ASYMPTOTIC STATES OF FRIEDMAN UNIVERSE AND THE ENERGY CONDITIONS USING THE PHASE VARIABLE METHOD

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The evolution of Friedman models with arbitrary pressure $p = p(\varepsilon, H)$ in the phase plane: Hubble function-energy density ε is presented. The conclusions are: critical points of the phase plane and their stability depend on energy conditions but not on the actual form of pressure; models with constant viscosity violate the condition $\varepsilon+p \geqslant 0$; the weak and the strong energy conditions are preserved if the viscosity coefficient is proportional to $\varepsilon^{1/2}$.

Introduction

Friedman models filled with dust matter and without cosmological constant ($\Lambda = 0$) are simply and naturally divided into two kinds:

1. closed models with limited life-time (they contain two singularities: the past one as the beginning and the future one as the end of the evolution, and the interval of time between them is finite),
2. open ($k = 0$, $k = -1$) models with unlimited life-time i.e., models containing one singularity at $t = 0$ and a regular state at $t \to \infty (t \to -\infty)$ in which $R \to \infty$, $H \to 0$, $\varepsilon \to 0$. (R, H, ε are respectively: the parameter of scale, the Hubble function and the energy density).

In a general case there is no such simple connection between the sign of curvature, the number of singular states and the limitation of time. For example, when $\Lambda > 0$ there exist closed models with an unlimited life-time. These models asymptotically behave in far future like a steady state universe. *If the life-time is unlimited we will call physical and geometrical features of the universe in the far future* (*the past*) *the future* (*past*) *asymptotic state.*

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Since the realistic equation of state for cosmic matter of extremely high density is still unknown [2, 6, 7] we will treat the Friedman equations as a system containing one arbitrary function, $p(\varepsilon, H)$.

A reasonable assumption will be investigated: all possible evolutions of the universe preserve $\varepsilon+p \geqslant 0$ for all times. We will notice the interesting fact, that in contrast to the exact solutions, the existence and character of asymptotic states, as well as the existence of singularities in Hawking's, Penrose's and Geroch's works, depend on energy conditions but not on the actual form of pressure.

1. Energy conditions

As shown by Hawking [2], Penrose and Geroch the energy conditions play an important role in singularity theorems. A weak energy condition holds at point P if the energy density is non-negative at P in all Lorentzian frames of reference ($T^0_0 \geqslant 0$). A dominant energy condition holds if the energy density dominates other components of energy momentum tensor $T^0_0 \geqslant |T^\mu_\nu|$ in all frames of reference. Strong energy condition requires that $R^0_0 \geqslant 0$ in all frames of reference.

Let us consider the universe filled with matter described by energy momentum tensor

$$T^\mu_\nu = \mathrm{diag}\,(\varepsilon, -p, -p, -p)$$

in a global comoving system of reference. Using elementary calculations one can prove the following: *if* $\varepsilon+p \geqslant 0$ *then each of the following inequalities is Lorentz-invariant*:

$$T^0_0 \geqslant 0, \qquad T^0_0 \geqslant |T^\mu_\nu|, \qquad R^0_0 \geqslant 0.$$

This means that if $\varepsilon+p \geqslant 0$ holds and if there is a frame of reference in which $T^0_0 \geqslant 0$ ($T^0_0 \geqslant |T^\mu_\nu|$, $R^0_0 \geqslant 0$) then $T^0_0 \geqslant 0$ ($T^0_0 \geqslant |T^\mu_\nu|$, $R^0_0 \geqslant 0$) holds in every frame of reference.

In a comoving system of reference energy conditions may be written in the following way:

$$\left.\begin{array}{ll} \text{weak} & \varepsilon \geqslant 0 \\ \text{dominant} & \varepsilon \geqslant |p| \\ \text{strong} & \varepsilon+3p-2\Lambda \geqslant 0 \end{array}\right\} \begin{array}{l}\text{plus condition of Lorentz} \\ \text{invariability } \varepsilon+p \geqslant 0.\end{array}$$

2. Friedman equations using the phase variable method

Using the Hubble function, $H = \dot{R}/R$, one can transform Friedman equations:

$$2\frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} - \Lambda = -p$$

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} - \frac{\Lambda}{3} = \frac{\varepsilon}{3} \tag{1}$$

to the dynamical form i.e., to the system of equations of the type $\frac{dx^k}{dt} = f^k(x^i)$ where f is time-independent [1], [4].

$$\dot{H} = -H^2 - \tfrac{1}{6}(\varepsilon + 3p - 2\Lambda),$$
$$\dot{\varepsilon} = -3H(\varepsilon + p). \tag{2}$$

The first of these equations can be recognized as the simple form of the Raychaudhuri equation for the congruence of the world lines of matter [2]. The second equation is equivalent to the differential law of conservation of energy-momentum. ε, H are chosen as phase variables. p is the generally unknown function of ε, H: $p = p(\varepsilon, H)$. Dependence of p on H signifies the existence of viscosity [7]. The evolution of the universe is described by a trajectory in the phase space and uniquely determined by the initial values of ε and H.

Let us now introduce the following notation: λ — set of all points of the phase plane at which Lorentz invariability of the energy conditions holds ($\varepsilon + p \geqslant 0$), S — set of all points of the phase plane at which: $\varepsilon + 3p - 2\Lambda \geqslant 0$. By simple calculations we obtain:

1. static ($H = 0$) critical points of system 2 are determined by the intersection of the ε-axis and the boundary, ∂S, of the S set;
   {static critical points} = {ε-axis} $\cap\, \partial S$,
2. nonstatic critical points of system 2 are determined by the intersection of the trajectory of the flat Friedman universe and the boundary of λ:
   {unstatic critical points} = {trajectory $k = 0$} $\cap\, \partial\lambda$.

As we know from the theory of dynamical systems every critical point corresponds to an asymptotic state of the universe.

Assumption 1: *The phase space does not contain trajectories along which Lorentz invariability of the energy conditions is broken in a finite value of time.*

In other words, according to the above assumption, there is no such situation that the universe starts with very regular initial conditions (for example, it fulfills all energy conditions — the energy density is non-negative, there is no heat transport faster than light, one geodesic attracts another in all frames of reference) and then in finite time something goes wrong and all these more or less realistic conditions are violated at a certain moment. Assumption 1 is not satisfied within the class of cosmological models with constant bulk viscosity [5].

From the theory of dynamical systems the above assumption is equivalent to the requirement that $\partial\lambda$ forms a trajectory in the phase space. Since $\dot{\varepsilon} = 0$ on $\partial\lambda$:

1. $\partial\lambda$ forms a trajectory $\leftrightarrow \varepsilon = \text{const}$ along $\partial\lambda$;
2. $dp/dH = 0$ on the trajectory $\partial\lambda$. Bulk viscosity is not important near $\partial\lambda$.

To determine the stability character of the critical points we have to linearize the left side of system 2

$$\begin{pmatrix} \dot{H} \\ \dot{\varepsilon} \end{pmatrix} = \begin{pmatrix} -2H_0 - \frac{1}{2}\left.\frac{\partial p}{\partial H}\right|_0 & -\frac{1}{6}\left.\frac{\partial}{\partial \varepsilon}\right|_0 (\varepsilon + 3p - 2\Lambda) \\ -3(\varepsilon+p)_0 - 3H_0 \left.\frac{\partial}{\partial H}\right|_0 (\varepsilon + p) & -3H_0 \left.\frac{\partial}{\partial \varepsilon}\right|_0 (\varepsilon + p) \end{pmatrix} \begin{pmatrix} H - H_0 \\ \varepsilon - \varepsilon_0 \end{pmatrix}, \tag{3}$$

where "0" denotes the value of the expression in the critical point. We obtain:

for static points

$$\det A = -\tfrac{1}{2}(\varepsilon+p)_0 \left.\frac{\partial}{\partial\varepsilon}\right|_0 (\varepsilon+3p-2\Lambda),$$

$$\mathrm{Tr}\, A = \tfrac{3}{2}\zeta_0,$$

for nonstatic points

$$\det A = 6H_0^2 \left.\frac{\partial}{\partial\varepsilon}\right|_0 (\varepsilon+p),$$

$$\mathrm{Tr}\, A = -H_0\left[2+3\left.\frac{\partial}{\partial\varepsilon}\right|_0 (\varepsilon+p)\right], \tag{4}$$

where A denotes the matrix of the linearised system 3, ζ denotes the coefficient of the bulk viscosity: $\zeta = -\dfrac{1}{3}\dfrac{\partial p}{\partial H}$.

In this way we have shown the following: *If assumption 1 holds and if* $\det A \neq 0$, *then the character of the critical points is predicted by the following rules*:

STATIC POINTS:

	saddle	stable knot/spiral	unstable knot/spiral
$\mathrm{sign} \left.\frac{\partial}{\partial\varepsilon}\right\|_0 (\varepsilon+3p-2\lambda)$	−	−	−
$\mathrm{sign}\, \zeta_0$		−	+

NONSTATIC POINTS:

	saddle	stable knot/spiral	unstable knot/spiral
$\mathrm{sign} \left.\frac{\partial}{\partial\varepsilon}\right\|_0 (\varepsilon+p)$	−	+	+
$\mathrm{sign}\, \mathrm{H}_0$		+	−

The type of the critical points, if it is saddle or a knot (a spiral), depends on whether the S and λ conditions hold "above" or "below" the boundary ∂S and $\partial\lambda$ respectively, on the phase plane. The stability of the knots (the spirals) depends on expansion or viscosity. Graphically this can be seen in Fig. 1.

The distinction between knots and spirals is generally not important because of the identity of the asymptotic states with which they correspond. However, one can easily check the fact that according to assumption 1 nonstatic critical points cannot be spirals.

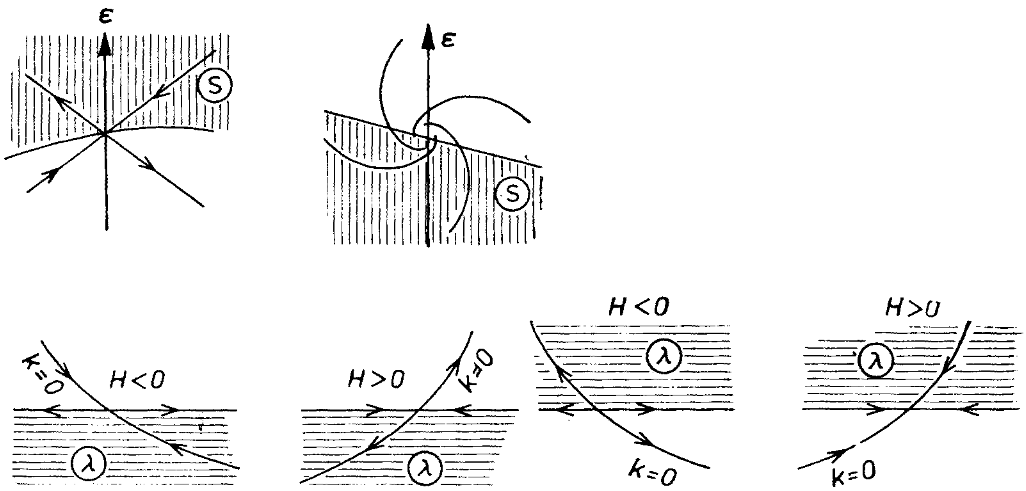

Fig. 1

2. *Some applications*

To illustrate now how the rules obtained in the previous section apply let us consider some particular examples.

2.1. The universe with non-vanishing cosmological constant filled with dust matter

In case $\Lambda > 0$ (Fig. 2a) we can identify the critical points with the well known particular solutions of Einstein equations:

— the stable knot ($\varepsilon = 0, H = \sqrt{\Lambda/3}$) — the expanding de Sitter model,

— the unstable knot ($\varepsilon = 0, H = -\sqrt{\Lambda/3}$) — the collapsing de Sitter model,

— the saddle point ($\varepsilon = 2\Lambda, H = 0$) — the static Einstein universe.

All solutions in the half-plane $\varepsilon \geqslant 0$ can be divided into four classes:

a) open models — with one regular and one singular end-point of the trajectory,
b) closed models satisfying strong energy conditions which have two singular end-points (singular oscillating models),
c) closed models violating strong energy conditions with one singular and one regular end-point,
d) closed non-singular models.

In the case $\Lambda = 0$ (Fig. 2b) all critical points converge to the point ($\varepsilon = 0$, $H = 0$). Closed models are double-singular, those unclosed are single-singular. In the case $\Lambda < 0$ (Fig. 2c) all models are double-singular. Critical points do not exist. All these results hold for matter with positive pressure.

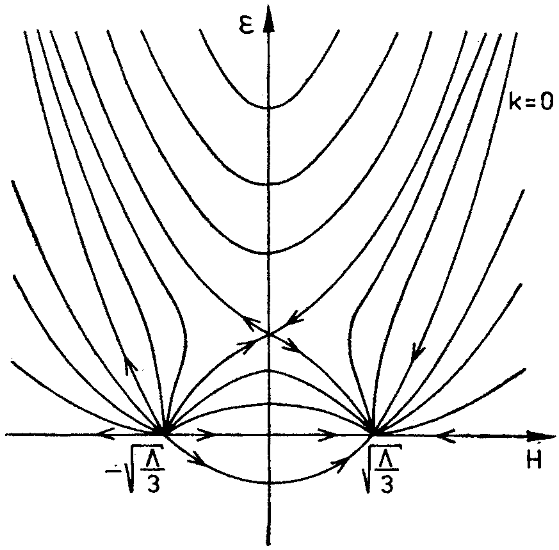

Fig. 2a

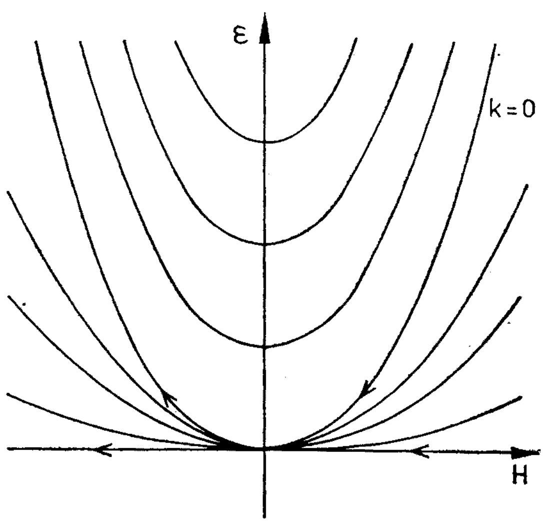

Fig. 2b

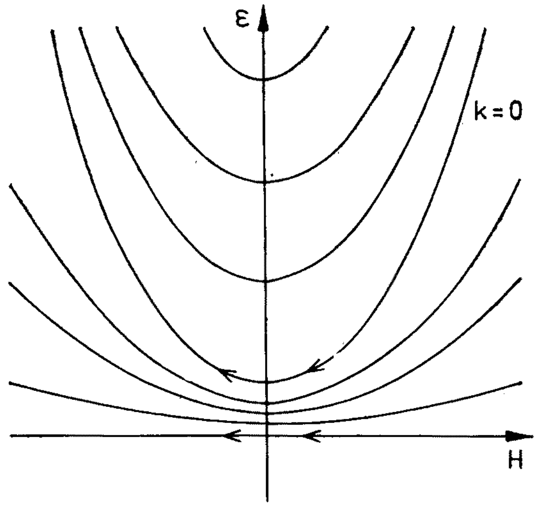

Fig. 2c

2.2. Matter filled universe with the equation of state $p = p(\varepsilon)$ where p is generally non-positive and the vanishing cosmological constant

In this case the strong energy condition may be violated because of negative pressure. If the condition $\varepsilon + p \geqslant 0$ holds for all values of $\varepsilon \geqslant 0$ some possible diagrams are presented in Fig. 3.

Example 1. The strong energy condition $\varepsilon + 3p \geqslant 0$ holds for all values of $\varepsilon \geqslant 0$. Qualitatively diagram (Fig. 3a) appears like the diagram for a universe with positive pressure.

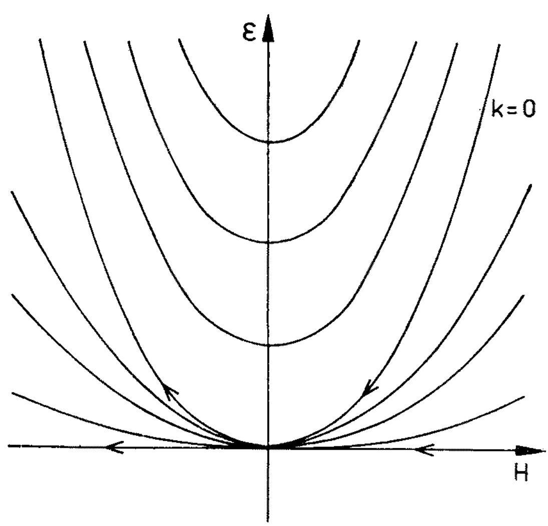

Fig. 3a

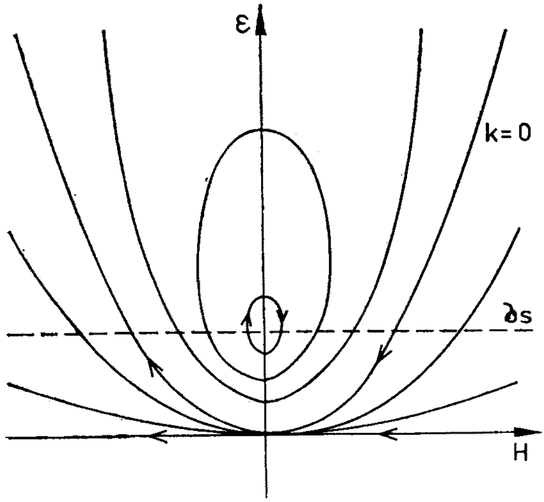

Fig. 3b

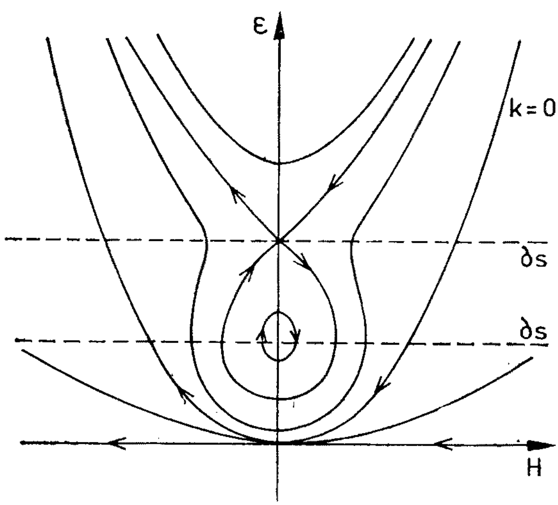

Fig. 3c

Example 2. A strong energy condition holds only for $0 \leqslant \varepsilon \leqslant \varepsilon_{max}$ (Fig. 3b). Every universe violates the strong energy condition. Double-singular models may not exist. Instead of this closed trajectories appear (cyclic non-singular closed models). The critical point represents the Einstein static universe like in a diagram for the dust matter, but from a stability point of view it has different features. Near this point trajectories form circles. The point itself may be understood as a stable point in the sense that the infinitesimal perturbations convert it into a circle trajectory.

Example 3. (Fig. 3c) The strong energy condition holds for $\varepsilon: 0 \leqslant \varepsilon \leqslant \varepsilon_1$ and $\varepsilon_2 \leqslant \varepsilon$. The second critical point (saddle) is also the static Einstein universe. Among closed models single-singular and non-singular, non-cyclic models appear, but they form a subset of measure zero.

Example 4. Suppose finally that $\varepsilon+p \geqslant 0$ only for $\varepsilon_{min} \leqslant \varepsilon \leqslant \varepsilon_{max}$ (Fig. 4). On the strength of assumption 1 our universe is described by one of the trajectories lying between ε_{min} and ε_{max} and no trajectory intersects $\partial\lambda$ in a finite value of time. Every model violates

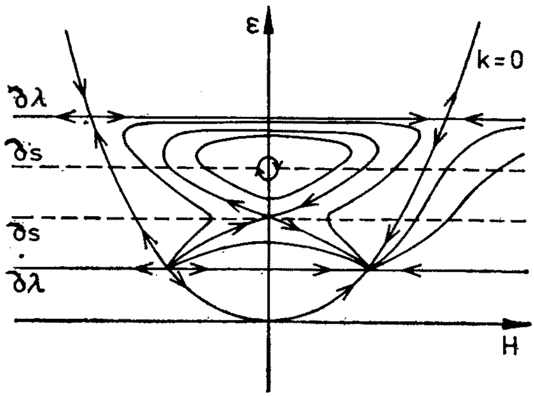

Fig. 4

the strong energy condition. Other energy conditions may be satisfied at all times. Only open models with the negative parameter of the curvature ($k = -1$) are singular, but in the singular point the energy density is finite. Similar diagrams such as in the previous examples (Fig. 3) can be obtained from the last one (Fig. 4) by changing $\varepsilon_{min} \to 0$, $\varepsilon_{max} \to \infty$.

2.3. The universe with pressure and viscosity

Viscous matter is generally described by state equation, $p = p(\varepsilon, H) \cdot \zeta = -\dfrac{1}{3}\dfrac{\partial p}{\partial H}$ is called the bulk viscosity coefficient. Assumption 1 implies that only the stability character of the static Einstein universe depends on ζ. Closed trajectories change into spirals which are stable for the negative viscosity and unstable for the positive viscosity. Viscous models have been widely discussed by Klimek, Heller and Suszycki [7].

3. Strong energy condition in phase plane

As has been pointed out the violation of the strong energy condition may be possible even in the case of realistic energy momentum tensor. A field of massive particles may violate the strong energy condition when the curvature radius, R, is less than 10^{-12} cm [2]. Hence, the requirement for fulfilling that condition during the entire evolution of the universe might be too strong. However, the strong energy condition plays an important role in the singularity theorems and it is interesting to see what structure the phase space has, when that condition holds.

Assumption 2. a) Assumption 1 holds. b) Condition $\varepsilon+3p-2\Lambda \geqslant 0$ cannot be violated along any trajectory in a finite time.

Assumption 2b is equivalent to the requirement that ∂S forms a trajectory in the phase space. Since on ∂S:

$$\dot{H} = -H^2, \quad \dot{\varepsilon} = -2(\varepsilon+\Lambda)H$$

the trajectory ∂S must be described by the equation having the form:

$$\varepsilon+\Lambda = cH^2,$$

where c is an arbitrary constant.

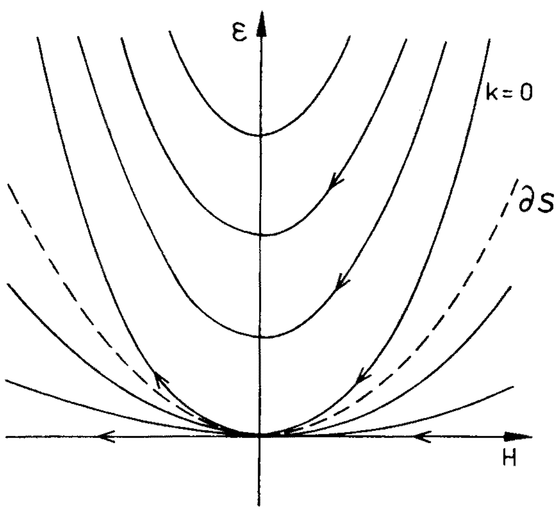

Fig. 5

For $\Lambda = 0$ assumption 2 implies a very simple structure of the phase plane (Fig. 5) with one critical point at $\varepsilon = 0$, $H = 0$. One can find that along the trajectory, ∂S, viscosity ζ is proportional to the square root of the energy density: $\xi \sim \sqrt{\varepsilon}$. Hence, the phase plane (with $\Lambda = 0$), in which the strong energy condition holds, favours the viscosity function of Bielinsky and Khalatnikov $\xi \sim \varepsilon^m$ where $m = 1/2$ [4].

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REFERENCES

[1] W. I. Arnold, *Równania różniczkowe zwyczajne*, PWN, Warszawa 1975, in Polish.
[2] S. W. Hawking, G. F. R. Ellis, *The Large Scale Structure of Space-Time*, Cambridge University Press 1973.
[3] S. Weinberg, *Gravitation and Cosmology*, John Wiley and Sons, Inc. 1972.
[4] V. A. Bielinsky, I. M. Khalatnikov, *Zh. Eksp. Teor. Fiz.* **69**, 401 (1975).

[5] M. Heller, *Acta Cosmologica* 7, 7 (1978); L. Suszycki, *Acta Cosmologica* 7, 147 (1978).
[6] V. Canuto, J. Lodenquai, *High Density Matter in the Universe in Many Degrees of Freedom in Particle Theory*, Plenum Press 1978.
[7] M. Heller, Z. Klimek, L. Suszycki, *Astrophys. Space Sci.* **12**, 205 (1973); M. Heller, Z. Klimek, *Astrophys. Space Sci.* **33**, L37 (1975); Z. Klimek, *Post. Astron.* **19**, 165 (1971) in Polish; Z. Klimek, *Acta Astron.* **25**, 79 (1975); Z. Klimek, unpublished, 1975.