EXTRACTION OF IGNORABLE VARIABLES FROM THE HAMILTONIAN OF THE YANG-MILLS FIELD

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The potentials $A_i^a(x)$ of the classical SU(2) Yang-Mills field in temporal gauge are expressed in terms of new variables $\omega_n^a(x)$ in a way permitting an explicit separation of ignorable (cyclic) variables in the Hamiltonian.

1. *Introduction*

In discussing a system which possesses some symmetry it is usually advantageous to choose the variables in a way which explicitly respects this symmetry. For instance in the case of a particle in a spherically symmetric potential it is simplest to use spherical variables r, ϑ and φ. The variables ϑ and φ are connected with a conserved quantity e.g. orbital momentum. The variables ϑ and φ "carry" rotations and they are (to some extent) ignorable variables.

The situation is similar in gauge theories, where one has to satisfy the conditions of the type of the Gauss law. In order to satisfy these conditions in a natural way we have to extract corresponding gauge invariant variables. Recently, in papers [1], [2] and [3] several ways have been proposed of defining gauge invariant variables and those which "carry" gauge transformations for Yang-Mills fields (YMF). In Refs. [1] and [3] the starting point has been the analogy between YMF and the electromagnetic field (EMF). As it is well known in the latter case the transverse fields are gauge independent whereas the longitudinal fields do depend on the gauge. The method used in Ref. [2], although similar to those of Refs. [1] and [3], can be used only in the case of the non-abelian YMF.

In this note we shall study the problem only for the case of classical fields, since this formulation can be very useful also for the more complicated case of quantised YMF. Some quantum aspects of YMF are studied in Refs. [1-4].

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In the next Section we shall describe YMF in the temporal gauge. It will be shown, that after subtracting the ignorable (gauge) degrees of freedom the remaining field is transverse both for the YMF and EMF cases. Comments and concluding remarks are presented in the last Section.

2. *Degrees of freedom of the classical YMF*

We shall start with the Hamiltonian formulation of the problem and we shall use the temporal (Hamiltonian) gauge $A_0^a = 0$ $(a = 1, 2, 3)$ in which the YMF is described by the Hamiltonian

$$H = \int d^3x[\tfrac{1}{2}(E_i^a)^2 + V(A)], \tag{1}$$

where $V(a) = \frac{1}{4}(\partial_i A_j^a - \partial_j A_i^a + \varepsilon^{abc} A_i^b A_j^c)^2$, A_i^a and E_i^a are canonically conjugated variables. The Hamiltonian (1) is invariant with respect to time-independent gauge transformations

$$A_i^a \to R^{ab} A_i^b - T^{ab} \partial_i \Omega^b, \qquad E_i^a \to R^{ab} E_i^b, \tag{2}$$

where R^{ab} is an orthogonal matrix

$$R^{ab} = (2K^2 - 1)\delta^{ab} + 2\Omega^a \Omega^b - 2K\varepsilon^{abc}\Omega^c$$

and

$$T^{ab} = (R^{ab} + \delta^{ab})/K, \qquad K = (1 - \Omega^a \Omega^a)^{1/2}.$$

The fields A_i^a are parametrised in the following way

$$A_i^a = r^{ab} B_i^b - t^{ab} \partial_i \omega_0^b, \tag{3}$$

where r^{ab} and t^{ab} depend on the ignorable variables ω_0^a in the same way as R^{ab} and T^{ab} depend on Ω^a. It is easy to show that fields B_i^a are unchanged by the gauge transformations provided

$$\omega_0^a \to \kappa \Omega^a + K \omega_0^a - \varepsilon^{abc} \omega_0^b \Omega^c, \tag{4}$$

where $\kappa = (1 - \omega_0^a \omega_0^a)^{1/2}$.

In this way, similarly to [1] and [3] we have separated the ignorable variables ω_0^a, which "carry" the gauge transformations. The variables are not fixed unambiguously and we have to specify the way in which fields B_i^b depend on the remaining six gauge invariant variables ω_α^a, $\alpha = 1, 2$. In Ref. [1] the fields B_i^b have been chosen as transverse $\partial_i B_i^b = 0$ and it has been shown that the decomposition in Eq. (3) is not unique. This has served as a motivation for Ref. [3] where the 9 variables B_i^b have been required to satisfy three general conditions of the type $F^a(B_i^b) = 0$. In Refs. [3] and [5] it has been shown, that in temporal gauge there are no supplementary conditions avoiding the problem of Gribov's ambiguities (provided that only fields regular at infinity are taken into account).

One might think, however, that there can exist other criteria prefering a specific choice of the conditions $F^a(B_i^b) = 0$. A hint for this can be obtained from the analogy with EMF. It is well known, that in the latter case we have to exclude from the potential the whole longitudinal part. This is equivalent to the minimization of the expression

$$(B, B) = \int d^3xB_i^2$$

with respect to λ, where λ is defined by $B_i = A_i - \partial_i\lambda$. As a straightforward extension of this requirement to the case of the YMF we can minimize the norm of the field B_i^b (see Eq. (3)) defined as

$$(B, B) = \int d^3x(B_i^b)^2. \tag{5}$$

From the orthogonality of r^{ab} and from the definition of t^{ab} it follows that $B_i^b = A_i^a r^{ab} + t^{ab}\partial_i\omega_0^a$. A straightforward but tedious calculation shows that the minimization of (5) with respect to ω_0^a leads to

$$t^{ab}\partial_i B_i^b = 0. \tag{6}$$

Taking into account the regularity of the matrix t^{ab} Eq. (6) implies that the fields B_i^b are transverse and we can put

$$B_i^b = e_i^\alpha \omega_\alpha^b, \tag{7}$$

where e_i^α are standard operators of the transverse polarization. In this way we arrived naturally at the Coulomb condition $\partial_i B_i^b = 0$. The Gribov's ambiguities result from the fact that, in general, function (5) possesses for specified A_i^a several local minima.

In the next step we have to find the momenta canonically conjugated to ω_n^a $(n = 0, 1, 2)$. These are calculated from the generating function

$$\phi(\omega_n^a, E_i^b) = \int d^3x(r^{ab}e_i^\alpha\omega_\alpha^b - t^{ab}\partial_i\omega_0^b)E_i^a$$

by using the standard relationship $\pi_n^a = \delta\phi/\delta\omega_n^a$. After some formal manipulations we obtain

$$\pi_0^a = t^{ac}G^c, \quad G^c = \partial_i E_i^c + \varepsilon^{abc}A_i^a E_i^b \tag{8}$$

$$\pi_\alpha^a = e_i^{*\alpha}F_i^a, \quad F_i^a = r^{ba}E_i^b; \tag{9}$$

where the asterisk denotes the conjugation with respect to the scalar product defined in Eq. (5).

The conserved quantities G^c are generators of gauge transformations (2). By inverting the matrix t^{ab} we get

$$G^c = -\tfrac{1}{2}\kappa\pi_0^c + \tfrac{1}{2}\varepsilon^{abc}\omega_0^a\pi_0^b. \tag{10}$$

As expected, G^c's depend only on ignorable variables. Let us note that formula (10) is general and independent of the choice of the dependence of B_i^a on ω_α^a (in the general case we have just to replace e_i^α by $\partial B_i^a/\partial\omega_\alpha^a$ in Eq. (9)).

Since, formally, the relationships between A_i^a and B_i^a and between E_i^a and F_i^a are the same as those given by gauge transformations (see Eqs. (3) and (9)) we have

$$H = \int d^3x[\tfrac{1}{2}(F_i^a)^2 + V(B)]. \tag{11}$$

Let us rewrite this Hamiltonian into new variables. According to the first of Eqs. (9) the transverse part of the field F_i^d is given as

$$F_i^{d,\mathrm{T}} = e_i^\alpha \pi_\alpha^d \tag{12}$$

and the longitudinal one can be determined from Eqs. (3), (8) and (9). It is easy to show that

$$F_i^{d,\mathrm{L}} = \partial_i D_{db}^{-1} Q^b, \tag{13}$$

where D_{db}^{-1} is the inverse of

$$D^{bd} = \Delta\delta^{bd} + \varepsilon^{bcd} B_i^c \partial_i \tag{14}$$

and

$$Q^b = G^c r^{cb} - \varepsilon^{bcd} B_i^c F_i^{d,\mathrm{T}}. \tag{15}$$

The quantities Q^c are gauge invariants, in fact the

$$G^e r^{ec} = \tfrac{1}{2}\,\kappa\pi_0^c + \tfrac{1}{2}\,\varepsilon^{abc}\omega_0^a\pi_0^b$$

do not change under gauge transformations.

Inserting $F_i^{d,\mathrm{T}}$ and $F_i^{d,\mathrm{L}}$ into Eq. (11) we get the final formula (obtained already in Ref. [1])

$$H = \int d^3x\left[\tfrac{1}{2}\,(\pi_\alpha^a)^2 + V(B) - \tfrac{1}{2}\,Q^a \Delta_{ab}^{-1} Q^b\right], \tag{16}$$

where $\Delta_{ab}^{-1} = D_{cd}^{*-1}\Delta D_{cd}^{-1}$ and Q^a has to be expressed by using Eqs. (15) and (10). The first and the second terms in the r.hs. of Eq. (16) describe a local self-interacting transverse field. The last term corresponds to the Coulomb-like interaction between the external charges (the first term in Eq. (15)) and the charges carried by the transverse field (the second term in Eq. (15)). In the free-field case $G^c = 0$ and the Hamiltonian is independent of the ignorable variables.

3. *Conclusions*

In the preceeding chapter we have arrived in a natural way at the decomposition of the YMF into the gauge invariant transverse part and the ignorable remainder carrying the gauge degrees of freedom: We have minimised a (naturally defined) norm of the field remaining after the subtraction of the gauge part. It is surprising, to some extent, that only the transverse part of the field remains after this procedure.

The transverse fields have interesting properties already on the classical level. Only the condition $\partial_i B_i^b = 0$ is satisfied; the effective charge Q^b is an algebraic function of the ignorable and transverse fields, not containing their derivatives. On the other hand our approach can be relevant also for the quantisation of the YMF: instead of introducing the constraints $F^a(B_i^b) = 0$ we minimize some additional functional, what may be more advantageous from both conceptual and computational points of view.

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