INCLUSIVE DISTRIBUTIONS IN EXTENDED LEADING LOGARITHMS

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The transverse momentum distribution of the partons is first studied in perturbative QCD in the leading logarithms approximation. The connection of the parton k_T with scaling violation is clarified. The inclusive cross sections for deep inelastic scattering and lepton pair production are then investigated in an extended leading logarithms approximation in which the relevant gluons are allowed to be emitted at arbitrary angles. In that way we obtain general expressions for the inclusive distributions that are valid for any detected transverse momenta q_T. The result for lepton-pair production is in a nonfactorizable form. However, upon integration over q_T it leads to $d\sigma/dM^2$ given by the factorizable Drell–Yan formula. For large q_T^2 it also has the usual parton model interpretation. Phenomenological implication of our results is also discussed.

1. *Introduction*

There has recently been some success in the experimental verification of the predictions of quantum chromodynamics [1] (QCD) in leptoproduction [2]. Thus the candidacy of QCD as a realistic theory of strong interactions has now a firmer basis than what theoretical appeal alone can provide. However, it is only the scaling violation of the total cross section in deep inelastic scattering (DIS) that has been tested. For such processes there are the powerful theoretical tools of operator product expansion and renormalization group analysis, which are not applicable to hadron-hadron collisions in general, or to the study of specific final states of any reaction in particular. Attention has therefore been turned to perturbation calculations in QCD and to the study of jets and inclusive reactions [3]. Initially, calculations were done for low-order diagrams, e. g. in e^+e^- annihilation [4], and lepton-pair production (LPP) [5]. More recently, extensions to higher orders have been considered [6, 7]. In almost all of these investigations it is the leading log (LL) approximation [8, 9] (i. e. leading in log Q^2) that has been used to make the calculations tractable. While the LL approximation is adequate for the study of cross sections integrated

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over the transverse momentum q_T of the detected particle, it is, however, necessary to go beyond LL for the calculation of q_T distribution. The study of such inclusive cross sections for arbitrary q_T is the main concern of this paper.

The purpose of our work is twofold. First, we consider the parton distribution function $G(x, k_T)$ as a function of the parton transverse momentum k_T, and relate it to the solution of the renormalization group equation. We shall describe $G(x, k_T)$ in perturbative QCD to infinite order but in LL approximation. Secondly, we consider inclusive cross sections for both DIS and LPP. Specifically, in the case of LPP in particular, we seek a general expression for the inclusive distribution that is valid for arbitrary values of the transverse momentum q_T of the dileptons. Complications arise because the generality of the range of q_T precludes the exclusive application either the LL or the hard-scattering approximations. The demand for a smooth transition between small and large values of q_T introduces a counting problem related to allowing certain gluon emission angles to be arbitrary. Our approximation is still in the leading order of log Q^2, but for the r-th moment of q_T^2, say, our result will be accurate to order $Q^{2r}(\log Q^2)^{n-1}$ for an n-th order diagram. In that sense our considerations constitute an improvement over the usual LL approximation [7].

The expression we obtain for the inclusive distribution of LPP is not of the factorizable form [10]. However, upon integration it reduces to the usual Drell–Yan factorizable formula [11] for $d\sigma/dM^2$. Moreover, for large q_T reactions our general distribution can also be simplified to the convolution of a product of universal distributions and hard--scattering cross sections. Its extrapolation to smaller values of q_T is, however, not valid.

We organize this paper in the following way. In Section 2 we discuss $G(x, k_T)$ in perturbative QCD and relate it to the solution of the evolution equation of Altarelli and Parisi [12]. We then consider DIS in Section 3 and derive the inclusive cross sections for the production of a quark, which can be defined experimentally as a jet. The result obtained there is then extended in Section 4 to LPP where two initial hadrons are involved. A counting problem is encountered and solved there. Our discussions in Section 3 and 4 will be for the simple case where gluons appear only as emitted partons (i. e. rungs in a ladder, for example). This is amended in Section 5 where both singlet and nonsinglet components of the distribution functions are taken into account. Finally, in Section 6 we give the concluding remarks.

2. *Renormalization group, leading logarithms and parton k_T*

In this section we describe the properties of the quark distribution $G(x, k_T)$ as derived from the LL calculations. Let us start from the connection between LL and RG approaches.

The standard technique for discussing DIS has been the renormalization group and operator product expansion [1]. Results of this approach are usually presented in terms of the moments of the parton distributions. For the non-singlet part of the moments they read

$$M_N(t) = M_N(0)\left(\frac{\alpha_0}{\alpha(t)}\right)^{A_N/2\pi b}, \qquad (2.1)$$

where

$$M_N(t) = \int_0^1 x^N G(x;t)dx, \qquad \alpha(t) = \frac{\alpha_0}{1+\alpha_0 bt}, \qquad t = \log\frac{Q^2}{Q_0^2}. \tag{2.2}$$

A_N is proportional to the anomalous dimension of the relevant operator and b is a constant. α_0 is the "coupling constant" ($\alpha_0 = g_0^2/4\pi$) renormalized at Q_0^2. For simplicity we discuss here the quark distributions only leaving the extension to include glouns to Section 5.

Equation (2.1) can be used to determine $G(x;t)$ by an inversion formula. It is also used by Altarelli and Parisi to derive equation which governs the t evolution of G, thereby specifying $G(x;t)$ itself once the initial conditions are given [12]. The master equation of Altarelli and Parisi provides a simple physical interpretation of the RG result (2.1) as being due to the emission of an infinite number of gluons.

Alternatively, we can expand (2.1) in α and infer from it the contributions coming from the emission of any fixed number of gluons. In view of the t dependence of $\alpha(t)$ in (2.1) we get

$$M_N(t) = M_N(0)\left\{1+\sum_{m=1}^{\infty}\frac{1}{m!}\left[\int_0^t \frac{\alpha(\tau)}{2\pi}d\tau\right]^m A_N^m\right\}. \tag{2.3}$$

The anomalous dimension A_N turns out to be the N-th moment of the Altarelli–Parisi probability function $P(z)$ for finding a quark with a momentum fraction z in a parent quark.

$$A^N = \int_0^1 z^N P(z)dz, \qquad P(z) = C_2(R)\left\{\frac{1+z^2}{(1-z)_+} + \tfrac{3}{2}\,\delta(1-z)\right\},$$

where $C_2(R)$ is the quadratic Casimir operator evaluated in the representation R for the quarks. The subscript $+$ denotes regularization of the singularity at $z = 1$ [12]. By inverting (2.3) we obtain

$$G(x;t) = G_0(x) + \sum_{n=1}^{\infty}\frac{1}{n!}\left[\int_0^t \frac{\alpha(\tau)}{2\pi}d\tau\right]^n P_n(x), \tag{2.4}$$

where

$$P_n(x) = \int dy G_0(y)\left[\prod_{i=1}^n dz_i P(z_i)\right]\delta(x-y\prod_{i=1}^n z_i)$$

and $G_0(x) \equiv G(x;0)$ contains all low Q^2 hadronic effects incalculable in perturbation theory.

We want to stress that the above discussion, which is the essence of the RG approach, makes no reference to the transverse momentum distribution. In all formulas k_T is integrated over and it is difficult to trace the effect of k_T on the final answer. In order to elucidate the role of parton transverse momentum we turn to the LL calculation which was proposed a long time ago [8, 9]. It has been shown that in Coulomb or planar gauge the important

n-th order skeleton diagram, for DIS, is the ladder diagram of Fig. 1 [6, 7, 9]. If the contribution of this diagram to the total cross section for the absorption of the virtual photon is expressed in the parton language, it reads

$$\sigma_n(x_b; Q^2) = \int dx G_n(x; t)\sigma_0(x_b)|_x, \tag{2.5}$$

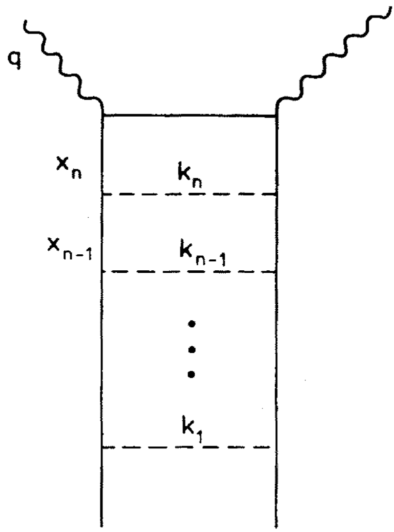

Fig. 1. Skeleton ladder diagram with solid line being quark or antiquark, dashed line gluon, and wavy line photon

where

$$G_n(x; t) = \frac{1}{n!}\left[\frac{\alpha_0}{2\pi}\int_{m^2}^{Q^2}\frac{dk_T^2}{k_T^2}\right]^n K_n(x), \tag{2.6}$$

m being the quark mass, and

$$K_n(x) = \int [\prod_{i=1}^{n} dz_i K(z_i)]\delta(x - \prod_{i=1}^{n} z_i), \quad K(z) = C_2(R)\frac{1+z^2}{1-z}, \tag{2.7}$$

and

$$\sigma_0(x_b)|_x = \frac{2\pi e^2}{xm\nu}\,\delta\left(1 - \frac{x_b}{x}\right), \quad x_b = \frac{Q^2}{2M_p\nu}. \tag{2.8}$$

The leading log Q^2 contribution to G_n comes from the region of ordered k_T's [13]

$$m^2 < k_{T1}^2 < \dots < k_{Tn}^2 < Q^2 \tag{2.9}$$

and this is the reason for the appearance of $\frac{1}{n!}$ in (2.6). In this approximation, the longitudinal and transverse momentum components decouple. The longitudinal degrees of freedom give $K_n(x)$ while the transverse ones factorize giving rise to exponentional-like series[1].

It was shown [9] that all other diagrams which give leading contributions can be obtained from the one in Fig. 1 by enriching it with all possible vertex and mass insertions as illustrated in Fig. 2. All required Green functions are themselves calculated in the LL.

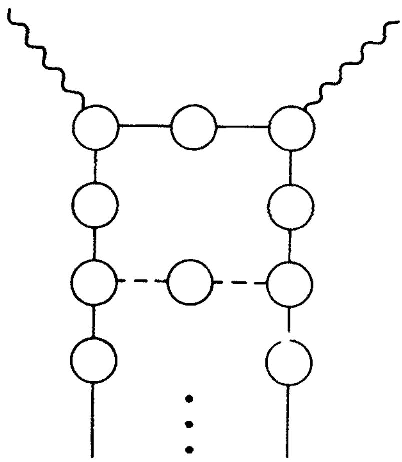

Fig. 2. Generalized ladder diagram with bubbles representing vertex corrections or mass insertions

All these insertions do not change the basic structure of (2.5) except that G_n is modified to

$$G_n(x;t) = \frac{1}{n!}\left[\int_0^t \frac{\alpha(k_T^2)}{2\pi}\frac{dk_T^2}{k_T^2}\right]^n P_n(x). \tag{2.10}$$

The important difference between (2.6) and (2.10) is that α_0 is replaced by

$$\alpha(k_T^2) = \frac{\alpha_0}{1+\alpha_0 b \log \frac{k_T^2}{m^2}}$$

and infrared divergences are regularized[2,3].

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[1] However, the longitudinal degrees of freedom factorize only after taking moments in x.

[2] We mention that the effect of insertions on the transverse degrees of freedom (i. e. the appearance of $\alpha(k_T^2)$) is well understood [9, 14]. On the other hand, it is not completely clear what set of diagrams give the proper regularization of the IR divergences in all orders. The answer to this question seems to depend on the method one chooses to investigate the problem [3, 21].

[3] The reason for appearance in (2.10) of the k_T^2 instead of the mass of the virtual quark, t_n say, is that in LL $t_n \sim k_{Tn}^2$ [9, 14].

Equation (2.10) establishes the equivalence of the LL and RG approaches to DIS. This fact was noticed independently by many authors [6, 7, 16]. From (2.10) we see that the dummy variable τ used in (2.3) and (2.4) has a simple interpretation, namely

$$\tau = \log \frac{k_T^2}{m^2}.$$

The advantage of LL calculations is that they provide much more physical insight into the mechanism of the generation of RG logarithms as we shall discuss below.

Extraction of the k_T distribution is now obvious. We write

$$G(x, k_T) = G_0(x, k_T) + \sum_{n=1}^{\infty} G_n(x, k_T).$$

Equations (2.9) and (2.10) suggest the nested integral form for G_n:

$$G_n(x, k_T) = P_n(x) \left\{ \frac{1}{2\pi} \frac{\alpha(k_T^2)}{k_T^2} \int_{m^2}^{k_T{}^2} \frac{dk_{Tn-1}^2}{k_{Tn-1}^2} \frac{\alpha(k_{Tn-2}^2)}{2\pi} \int_{m^2}^{k_{Tn-1}{}^2} \frac{dk_{Tn-2}^2}{k_{Tn-2}^2} \frac{\alpha(k_{Tn-2}^2)}{2\pi} \cdots \int_{m^2}^{k_{T2}{}^2} \frac{\alpha(k_{T1}^2)}{2\pi} \frac{dk_{T1}^2}{k_{T1}^2} \right\}. \tag{2.11}$$

The connection with Q^2 dependent distribution given in (2.4) is then consistent with

$$G(x; Q^2) = \int_{m^2}^{Q^2} G(x, k_T) dk_T^2. \tag{2.12}$$

Equation (2.12) is of central importance for our latter considerations.

$G_n(x, k_T)$ can be rewritten as

$$G_n(x, k_T) = P_n(x) \frac{\alpha(k_T^2)}{2\pi k_T^2} \frac{1}{(n-1)!} \left[\frac{1}{2\pi b} \log \left(1 + \alpha_0 b \log \frac{k_T^2}{m^2} \right) \right]^{n-1}. \tag{2.13}$$

Thus $G(x, k_T)$ is a slowly decreasing function of k_T at large k_T, a behaviour which results in a d vergent integral over k_T^2. It is precisely this divergence which gives the Q^2 dependence of $G(x;t)$. In other words, scaling violation in DIS is a manifestation of the fact that the transverse momenta of the partons are not limited. In that connection a direct phenomenological investigation of the k_T dependence may be more revealing than the Q^2 dependence of the integrated cross section.

Emergence of leading logarithms can be discussed in terms of integrals over either k_T or θ, the angle of the gluon momentum relative to the incident hadron. In terms of θ a logarithm arises from each integration, over $\cos\theta$, of a term behaving as $(E–P\cos\theta)^{-1}$ at small θ. In that sense one associates leading logarithms with collinear or narrow gluons. In terms of k_T, on the other hand, logarithms come from the large k_T (relative to the quark mass) part of the integral. That range can be large and yet still consistent with θ

small when Q^2 is large. It corresponds to $k_T^2 < \varepsilon Q^2$ for arbitrarily small (but fixed when $Q^2 \to \infty$) ε. In either language we shall regard LL as arising from gluons emitted in a narrow cone.

We remark that Lam and Yan [17] have first considered the k_T distribution in QCD. However, their study is based on a conjectured form for a generalized Altarelli–Parisi's evolution equation, and is therefore somewhat different from ours.

As a final note on $G(x, k_T)$ we emphasize that while the LL result given in (2.11) is adequate for calculating σ_{tot} (always in the large Q^2 limit), it is definitely inadequate for computing the moments of k_T^2. To illustrate the point, consider a typical form for G expressed in the θ variable

$$G(\theta) = \frac{A}{1-\cos\theta} + B. \tag{2.14}$$

In the LL approximation the constant term is ignored; indeed, upon integration over the phase space, i.e. $d\cos\theta$, its contribution is negligible compared to the narrow cone divergence. However in computing $\langle k_T^2 \rangle$, the integral must be weighted by $\sin^2\theta$, and the importance of the last term is self-evident.

Thus for an inclusive distribution that is reliable for calculating σ_{tot} as well as any moment of the transverse momentum, we must go beyond LL. It turns out that for the latter to be computed accurately up to its own LL approximation it is only necessary to allow the gluon immediately adjacent to the photon vertex to be emitted at an arbitrary angle. We shall call this the extended leading log approximation. Its adequacy is due to the order of k_{Ti} in (2.9) and its consequence is that for a given n-th order amplitude M_n corresponding to Fig. 1, one has

$$[k_{Ti}^2]/[k_{Ti-j}^2] = O(\log^j Q^2/m^2), \tag{2.15}$$

where

$$[k_{Ti}^2] \equiv \int d\Phi_n |M_n|^2 k_{Ti}^2, \quad i = 1, \ldots, n; \quad j = 1, \ldots, i-1; \tag{2.16}$$

Φ_n being the phase space for n gluons.

In the following sections we shall discuss the procedure for calculating the inclusive cross sections in extended LL approximation.

3. *Deep inelastic scattering*

It is both instructive and meaningful to consider the inclusive cross section, $f^0 d^3\sigma/df^3$, for the production of a quark with momentum f^μ in DIS. We do this even though quark is not usually regarded as a detectable particle. However, appropriate jet analyses can in principle specify $\vec{f}$. As we have already stated, we shall consider skeleton diagrams only. Our aim is to generalize the ladder diagram (Fig. 1) in which all gluon corrections are narrow, in such a way that that the innermost gluon (next to the photon vertices) can be emitted at an arbitrary angle. Apart from the phenomenological utility of the result, the investigation is also a prelude to our discussion on LPP in the next section.

The total cross section for virtual photon in DIS is, in the LL approximation,

$$\sigma = \int dx G(x; t)\sigma_0(x_b)|_x, \tag{3.1}$$

where the quantities in the integrand are given in (2.4) and (2.8). For later convenience, we represent (3.1) by a simplified diagram shown in Fig. 3, where the black dot denotes $G(x; t)$ and stands for a sum over n of all ladder diagrams depicted in Fig. 1. We stress

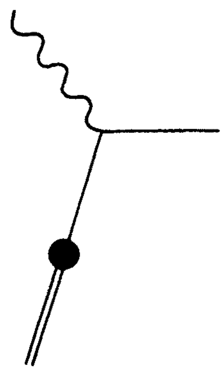


Fig. 3. Graphical representation of Eq. (3.1). Double line stands for hadron; black dot represents $G(x; t)$

that the graph in Fig. 3 represents probability, not an amplitude, and that only narrow gluon corrections are included in the black dot. Schematically, what we do now is to extract a gluon from the black dot and to liberate it from the narrow cone restriction, as indicated in Fig. 4.

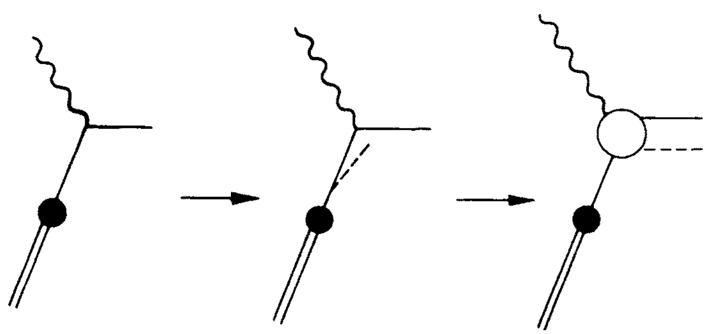


Fig. 4. Schematic representation of the procedure to extract a narrow gluon emission next to the photon vertex and then to generalize it to arbitrary angle

Analytically, it is more convenient to begin with σ_n for a fixed n-gluon process. From (2.5) and (2.6) we write

$$\sigma_n(x_b; t) = \int_{x_b}^{1} dx_n K_n(x_n) \frac{1}{n!}\left(\frac{\alpha_0}{2\pi} t\right)^n \sigma_0(x_b)|_{x_n}. \tag{3.2}$$

By $\sigma_0(x_b)|_{x_n}$ we denote the elementary (no gluon corrections) cross section for the absorption of the virtual photon by a quark with the momentum fraction x_n.

To extract the n-th gluon, we need to factorize the integrals over x_n and k_{Tn}^2. From (2.7) we have

$$K_n(x_n) = \int_{x_n}^{1} \frac{dx_{n-1}}{x_{n-1}} K_{n-1}(x_{n-1}) K(x_n/x_{n-1}), \tag{3.3}$$

while for k_{Tn}^2 we observe that

$$\frac{1}{n!} t^n = \int_{m^2}^{Q^2} \frac{1}{(n-1)!} [\log k_{Tn}^2/m^2]^{n-1} k_{Tn}^{-2} dk_{Tn}^2. \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.2), we obtain

$$\sigma_n(x_b; t) = \int_{x_n}^{1} dx_{n-1} K_{n-1}(x_{n-1}) \frac{1}{(n-1)!} \left(\frac{\alpha_0}{2\pi}\right)^{n-1} \int_{\Phi_2} \tau_n^{n-1} d\sigma_1^{LL}|_{x_{n-1}}, \tag{3.5}$$

where

$$\tau_n = \log k_{Tn}^2/m^2 = \log f_T^2/m^2,$$

$$d\sigma_1^{LL}(x_b; t)|_{x_{n-1}} = \frac{\alpha_0}{2\pi} \frac{1}{x_{n-1}} K\left(\frac{x_n}{x_{n-1}}\right) \sigma_0(x_b)|_{x_n} dx_n k_{Tn}^{-2} dk_{Tn}^2.$$

$d\sigma_1^{LL}|_{x_{n-1}}$ denotes now the differential cross section, for the scattering of γ^v off the $(n-1)$-th quark with the bremsstrahlung of one gluon, calculated in the narrow cone only. $\int_{\Phi_2}$ means integration over the two-body final state (f, k_n). Summing (3.5) over n we obtain in LL

$$\sigma - \sigma_0 \equiv \sum_{n=1}^{\infty} \sigma_n = \int_{x_b}^{1} dx \int_{\Phi_2} G(x; \tau) d\sigma_1^{LL}|_x. \tag{3.6}$$

Finally, by allowing the quark production to be at any angle, the more general expression for the inclusive cross section is

$$f^0 \frac{d\sigma}{d^3 f} = \int dx G(x; \tau) f^0 \frac{d\sigma_1}{d^3 f}\bigg|_x, \tag{3.7}$$

where $f^0 \dfrac{d\sigma_1}{d^3 f}$ is exact (not LL) cross section for $\gamma^v q \to qg$. Equation (3.7) is the main result of this section. It is consistent with LL but generalized to allow one gluon to be emitted at arbitrary angle without double counting in a sense to be described below.

Equation (3.7) is almost the same as what one would naively expect from the parton model considerations. However there is one very important difference, which is closely related to the underlying LL dynamics: namely, the parton distribution is controlled, in the sense of scaling violation, by the outgoing quark transverse momentum rather than by Q^2. This difference is clearly seen in Eq. (3.6) which is not of the parton model type. The reason for this is the following. As was argued in Section 2 we have to go beyond the leading logarithms if we want to calculate the k_T moments. In other words the universal distribution function $G(x, k_T)$ is not sufficient and it must be supplemented by a non-universal piece coming from the last loop. Nevertheless the universal part is restored by going backwards one step in the parton hierarchy. But the price we must pay is the loss of the simple parton picture. Now the transverse momenta in the *final* state are not decoupled from other degrees of freedom and cannot be independently integrated over to give Q^2 dependent *initial* density. Indeed, $G(x; \tau)$ cannot be replaced by $G(x; t)$ and taken in front of the $d\Phi_2$ integration when $d\sigma_1^{\mathrm{LL}}$ is replaced by $d\sigma_1^{\mathrm{exact}}$ in (3.6). Otherwise, upon integration over k_T^2 one would get an extra unwanted $\log Q^2$ factor, arising from double counting the scaling violation effect.

One recovers the parton model interpretation at the level of differential cross sections (3.7), but, as was mentioned above, parton distributions depend parametrically, in the sense of scaling violation, on f_T. Technical reason for this is that in LL k_T's of the emitted gluons are ordered. The Q^2 dependence comes from the upper limit on the largest k_T in a ladder. Now, since we have moved one step backwards, in the parton hierarchy, the upper limit for the k_T's of all other gluons becomes $k_{Tn} = f_T$. This variable should be regarded as the parameter controlling scaling violations in inclusive processes.

The reader may have noticed peculiarity in our notation, namely the use of semicolons in the G functions. It is because we want to avoid any possible confusions of the usage of k_T (1) as a parameter on which the x distributions depend, in a sense of scaling violation, and (2) as an actual transverse momentum of a quark with longitudinal momentum fraction x. In the former case we write $G(x; \tau)$ or $G(x; k_T)$ while in the latter we use $G(x, k_T)$. The connection between the two is as in (2.12).

$$G(x; \tau) = \int_{m^2}^{k_T{}^2} dk_T'^2 G(x, k_T'), \qquad \tau = \log \frac{k_T^2}{m^2}. \tag{3.8}$$

Therefore τ in (3.6) must not be regarded as the logarithm of the transverse momentum of the quark whose momentum fraction is x.

It is easily seen from (2.11) that $G(x; t)$ satisfies the following integral equation (including generalized ladders)

$$G(x; t) = G_0(x) + \int_x^1 \frac{dy}{y} P\left(\frac{x}{y}\right) \int_{m^2}^{Q^2} \frac{\alpha(k_T^2)}{k_T^2} \frac{dk_T^2}{2\pi} G(y; \tau). \tag{3.9}$$

As for x and k_T distribution, it can be inferred from (3.9) with the help of (3.8) yielding

$$G(x, k_T) = G_0(x, k_T) + \frac{1}{2\pi} \frac{\alpha(k_T^2)}{k_T^2} \int_x^1 \frac{dy}{y} P\left(\frac{x}{y}\right) \int_{m^2}^{k_T{}^2} dk_T'^2 G(y, k_T'). \tag{3.10}$$

We can recover the RG results by taking the x moments of (3.9). Then we obtain an integral equation for $M^N(t)$:

$$M^N(t) = M^N(0) + \frac{1}{2\pi} A^N \int_{m^2}^{Q^2} \frac{\alpha(k_T^2)}{k_T^2} M^N(\tau) dk_T^2, \tag{3.11}$$

which is equivalent to the Altarelli–Parisi differential equation [12]

$$\frac{dM^N(t)}{dt} = \frac{\alpha(t)}{2\pi} A^N M^N(t). \tag{3.12}$$

In the spirit of graphical representation, as in Figs 3 and 4, (3.9) is shown in Fig. 5, where the small open circle denotes $G_0(x)$. This will prove useful in the next section.

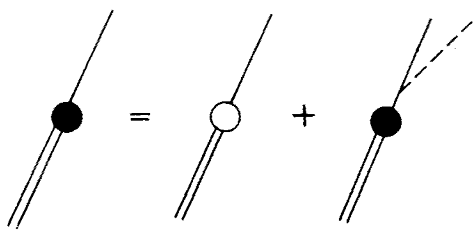


Fig. 5. Graphical representation of the integral equation, (3.9), for $G(x; t)$. The open circle represents $G_0(x)$

So far we have ignored infrared divergences. In particular we have not replaced $K(z)$ by $P(z)$ for the innermost gluon emission; consequently the integration over Φ_2 in (3.7) is infrared divergent. This divergence is not important when calculating the moments of k_T. When one wants to obtain total cross sections, however, the contributions from virtual gluons (i.e. "ivy diagrams" [3]) must be added in order to get meaningful result.

Finally, let us also comment on the generalized ladders, which must be considered in order to get complete answer. As we have mentioned in Section 2, the inclusion of vertex corrections and mass insertions have the effect of replacing α_0 by $\alpha(k_T^2)$ and $K(z)$ by $P(z)$ in formulas for total cross sections. We assume that the effect on the inclusive distributions is the same.

4. Lepton pair production

To apply the formalism developed in Section 3 to the LPP, we start with the LL result for the total cross section for fixed mass M of the lepton pair [7]

$$\frac{d\sigma^{LL}}{dM^2} = \int dx d\bar{x} G(x; t) G(\bar{x}; t) \left. \frac{d\sigma_0}{dM^2} \right|_{x\bar{x}}, \tag{4.1}$$

where $t = \log M^2/m^2$, and $d\sigma_0/dM^2|_{x\bar{x}}$ is the elementary cross section (i.e. no gluon corrections) for the annihilation of $q(x)$ and $\bar{q}(\bar{x})$ into γ. All narrow gluon corrections are included in the G functions. Equation (4.1) is depicted in Fig. 6.

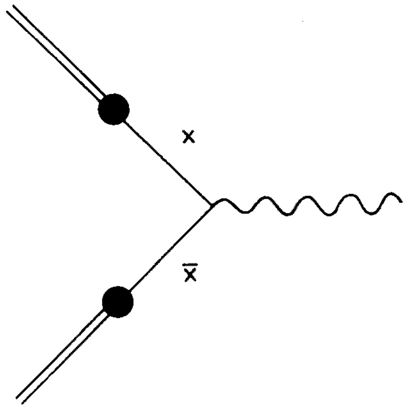


Fig. 6. Graphical representation of the Drell–Yan formula in leading logarithms

Now let us exhibit the innermost gluons explicitly, as shown graphically in Fig. 7. We repeat the steps leading from (3.1) to (3.7). All details of the calculation are the same as before. The main difference is that we now have to do it for *both* q and $\bar{q}$. The result is

$$\frac{d\sigma^{LL}}{dM^2} = \int dx d\bar{x} \left\{ G_0(x) G_0(\bar{x}) \frac{d\sigma_0}{dM^2}\bigg|_{x\bar{x}} + G_0(x) \int_{\Phi_2} G(\bar{x};\bar{\tau}) \frac{d\sigma_{1L}^{LL}}{dM^2}\bigg|_{x\bar{x}} \right.$$

$$\left. + G_0(\bar{x}) \int_{\Phi_2} G(x;\tau) \frac{d\sigma_{1R}^{LL}}{dM^2}\bigg|_{x\bar{x}} + \int_{\Phi_3} G(x;\tau) G(\bar{x};\bar{\tau}) \frac{d\sigma_2^{LL}}{dM^2}\bigg|_{x\bar{x}} \right\}. \tag{4.2}$$

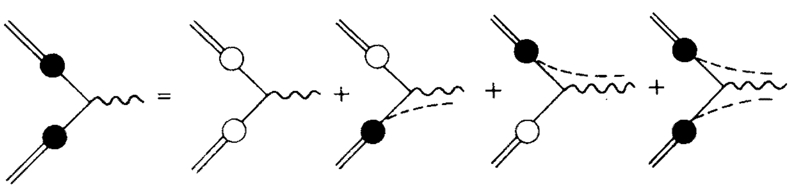


Fig. 7. Application of Fig. 5 to Fig. 6. The gluons are depicted to be in narrow cones

$\sigma_{1L(R)}^{LL}$ is, calculated in LL, the cross section for $q\bar{q}$ annihilation with bremsstrahlung of a gluon from a left (right) moving parton[4]. σ_2^{LL} is an analogous cross section but for the bremsstrahlung of two gluons from both partons. In terms of Feynman diagrams (in the Coulomb or planar gauge) we write

$$\sigma_{1L} \propto |M_L|^2, \quad \sigma_{1R} \propto |M_R|^2, \quad \sigma_2 \propto |M_{LR}|^2, \tag{4.3}$$

[4] Note that this statement is meaningful only in LL and in the Coulomb (or planar) gauge.

where the three M-amplitudes are shown, respectively, in Fig. 8 (a, b, c). The LL approximations of the σ's are for the case where all the gluons are emitted in narrow cones. It is seen from (4.2) or Fig. 7 that because there are *two* partons in the initial state, conse-

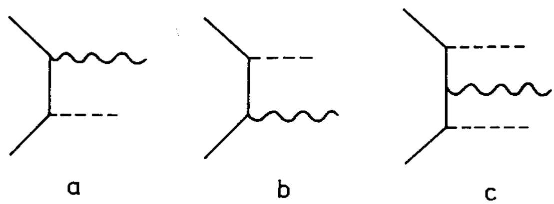


Fig. 8. Elementary $q\bar{q}$ annihilation amplitudes: (a) M_L, (b) M_R, (c) M_{LR}

quently *two* ladders, we are led to consider a process involving two emitted gluons as well as the bremsstrahlung of one gluon only.

To proceed, we allow all inner gluons to be emitted at arbitrary angles and replace σ^{LL} in (4.3) by the exact Born cross sections. However, the latter are not merely the ones given in (4.3) because of the existence of interference terms that are negligible in LL. Such terms must be included even in Coulomb gauge in extended LL. The problem is to guarantee no double counting for arbitrary values of q_T.

To that end we start from a set of all relevant Feynman diagrams, reorganize it according to (4.2), and read off the appropriate matrix elements for $\sigma_{1R(L)}$ and σ_2. Since perturbative QCD does not apply to the hadronic bound state problem, we shall factor out the G_0 function and ignore them temporarily. For the moment, initial state refers to quarks. The matrix element for the emission of k gluons, in a ladder approximation, has the structure

$$M_k = \sum_{\substack{m+n=k \\ m,n \geqslant 0}} D_{m,n}, \tag{4.4}$$

where $D_{m,n}$ is the amplitude for emission of m gluons from a right, and n from a left, moving parton, shown in Fig. 9. Let us denote

$$A_k = |M_k|^2 = A_k^{(1)} + A_k^{(2)}, \tag{4.5}$$

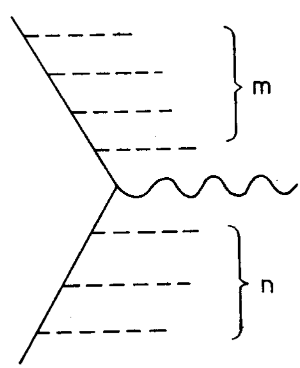


Fig. 9. Annihilation amplitude with $k = m+n$ gluons

where $A_k^{(1)}$ contains all diagonal terms and $A_k^{(2)}$ all possible interference terms. In LL the latter is negligible and the former can be simplified. For ease of generalization later, we write $A_k^{(1)}$ in the following form (still in LL):

$$A_k^{(1)} = \sigma_0 V_1 F_{k-1} + F_{k-1} V_1 \sigma_0 + \sum_{n=0}^{k-2} F_n (V_1 \sigma_0 V_1) F_{k-n-2}, \tag{4.6}$$

where we have introduced the notation

$$|D_{0,k}|^2 = \sigma_0 V_1 F_{k-1}$$

with V_1, which includes all factors connected with the innermost rung, written explicitly. F_{k-1} denotes the remaining ladder having $k-1$ rungs. The ordering in the last term of (4.6) emphasizes the double ladder structure of the amplitude, $n+1$ rungs on one side of the photon vertex, and $k-n-1$ on the other. All factor connected with the photon are lumped in σ_0. If we integrate over the phase space and sum over k, we recover (4.2). Extended LL is then simply to allow the gluon in V_1 to be emitted at any angle.

Now we turn to the interference term $A_k^{(2)}$ which we rewrite as follows

$$A_k^{(2)} = \sum_{m+n=k} \sum_{\substack{m'+n'=k \\ m' \neq m \\ n' \neq n}} D_{m,n} D^*_{m',n'}. \tag{4.7}$$

To select leading contribution from (4.7), we retain only terms whose contribution to the r-th moment of q_T^2 is of order $M^{2r} \log^{k-1}(M^2)$. The $\log^{k-1}(M^2)$ factor arises from $k-1$ narrow gluons, while M^{2r} comes from the large angle part of a gluon closest to the photon vertex, but not already included in $A_k^{(1)}$. The latter should clearly include the crossed terms between the two diagrams in Fig. 8(a) and (b). Explicit calculation of the diagrams in Fig. 10 indeed yields results of order M^{2r} to the r-th moment of q_T^2. In those diagrams

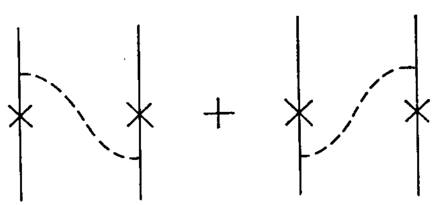
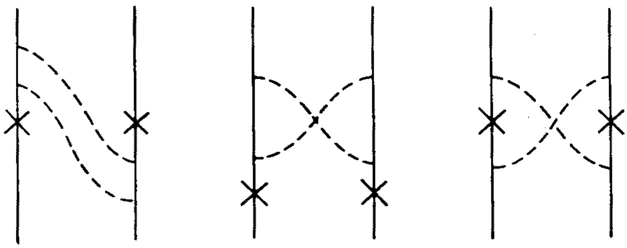


Fig. 10. One-gluon crossed diagrams with the crosses indicating the locations of the photon vertices

the crosses denote the location of the photon vertices. It is easy to see that any other crossed diagrams would involve either (1) more than one gluon line going across the photon vertices, as illustrated in Fig. 11(a), or (2) gluon lines crossing themselves, such as the ones shown in Figs 11(b) and (c). However, all such diagrams contribute to the moments of q_T^2 in lower order in $\log M^2$. In particular, the three diagrams in Fig. 11 contribute to $d\sigma/dM^2$ as $(\log M^2)^0$ and to the r-th moment of q_T^2 as $M^{2r}(\log M^2)^0$, as can be demonstrated by explicit calculations. They are therefore negligible in extended LL when compared to the contributions from diagrams such as those in Fig. 12, which are a power of log M^2 higher, and are included already in (4.7) in connection with Fig. 10. This argument can be generalized to any gluon diagram with any gluon lines crossing;



a b c

Fig. 11. Two-gluon crossed diagrams

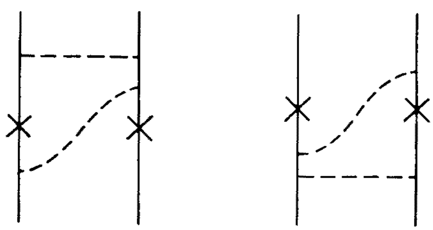


Fig. 12. Two-gluon mixed diagrams

their contribution is of order $(M^2)^r \log(M^2)^{k-2}$ or lower, and therefore not leading. Hence, we conclude that (4.7) can be reduced to

$$A_k^{(2)} = \sum_{m=0}^{k-1} D_{m,k-m} D^*_{m+1,k-m-1} + \sum_{m=1}^{k} D_{m,k-m} D^*_{m-1,k-m+1} \tag{4.8}$$

in which no gluons are allowed to cross each other. The above expression corresponds to sandwiching the diagrams in Fig. 10 between the ladder, which collectively have $k-1$ rungs. It then follows that (4.8) can alternatively be written as

$$A_k^{(2)} = \sum_{m=0}^{k-1} F_m \sigma_{1C} F_{m-k-1}, \tag{4.9}$$

where σ_{1C} is defined to consist of the two crossed diagrams in Fig. 10.

Having derived (4.6) and (4.9) in the extended LL, we now integrate them over the k-gluon phase space Φ_k and sum over k, obtaining

$$q^0 \frac{d^4\sigma}{dM^2 d^3q} = \sum_{k=0} \int d\Phi_k A_k = \int (LG+GR) + \int GCG, \tag{4.10}$$

where

$$L = \sigma_0 V_1, \quad R = V_1\sigma_0, \quad C = V_1\sigma_0 V_1 + \sigma_{1C}. \tag{4.11}$$

Equations (4.10) and (4.11) are formal; for example, the hadronic nonperturbative part G_0 that guards each incident particle is implied but not exhibited explicitly, and the integration variables not specified. To establish a parallelism with the LL expression (4.2), we write (4.10) as

$$d(d\sigma/dM^2) = \int dx d\bar{x}[G_0(x)G_0(\bar{x})d\sigma_0 + G_0(x)G(\bar{x};\tau_q)d\sigma_{1L} + G(x;\tau_q)G_0(\bar{x})d\sigma_{1R}$$

$$+ G(x;\tau_q)G(\bar{x};\tau_q)d\sigma_{1C} + G(x;\tau)G(\bar{x};\bar{\tau})d\sigma_2] \tag{4.12}$$

where $\tau_q = \log q_T^2/m^2$,

$$d\sigma_{1P} = \frac{1}{\text{flux}}|M_P|^2 d\Phi_2, \quad \text{P = L, R}, \quad d\sigma_{1C} = \frac{1}{\text{flux}}(M_L M_R^* + M_R M_L^*)d\Phi_2,$$

$$d\sigma_2 = \frac{1}{\text{flux}}|M_{LR}|^2 d\Phi_3. \tag{4.13}$$

Equation (4.12) is the main result of this paper. It is the differential form of (4.2) improved over LL. The inclusive cross section $q^0 d^4\sigma/dM^2 d^3q$ can be obtained from it by leaving the photon momentum unintegrated. We have not included the (primordial) k_T distribution in G_0 because our consideration on k_T has been entirely in the context of perturbative QCD. However, the primordial component can be inserted by hand in (4.12) for phenomenological applications.

A number of remarks on (4.12) are in order. The appearance of τ_q, τ, and $\bar{\tau}$ in the G functions instead of log (M^2) is as expected from our previous result on DIS. The G functions are universal distributions associated with hadrons, independent of the specific process. But they are parametrized by the τ variables, which depend on the nature of the subprocesses described in (4.13). Consequently, the longitudinal and transverse degrees of freedom are dynamically coupled, a phenomenon that departs from the simple picture of the naive parton model.

An essential character of (4.12) is that it is not factorizable into an overall product of two universal G functions. Any attempt to do so would result either in double counting or in missing some terms in some kinematical region. It can, however, be simplified for specific purposes. For total cross section (integrated over $\vec{q}$), LL is sufficient, so (4.12) reduces to (4.2), which is equivalent to (4.1) in the factorized form. For computing (different from zeroth) moments of q_T^2, the first term in the integral of (4.12) is negligible, and the last term can be separated into a sum of two terms, each corresponding to one of two gluon angles in Fig. 8(c) being small, the other arbitrary. These two terms then combine with the second and third terms in the integrand to render the whole sum factorizable, i.e.

$$d(d\sigma/dM^2) = \int dx d\bar{x} G(x;\tau_q)G(\bar{x};\tau_q)d\sigma_{1T}, \tag{4.14}$$

where $d\sigma_{1T} = d\sigma_{1L}+d\sigma_{1R}+d\sigma_{1C}$. Indeed, apart from the appearance of τ_q, (4.14) is the intuitive formula used by many for phenomenological calculations [18]. The derivation of (4.14) relies on the crucial approximation that we can double count the contribution from the last term of (4.12) in the region where both gluon angles are small. While that region is excluded when q_T^2 is large, and makes non-leading contribution to the calculation of the moments of q_T^2, it is of central importance when q_T^2 is small and therefore crucial to the calculation of total cross section. Thus an inclusive cross section based on (4.14) is not reliable (roughly double counts) when q_T^2 is extrapolated from $O(M^2)$ to the narrow cone region, $m^2 < q_T^2 < \varepsilon M^2$.

Present experimental data on $d^3\sigma/dMdyq_T^2$ are for $q_T^2 \ll M^2$ so phenomenology using (4.14) is not justified [19]. In most phenomenological work [18] scaling violation is ignored, corresponding to setting G to G_0 in (4.12) and (4.14); it implies neglecting of the last term of (4.12), which is not warranted for any value of q_T^2. Thus for an inclusive cross section that is generally valid for all q_T^2, one must use (4.12) and recognize the nonfactorizability.

5. *Generalization to include gluons*

In the preceding sections we have considered only quarks emitting gluons, but not gluons creating $q\bar{q}$ pairs, or the self-coupling of gluons. We now generalize the results of Section 4 to such possibilities in QCD.

Since the diagrams involving quartic self-coupling of the gluons do not contribute to LL, the introduction of $g \to q\bar{q}$ and $g \to gg$ vertices does not change the topology of the ladder diagrams already considered. Hence, we need only generalize the formalism described in the previous section to matrices, giving our attention mainly to the hard processes involving large angles. We then elevate G to a two-component column matrix G_i, where i = Q and G, denoting quark and gluon channels, respectively. We shall, for brevity, adopt the simplified notation of (4.10) and suppress the symbols for phase space integration.

Following the same considerations as in Section 4, we arrive at the matrix equation

$$d\sigma = G_0^T\Sigma_0 G_0+G_0^T LG+G^T RG_0+G^T ZG, \tag{5.1}$$

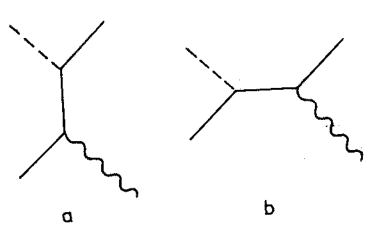


Fig. 13. Compton-like Born amplitudes

where T denotes transpose and $(\Sigma_0)_{ij} \equiv \sigma_0\delta_{iQ}\delta_{jQ}$. L, R, and Z are now 2×2 matrices to be described. The QQ elements of these matrices are, of course, just the ones given in (4.11) since the photon does not couple to the gluons directly. The other elements of L and R

are obvious, since they follow straightforwadly from the nature of the Born amplitudes for quark-gluon scattering shown in Fig. 13. The results are shown in Fig. 14. The square of the amplitude in Fig. 13(b) does not contribute to LL even if q_T is small, so it should appear in Z. It is not difficult to determine all the terms that contribute to extended LL;

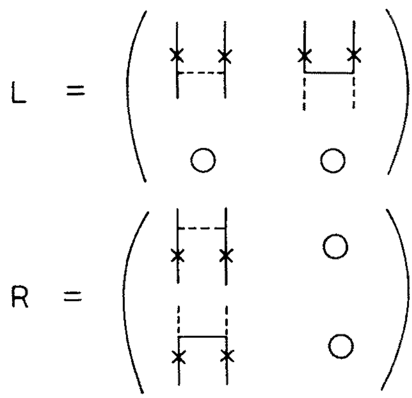


Fig. 14. Graphical representation of the matrices L and R

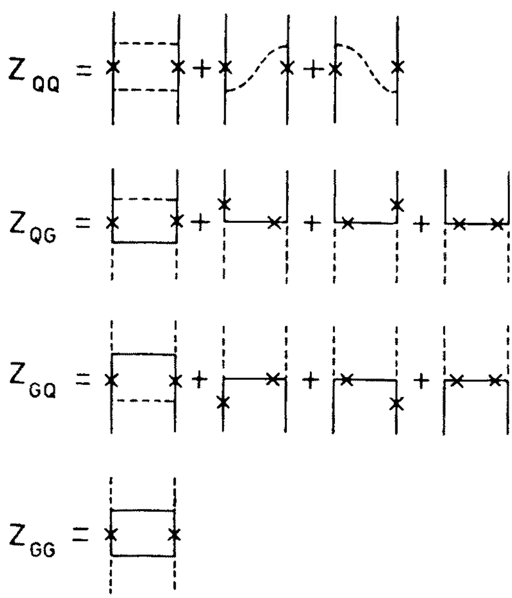


Fig. 15. Graphical representation of the matrix elements of Z

the result for Z is given in Fig. 15. Note that Z corresponds to $\sigma_{1C}+\sigma_2$ in the notation of Section 4. Different parameters controlling the scaling violations in the two adjacent G functions should be used in (5.1) for the one and two parton cases, as is explained in Section 4.

It is obvious that in the LL approximation where small $q_T^2(\sim \varepsilon M^2)$ is dominant, (5.1) reduces to the naive formula, similar to the one in Fig. 7 but in matrix form. For large q_T^2 ($\sim M^2$), on the other hand, it takes the form

$$d\sigma = G^T T G, \tag{5.2}$$

where

$$T_{QQ} = |\text{Fig. 8(a)+(b)}|^2 d\Phi_2/\text{flux}, \quad T_{GQ} = |\text{Fig. 13(a)+(b)}|^2 d\Phi_2/\text{flux}, \quad T_{GG} = 0.$$

This is the intuitive result that generalizes (4.14), but only for $q_T^2 = O(M)$. For arbitrary q_T^2 (5.1) should be used.

6. *Conclusion*

We have investigated the problems related to the transverse momenta of both the partons and the detected "particle". For the former we have studied the parton distribution function in the LL approximation, while for the latter we derived the general expression for the inclusive cross section in the leading power of log Q^2 that allows the detected particle to have arbitrary transverse momentum.

The distribution function $G(x, k_T)$ is required to satisfy the condition that upon integration over k_T it yields the usual $G(x; Q^2)$ whose Q^2 dependence is prescribed by the operator product expansion and renormalization group analyses. Our result reveals how the scaling violation is related to the k_T dependence of the G function. It is mainly controlled by the k_T behaviour in the region where k_T^2 is large compared to the quark (mass)^2 (or the primordial k_{T0}^2).

In LL the transverse momenta of the partons in a ladder are ordered. If we integrate $G(x_{n-1}, k_{Tn-1})$ over k_{Tn-1}^2, we obtain $G(x_{n-1}; \log k_{Tn}^2)$. Thus the scaling violation is controlled by the k_{Tn}^2 of the next generation of partons, in the sense of ladders as well as the Kogut–Susskind picture [20] of parton hierarchy. In total cross sections, such as νW_2 in DIS or $d\sigma/dM^2$ in LPP, the parton model is valid at the deepest level of that hierarchy. Because only the k_T^2 of the final generation is bounded by Q^2, scaling violations for total cross sections are controlled by Q^2 in LL. In inclusive distributions, on the other hand, the parton model is recovered at the $(n-1)$-th level; hence, scaling violation is governed by k_{Tn}^2. This transverse momentum is controlled by that of the detected particle and *not* by Q^2. The distinction is significant and testable experimentally.

In LPP we have further obtained a general expression for the inclusive cross section, valid for arbitrary q_T. It gives rise to the factorizable forms in two limiting cases. After integration over q_T^2, which is dominated by small q_T^2 ($\ll M^2$), it reproduces the LL result for $d\sigma/dM^2$, as well as $q^0 d^2\sigma/dM^2 dq_L$. For large $q_T^2(\sim M^2)$ it can be reduced to the familiar parton model form for inclusive distributions. The general formula that interpolates between the small and large value of q_T^2 is not factorizable. The complication arises because we have properly taken into account (without double counting) the set of ladder diagrams that describe the correction due to the parton emitted in a narrow cone around each incident hadron. Those partons have k_T^2 much smaller than M^2 but it can be large compared to the primordial $k_{T_0}^2$. Phenomenologically, that is just the region where the dilepton data show features that cannot be accommodated by hard-collision calculations [19]. The application of our general formula to the high M^2 data should be very interesting and revealing.

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