text{ for } \varepsilon = +1). \tag{23}$$

These energy levels exist if $0 < \alpha^2 < (2j+1)^2$. The familiar Coulomb $\alpha = 1/137$ lies in this range. In the case of $p < 0$ the condition (22) can be fulfilled both for $\varepsilon = +1$ (if $n_r > 0$) and $\varepsilon = -1$ (if $n_r = 0$), the latter being excluded by the normalization condition at $r \to \alpha/E$ (since here $E > 0$), so that we get the new formula

$$E = \frac{2m}{\sqrt{1+\left[\dfrac{\alpha/2}{n_r - \sqrt{(j+1/2)^2-\alpha^2/4}}\right]^2}} \quad (n_r = 1, 2, \ldots \text{ for } \varepsilon = +1). \tag{24}$$

Since in the case of $p < 0$ the regularity condition $r\psi^{\pm} \to 0$ at $r \to 0$ requires $-1/2 < p < 0$, the new energy levels exist if $(2j+1)^2-1 < \alpha^2 < (2j+1)^2$. Eventually, in the case of $p = 0$ the bound state condition (22) can be satisfied both for $\varepsilon = +1$ (if $n_r \geqslant 0$) and $\varepsilon = -1$ (if $n_r = 0$), leading to the formula

$$E = \frac{2mn_r}{\sqrt{n_r^2+\alpha^2/4}} \quad (n_r = 0, 1, 2, \ldots \text{ for } \varepsilon = +1,\ n_r = 0 \text{ for } \varepsilon = -1). \tag{25}$$

These energy levels exist if $\alpha = 2j+1$. For $n_r = 0$ formulae (23) and (25) take the same form

$$E = \frac{2m\sqrt{(2j+1)^2-\alpha^2}}{2j+1}. \tag{26}$$

In particular, if $\alpha = 2j+1$ Eq. (26) gives $E = 0$ for $\varepsilon = +1$ and $\varepsilon = -1$, showing the existence of a massless Coulombic bound state, irrespectively of the sign $\varepsilon = \pm 1$ of Coulombic potential.

Concluding our discussion of Eq. (20), we can see that in the case of $\alpha^2 \leqslant (2j+1)^2-1$ (i.e. $\alpha \leqslant \sqrt{3}$ for $j = 1/2$) Coulombic bound states exist only for attraction, corresponding to the Sommerfeld formula (23). In the case of $(2j+1)^2-1 < \alpha^2 < (2j+1)^2$ (i.e. $\sqrt{3} < \alpha < 2$ for $j = 1/2$), besides those related to Eq. (23), new Coulombic bound states appear for attraction, corresponding to formula (24). Finally, in the case of $\alpha = 2j+1$ (i.e. $\alpha = 2$ for $j = 1/2$) there are Coulombic bound states both for attraction (if $n_r = 0, 1, 2, \ldots$) and repulsion (if $n_r = 0$), corresponding to formula (25). In particular, we have shown that, *if only the coupling constant* α *assumes one of its critical values*

$$\alpha = 2j+1 = 2, 4, 6, \ldots \tag{27}$$

defined by the equation $p = 0$, *there exits for the system of one Dirac and one Klein–Gordon particle of equal non-zero masses a massless stable Coulombic bound state with the appropriate spin* $j = 1/2, 3/2, 5/2, \ldots$ *and parity* $P = \eta(-1)^{j-\varepsilon/2}$. This is true irrespectively of the sign $\varepsilon = \pm 1$ of Coulombic potential. For the first critical $\alpha = 2$ we have here $j = 1/2$ and $P = \eta(-1)^{(1-\varepsilon)/2}$.

It may be instructive to compare various one- and two-body wave equations with respect to their Coulombic behaviour at $r \to 0$. This is done in Table I. We can see from this Table that only for the Klein–Gordon and double Klein–Gordon equations the lower sign at p is allowed for the familiar Coulomb $\alpha = 1/137$ (if $l = 0$), since then $0 < p^2 < 1/4$

and hence $r\psi \to 0$ at $r \to 0$ [5]. A massless stable bound state with $l = 0$ turns out to exist here [5]. For the Dirac and double Dirac equations the lower sign at p is excluded for all values of α if we restrict ourselves to the conventional solutions which are not only normalizable at $r \to 0$ but also satisfy the more restrictive regularity condition $r\psi \to 0$ at $r \to 0$ (cf., however, Ref. [6]). Finally, for the Dirac–Klein–Gordon equation

TABLE I

Coulombic behaviour at $r \to 0$ of various one- and two-body wave equations

Equation	$r\psi$ at $r \to 0$	p	First critical α
Klein–Gordon	$r^{p+1/2}$	$\pm\sqrt{(l+\frac{1}{2})^2-\alpha^2}$	$\frac{1}{2}$
Dirac	r^p	$\pm\sqrt{(j+\frac{1}{2})^2-\alpha^2}$	1
Double KG	$r^{p+1/2}$	$\pm\sqrt{(l+\frac{1}{2})^2-\frac{\alpha^2}{4}}$	1
Double Dirac	r^p	$\pm\sqrt{j(j+1)+1-\frac{\alpha^2}{4}}$	2
Dirac–KG	$r^{p+1/2}$	$\pm\sqrt{(j+\frac{1}{2})-\frac{\alpha^2}{4}}$	2

conjectured in the present note, the lower sign at p is allowed only for $\alpha > \sqrt{3}$ (if $j = 1/2$) because then $0 \leqslant p^2 < 1/4$ or $p^2 < 0$ and hence $r\psi \to 0$ at $r \to 0$. A massless stable bound state with $j = 1/2$ appears here for the critical $\alpha = 2$. We can see that only here *the requirement of existence of a massless stable Coulombic bound state determines the coupling constant* α (*to be critical*). It is, of course, a coupling constant of a very strong interaction mediated between the massive Dirac and Klein–Gordon particles by some massless vector particles.

In conclusion, we would like to stress that in the case of *equal non-zero* masses the wave equation (3) (conjectured for the system of one Dirac and one Klein–Gordon particle) implies the existence of a *massless* stable Coulombic bound state with $j = 1/2$ *just* for the *critical* $\alpha = 2$. Remarkably enough, such a bound state would appear for every *unbroken supersymmetric* massive pair of one Dirac and one Klein–Gordon particle if only they interacted mutually through a critical Coulombic field. One may wonder whether neutrinos of different kinds are not such critical Coulombic bound states of various unbroken supersymmetric pairs of some subelementary constituents.

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