# NONLINEAR VISCOUS COSMOLOGY

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We treat the galactic fluid, in the early epochs of the Universe, as a nonlinear Stokesian fluid. Some general comments on fluids with viscosity are made. Then we examine an exact solution and give a qualitative analysis for a Bianchi type-I cosmological model generated by a fluid whose viscosity depends quadratically on the deformation tensor.

## 1. Introduction

In the ambitious program to create a coherence global description of the Universe, cosmologists have been led to the analysis of some idealized configurations of the Cosmos. Starting with the assumption that Einstein's General Relativity is actually the better theory of gravity, the essential task is to present a coupled system constituted by a given distribution of the energy content in the Universe and the corresponding geometrical structure of the space-time. Due to its high degree of complexity one can achieve a good answer to this problem only through successive idealized schemes.

The most acceptable cosmological model at present is Friedmann's homogeneous and isotropic expanding Universe. This model assumes a continuous fluid description for the galactic matter plus radiation. Actually, the great majority of cosmical models deal with a perfect fluid behaviour. This behaviour has the great advantage of simplicity and besides, it seems to be in good agreement with current experimental cosmical observations [1].

Recently, however, the idea that Cosmology can go beyond the investigation of our present equilibrium era [1] has led some authors [2–5] to try to incorporate dissipative

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terms in the energy-momentum tensor of the galactic fluid. Although such a study is still at its beginning it seems conceivable that viscosity effects will play an important role in Cosmology.

In 1968 Misner [2] suggested that neutrino viscosity could be an efficient mechanism by means of which arbitrary initial anisotropy dies away rapidly as the Universe expands.

A simple phenomenological description of such a process is given by the Cauchy linear expression which relates the anisotropic pressure $\pi^{\mu}{}_{\nu}$ with the shear deformation tensor $\sigma^{\mu}{}_{\nu}$. The second viscosity coefficient has been used by Klimek [3] and later by Murphy [4] in order to create a homogeneous and isotropic cosmological model without singularity. The effect of the first and second viscosity coefficients on the cosmological singularity have been qualitatively investigated by Belinski and Khalatnikov [5]. Grischuck has considered the possibility of describing the particle creation mechanism as a viscous effect in a non-stationary Universe. The purpose of the present work is to initiate a program of systematic analysis of viscous cosmological models of a more general type than those which have been examined so far. In section 2 we review the description of a fluid by Stokes definition and introduce the general principle of viscosity. Such a principle relates the kinematical and the dynamical parameters of the fluid.

In section 3 we develop a specific example of a nonlinear viscous model and its influence on the metric properties. We analyse an exact solution and the qualitative features of a quadratic Stokesian fluid. We end with section 4 in which some general comments on nonlinear viscous cosmological models are set up.

## 2. *Stokes fluidity*

The energy-momentum tensor of a viscous fluid, without heat conduction, is given by

$$T_{\mu\nu} = \varrho V_{\mu} V_{\nu} - p h_{\mu\nu} + \pi_{\mu\nu}, \tag{1}$$

where $h_{\mu\nu} \equiv g_{\mu\nu} - V_{\mu} V_{\nu}$ is the projector on the 3-dimensional rest-frame of the observer co-moving with the fluid velocity $V_{\mu}$ [6]; $\pi_{\mu\nu}$ is the anisotropic pressure responsible for viscosity effects. The tensor $\pi_{\mu\nu}$ is symmetric, trace-free and orthogonal to $V^{\mu}$. In the analysis of the evanescence of some eventual primordial anisotropy some authors [2] made the hypothesis that the anisotropic pressure should be approximated by a linear relation to the dilatation tensor $\theta^{\alpha}{}_{\beta}$. This hypothesis is a special case of a more general fluidity principle which has been set up by Stokes. This principle is based on the fundamental assumption which states that the fluid dynamic quantities (anisotropic pressure, heat conduction) are related to the kinematical ones (dilatation, vorticity). Such a relation, which is the support of any phenomenological description of the fluid behaviour, will be called the Principle of Generalized Viscosity. The first typical simple example of such a principle is given by the Stokes fluidity definition [8], which states that the stress tensor of a fluid is a continuous function of the dilatation tensor $\theta^{\mu}{}_{\nu}$. By taking the time-like velocity vector $V^{\mu} = \delta^{\mu}{}_{0}$ we can write

$$\pi^{i}{}_{j} = \phi_0 \delta^{i}{}_{j} + \phi_1 \theta^{i}{}_{j} + \phi_2 \theta^{i}{}_{k} \theta^{k}{}_{j} \tag{2}$$

(latin indices vary in the domain {1, 2, 3}. Greek indices run through 0, 1, 2, 3) in which $\phi_0$, $\phi_1$ and $\phi_2$ are polynomials in the principal invariants of the matrix $\theta^i{}_j$, that is, the scalars $I$, $II$ and $III$ given by

$$I \equiv \theta^i{}_i = \theta, \tag{3a}$$

$$II = \tfrac{1}{2}(\theta^2 - \theta^i{}_j\theta^j{}_i), \tag{3b}$$

$$III = \tfrac{1}{6}\,\theta^i{}_h\theta^j{}_l\theta^k{}_m\varepsilon_{ijk}\varepsilon^{hlm}, \tag{3c}$$

in which $\varepsilon_{ijk}$ is the completely anti-symmetric Levi–Civita symbol.

As a second example we consider a non-Stokesian fluid in which the stress tensor is a function of the vortex matrix $\Omega^i{}_j \equiv \omega^i\omega_j - \dfrac{\omega^2}{3}\,h^i{}_j$ constructed with the vorticity vector $\omega_i$. This vector is related to the vorticity tensor $\omega_{\mu\nu} = \tfrac{1}{2}\,h_{[\mu}{}^{\lambda}h_{\nu]}{}^{\varepsilon}V_{\lambda/\varepsilon}$ through the expression

$$\omega^i = -\tfrac{1}{2}\,\frac{1}{\sqrt{-g}}\,\varepsilon^{ijk}\omega_{jk}.$$

A single bar means the simple derivative, a double bar means covariant derivative; the symbol [ ] means anti-symmetrization.

In this case we can write, for instance

$$\pi^i{}_j = \alpha\Omega^i{}_j. \tag{4}$$

Let us point out here that, as it has been remarked previously by Sommerfeld [7] and others, the dependence of the stress on the vorticity is possible only in a quadratic regime, at least.

In each of these cases, the presence of viscosity induces new properties which are worthy of investigation. Among these, the question of entropy non-conservation induced by a non-null characteristic function $\phi = \pi^i{}_j\theta^j{}_i$ deserves special interest. From conservation of the stress energy-tensor projected in the rest-frame of the observer $V^\mu$ we obtain [6]

$$\dot{\varrho} + (\varrho + p)\theta - \phi = 0 \tag{5}$$

in which a dot means derivative in the $V^\mu$-direction; the generalized Navier–Stokes equation is given by:

$$(\varrho + p)\dot{V}^\alpha - p_{|\mu}h^\mu{}_\alpha + \pi^{\mu\nu}{}_{||\nu}h_\mu{}^\alpha = 0. \tag{6}$$

The characteristic function $\phi$ is a measure of the time-variation of the entropy (as one can obtain from (5)) and sets some restrictions on the acceptable values of the polynomials $\phi_0$, $\phi_1$ and $\phi_2$ through the conditions imposed on it by the second law of Thermodynamics. Thus, $\phi$ must either vanish (entropy conservation) or be positive (see Table I).

The above phenomenological equations are set up in the belief that they can give us a better understanding of how anisotropy could die away as the Universe expands. The influence of the anisotropic pressure on the evolution of the shear can be investigated by the equations of motion of the Kinematical quantities and knowledge of the electrical

TABLE

| $\pi^i{}_j$ | $\Phi = \pi^i{}_j\theta^j{}_i$ | Constraints |
|---|---|---|
| $\alpha\theta^i{}_j+\beta\theta\delta^i{}_j$ ($\alpha$, $\beta$ are constants) | $(\alpha+\beta)\theta^2-2\alpha II$ | $\alpha+3\ \beta = 0$ |
| $a\theta^2\delta^i{}_j+b\theta\theta^i{}_j+c\theta^i{}_k\theta^k{}_j$ ($a, b, c$ are constants) | $(a+b+c)\theta^3-(2b+3c)\theta II+3cIII$ | $(3a+b+c)\theta^2-2cII = 0$ |
| $f\delta^i{}_j+g\theta^i{}_j+h\theta^i{}_k\theta^k{}_j$ ($f, g, h$ are polynamials in $I$, $II$ and $III$) | $f\theta+g(\theta^2-2II)+h(\theta^3+3III-3\theta II)$ | $3f+g\theta+h(\theta^2-2II) = 0$ |
| $\lambda\omega^i{}_j+\eta\omega^2\delta^i{}_j$ ($\lambda, \eta$ constants) | $\lambda\theta^{ij}\omega_i\omega_j+\eta\theta\omega^2$ | $\lambda+3\eta = 0$ |

One can extract the conditions imposed by the second law of Thermodynamics by simple examination of this Table. For instance, one obtains if $\alpha > 0$, $\theta^2 > 3\ II$ or if $\alpha < 0$, $\theta^2 < 3\ II$; for a nonexpanding fluid we have $\frac{g}{h} \leqslant \frac{3}{2}\frac{III}{II}$; if the second invariant vanishes, we must have $\frac{b}{h} < \frac{2}{3}\frac{\theta}{III}$ and so on.

part $E_{\mu\nu} = -C_{\mu\alpha\nu\beta}V^\alpha V^\beta$ of the Weyl tensor $C_{\alpha\beta\mu\nu}$ as it has been discussed previously by some authors [6].

If we call $\mu \equiv \sigma^i{}_j\sigma^j{}_i$, in the case in which there is neither acceleration nor rotation, the equation of evolution of shear reduces to

$$\dot{\mu}+\tfrac{4}{3}\,\theta\mu = -6\delta III+E^i{}_j\sigma^j{}_i \tag{7}$$

in which we have set $\pi^i{}_j = (1-\delta)(\sigma^i{}_k\sigma^k{}_j-\frac{1}{3}\,\sigma_{lm}\sigma^{lm}h^i{}_j)$ in which $\delta$ is a constant. Thus, the vanishing of $\mu$, as time goes on, depends crucially on the determinant of the dilatation matrix and on the Electric tensor $E^i{}_j$. These quantities should be analysed in each specific model, once very little can be said in generic terms. Thus, we cannot say that the above mechanism is efficient enough to give an explanation of the high value of isotropy of our Universe for an arbitrary initial configuration.

### 3. *Einstein's equations for Bianchi type-I and the quadratic Stokesian regime*

In order to gain some insight on the influence of the nonlinear Stokesian regime on the behaviour of the gravitational field, we will examine here a specific geometry which represents a homogeneous but anisotropic expanding Universe of type-I in the Bianchi classification.

The infinitesimal element of lenght is given by

$$ds^2 = dt^2-a^2(t)dx^2-b^2(t)dy^2-c^2(t)dz^2. \tag{8}$$

Choosing a co-moving observer with the fluid velocity, $V^\mu = \delta^\mu{}_0$, the non-null components of the stress energy tensor are the following:

$$T^0{}_0 = \varrho, \quad T^i{}_j = -p\delta^i{}_j + \alpha\theta^2\delta^i{}_j + \beta \cdot \theta\theta^i{}_j + \gamma\theta^i{}_k\theta^k{}_j, \tag{9}$$

where $\alpha$, $\beta$ and $\gamma$ are, for the time being, arbitrary constants.

Before proceeding in this analysis, it seems worthwhile to make the following remark. In the discussion of viscosity, cosmologists have limited their analysis to the linearized case of Stokes fluid. Although it could be a difficult task to elaborate models by means of which one could evaluate the value of the generalized second-order coefficients of viscosity, there is no a priori reason to reject its presence. Further, in the region near the singularity the nonlinear regime could dominate and eventually give a better approximation of the cosmic fluid.

The entrance of the Universe in a full nonlinear era, in Stokes expansion, has the effect of changing radically the early features of the Cosmos. For instance, it has been argued by many cosmologists that in the early epochs of the Universe matter should be gravitationally unimportant. Thus, going back in time one should enter a region (called vacuum stage) where gravitation is sustained by itself. If this is true, then the properties of the singularity in such models are independent of matter behavior. This has been proved by Lifshitz et al. for an equilibrium era in which matter was treated as a perfect fluid. However, it is easy to show that such vacuum stage does not exist in a quadratic regime for the Stokes fluid. The reason for this is simple: the matter terms are no more negligible, in Einstein's equations, by comparison with pure gravity terms, e.g. the Ricci tensor, as we will see next.

The dilatation tensor $\theta^i{}_j$ is diagonal. The unique non-null terms are $\theta^1{}_1 = \dot{a}/a$, $\theta^2{}_2 = \dot{b}/b$, $\theta^3{}_3 = \dot{c}/c$. Einstein's equations are given by:

$$\frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{\dot{a}}{a}\frac{\dot{c}}{c} + \frac{\dot{b}}{b}\frac{\dot{c}}{c} = \varrho, \tag{10a}$$

$$\frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} + \frac{\dot{b}}{b}\frac{\dot{c}}{c} = -p + \alpha\theta^2 + \beta\theta\frac{\dot{a}}{a} + \gamma\left(\frac{\dot{a}}{a}\right)^2, \tag{10b}$$

$$\frac{\ddot{a}}{a} + \frac{\ddot{c}}{c} + \frac{\dot{a}}{a}\frac{\dot{c}}{c} = -p + \alpha\theta^2 + \beta\theta\frac{\dot{b}}{b} + \gamma\left(\frac{\dot{b}}{b}\right)^2, \tag{10c}$$

$$\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\dot{a}}{a}\frac{\dot{b}}{b} = -p + \alpha\theta^2 + \beta\theta\frac{\dot{c}}{c} + \gamma\left(\frac{\dot{c}}{c}\right)^2, \tag{10d}$$

The parameters $\alpha$, $\beta$ and $\gamma$ must satisfy the relation for constraints:

$$(3\alpha+\beta)\theta^2 + \gamma\left[\left(\frac{\dot{a}}{a}\right)^2 + \left(\frac{\dot{b}}{b}\right)^2 + \left(\frac{\dot{c}}{c}\right)^2\right] = 0. \tag{11}$$

From the above set of equations, we may infer that in the case of a power law dependence of the anisotropic functions $a$, $b$ and $c$ on time, there is no possibility of a vacuum stage.

We will now discuss these equations from two very different points of view. Here, we give a special exact solution of these equations in order to have some feeling for their properties, and later we will turn to a qualitative analysis of equations (10) using the standard techniques of qualitative investigation of ordinary nonlinear differential equations.

Let us set

$$a(t) = e^{\mu t} t^{A}, \qquad b(t) = e^{\nu t} t^{B}, \qquad c(t) = e^{\eta t} t^{C}, \tag{12}$$

in which $\mu$, $\nu$, $\eta$ and $A$, $B$, $C$ are constants.

Using the ansatz (12) in equations (10) we obtain

$$\mu\nu+\mu\eta+\nu\eta+[A(\nu+\eta)+B(\mu+\eta)+C(\mu+\nu)]t^{-1}+(AB+AC+BC)t^{-2} = \varrho, \tag{13a}$$

$$\nu^2+\eta^2+\nu\eta+[2\nu B+2\eta C+\eta B+\nu C]t^{-1}+[B(B-1)+C(C-1)+BC]t^{-2} = -p+\alpha\Phi^2$$
$$+\beta\Phi\mu+\gamma\mu^2+[2\alpha\Phi\Omega+\beta\mu\Omega+\beta\Phi A+2\mu\gamma A]t^{-1}+[\alpha\Omega^2+\beta\Omega A+\gamma A^2]t^{-2}, \tag{13b}$$

$$\nu^2+\mu^2+\nu\mu+[2\nu B+2\mu A+\mu B+\nu A]t^{-1}+[A(A-1)+B(B-1)+AB]t^{-2} = -p+\alpha\Phi^2$$
$$+\beta\Phi\eta+\gamma\eta^2+[2\alpha\Phi\Omega+\beta\eta\Omega+\beta\Phi C+2\eta\gamma C]t^{-1}+[\alpha\Omega^2+\beta\Omega C+\gamma C^2]t^{-2}, \tag{13c}$$

$$\eta^2+\mu^2+\eta\mu+[2\mu A+2\eta C+\eta A+\mu C]t^{-1}+[A(A-1)+C(C-1)+AC]t^{-2} = -p+\alpha\Phi^2$$
$$+\beta\Phi\nu+\gamma\nu^2+[2\alpha\Phi\Omega+\beta\Omega\nu+\beta\Phi B+2\nu\gamma B]t^{-1}+[\alpha\Omega^2+\beta\Omega B+\gamma B^2]t^{-2}, \tag{13d}$$

in which we have set $\theta = \Phi+\Omega t^{-1}$; that is $\Phi = \mu+\nu+\eta$ and $\Omega = A+B+C$.

Let us take the very special situation in which the coefficients of viscosity are such that

$$3\alpha+\beta = 0, \tag{14a}$$

$$\gamma = 0. \tag{14b}$$

Then we look for a solution which has $\Phi = 0$ and $A = B = C$, in order to obtain the simplest generalization of Friedmann's Cosmos with Euclidean section. We set:

$$\varrho = \varrho_0+\varrho_1 t^{-1}+\varrho_2 t^{-2}, \tag{15a}$$

$$p = p_0+p_1 t^{-1}+p_2 t^{-2}, \tag{15b}$$

where $\varrho_0$, $\varrho_1$, $\varrho_2$, $p_0$, $p_1$, $p_2$ are constants.

Then a straightforward calculation gives, for the compatibility of equations (13) with our ansatz (12) the values

$$p = \varrho, \qquad \alpha = 1/3, \qquad \beta = -1, \tag{16}$$

$A = 1/3$, $\varrho_0 = -\mu^2-\nu^2-\nu\mu$, $\varrho_1 = 0$, $\varrho_2 = 1/3$.

Let us make some comments on this solution. The density is not strictly positive for all time $t$. Indeed, we have $\varrho = \varrho_0+1/3t^{-2}$. It is positive definite only for those times in the range $0 < t < \frac{1}{3}\dfrac{1}{|\varrho_0|} \equiv t_c$. The domain of positivity, measured by the value of $t_c$, depends on the anisotropy. Indeed, the smaller the anisotropy (that is, $\varrho_0$) the bigger $t_c$ is.

The unique non-vanishing components of the dilatation tensor are:

$$\theta_1^1 = \mu+\tfrac{1}{3}t^{-1}, \quad \theta_2^2 = \nu+\tfrac{1}{3}t^{-1}, \quad \theta_3^3 = \eta+\tfrac{1}{3}t^{-1}.$$

Finally, let us remark that the characteristic function equals $2\varrho_0 t^{-1}$ and thus is a negative increasing function of time. The Universe evolves in the direction of decreasing time, in accordance with the second law of Thermodynamics. We face here the same situation as encountered in [5], of break of invariance with respect to time inversion, due to the effects of viscosity.

Let us turn now to a difference approach [9] of the investigation of the set of equations (10), which consists in the reduction of equations (10) into an autonomous planar system. This technique has been used many times in the literature in similar situations [5, 10].

In order to simplify our analysis we will limit the anisotropy to a plane assuming $b = c$, for instance. Then we define two new variables $U$ and $V$ by the relations:

$$U = \frac{\dot{a}}{a}, \tag{17a}$$

$$V = \frac{\dot{b}}{b}. \tag{17b}$$

In the new variables, the system of Einsten's equations (10) reduces to

$$2V(U+V) = \varrho, \tag{18a}$$

$$\dot{U} = -U^2+\tfrac{1}{2}V^2-UV-\frac{p}{2}+2L_{\mathrm{I}}(U, V), \tag{18b}$$

$$\dot{V} = -\tfrac{3}{2}V^2-\tfrac{1}{2}p-L_{\mathrm{I}}(U, V), \tag{18c}$$

in which

$$L_{\mathrm{I}} \equiv \alpha(U+2V)^2+\beta(U+2V)V+\gamma V^2.$$

We assume that there is an equation of state $p = \lambda\varrho$, where $\lambda$ is a constant. Let us simplify our discussion further by assuming that conditions (14) hold.

The existence of a plane of anisotropy has the effect of reducing our system of equations to the set (18) which defines an autonomous planar system of differential equations. Indeed we have

$$\dot{U} = R(U, V), \tag{19a}$$

$$\dot{V} = \tau(u, V), \tag{19b}$$

with

$$R(U, V) \equiv (2\alpha-1)U^2+\left[\frac{1-\lambda}{2}-4\alpha\right]V^2+(2\alpha-1-\lambda)UV, \tag{20a}$$

$$\tau(U, V) \equiv \left(2\alpha-\tfrac{3}{2}-\frac{\lambda}{2}\right)V^2-\alpha U^2-(\lambda+\alpha)UV. \tag{20b}$$

The origin, in the $(U, V)$ plane is an isolated singularity for the system (19). Both $R$ and $\tau$ are homogeneous functions, of degree two, on $U$ and $V$. Thus, in order to analyse the properties of the above system (19) it is convenient to change to polar coordinates $r, \phi$ defined by $U = r\cos\phi$, $V = r\sin\phi$. Thus, (19) becomes

$$\dot{r} = r^2 R[\phi], \tag{21a}$$

$$\dot{\phi} = rT[\phi], \tag{21b}$$

where $R[\phi]$ and $T([\phi]$ are given by

$$R[\phi] = \sin^3\phi\left(2\alpha - \tfrac{3}{2} - \frac{\lambda}{2}\right) + (2\alpha - 1)\cos^3\phi + (\alpha - \lambda - 1)\sin\phi\cos^2\phi$$

$$-[\tfrac{3}{2}\lambda + 5\alpha - \tfrac{1}{2}]\sin^2\phi\cos\phi, \tag{22a}$$

$$T[\phi] = -\alpha\cos^3\phi - \left[\frac{1-\lambda}{2} - 4\alpha\right]\sin^3\phi + \left(\frac{\lambda - 1}{2}\right)\cos\phi\sin^2\phi$$

$$-(3\alpha + \lambda - 1)\cos^2\phi\sin\phi. \tag{22b}$$

In the investigation of the behaviour of the trajectories of the system (21) in the $(r, \phi)$ plane the particular solutions which pass through the (singular) origin are determined by the real roots of the equation $T[\phi] = 0$. We examine two special cases.

Case $(i)$: $\lambda = 1 - 6\alpha$; case $(ii)$: $\lambda = 1 - 8\alpha$. First, let us consider case $(i)$. A simple inspection of the set (19) shows that there is a unique characteristic line which makes an angle of 45° with the $U$-axis. This line, whose equation $U = V$, is nothing but a Friedmann

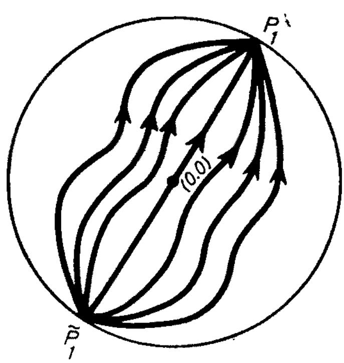


Fig. 1. Conformal representation of the characteristics of the plane autonomous system (19) in the special case $\lambda = 1 - 6\alpha$. The arrows point in the direction of increasing $t$

isotropic solution. The behaviour of the trajectories (of the solutions) are given in figure 1 in which we have made a conformal mapping in order to represent the infinite as the boundary of a circle. The curves go approximatively parallel to Friedmann's solution with a small (not catastrophic) attraction near the origin.

There are two singular points at infinity which are the contact points of the Friedmann solution with the boundary, represented by points $P_1$ and $\tilde{P}_1$ of figure 1. These two points are two-tangent nodes for the trajectories of our system. Thus, all solutions start at $\tilde{P}_1$ past infinity, and end at $P_1$, future infinity. The solution passes through a region in which the total energy is negative that is $U(2U+V) < 0$. However the solution does not rest at this region for long periods of time. Note that there is no possibility of interchanging the axis of expansion/contraction. Both axes begin contracting, pass through a region of minimum and then both start to expand. This general behaviour is stable for those perturbations which do not destroy the constraint relation between $\lambda$ and $\alpha$.

Let us turn now to discuss case (*ii*) $\lambda = 1-8\alpha$. The roots of $T[\phi]$ are given by the solution of $\cos\theta_0 = 0$; $\cos\theta_1 = \sin\theta_1$; $\cos\theta_2 = 4\sin\theta_2$. Contrary to the previous situation, here the behaviour of the trajectories depend not only on the relation between $\lambda$ and $\alpha$ but also on the value of $\alpha$ itself. From the systematics of qualitative analysis for a homogeneous system we know that the behaviour of the characteristics depends on the sign of $R[\phi]$ in the neighborhood of the invariant rays $\phi_0 = \theta_0$, $\phi_1 = \theta_1$, $\phi_2 = \theta_2$. This sign, as one sees from (22a) depends on the sign of the difference $\alpha - 7/24$. Now, from the natural

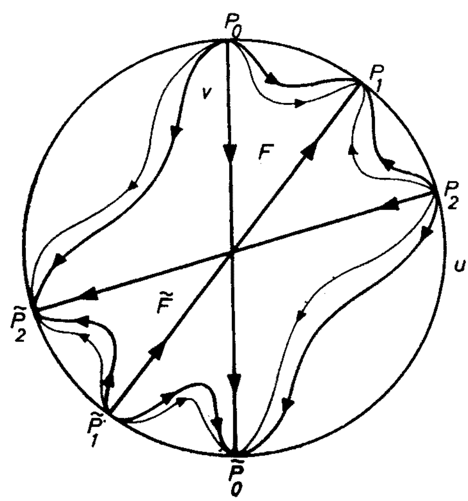


Fig. 2. Conformal representation of the characteristics of the plane autonomous system (19) in the special case $\lambda = 1-8\alpha$, $\alpha < \frac{1}{8}$. The arrows point in the direction of increasing $t$. *FF* is Friedmann's solution

limit on $\lambda$, we obtain $0 < \alpha < 1/8$. In figure 2 we represent the conformal mapping of infinity into a circle. The contact points of the axis $\phi_0 = \theta_0$, $\phi_1 = \theta_1$ and $\phi_2 = \theta_2$ with the circle are the singular points at infinity, which we represent by $P_0$, $P_1$, $P_2$ and its symmetrically related points $\tilde{P}_0$, $\tilde{P}_1$, $\tilde{P}_2$. All these points are two-tangent nodes. The trajectory $P_1$, $\tilde{P}_1$ represents the Friedmann solution. As trajectories in the planar system cannot be crossed, figure 2 tells us that only in the $P_0P_2$ and $\tilde{P}_0P_2$ regions the phenomenon of alternation role of the axis — expansion turned into contraction and/or vice-versa — can occur. Thus, in our case the expanding (contracting) plane $Y$-$z$ can turn into a contracting (expanding) era, although the $x$-axis cannot change the sign of its expansion (or contraction). Particularly interesting is the behaviour of the model under perturbations of the Friedmann solution. In the region inside the arc $\tilde{P}_0\tilde{P}_1\tilde{P}_2$, perturbations of the Friedmann solution are unstable; in the region $P_0P_1P_2$ perturbations of Friedmann solution are stable. Such time asymmetry of the behaviour of the above model Universe is a consequence of the viscosity effects, as has been pointed out previously by some authors [5].

We would like to call attention of the reader to the high degree of instability in (Friedmann) $\tilde{F}$-solution. A small departure from it may be responsible for the model to annihilate at $\tilde{P}_2$ or $\tilde{P}_0$. If $\tilde{F}$ solution happens to occur, then its trajectory up to the point 0 must not be perturbed in order for the solution to enter the $F$-region. Thus, it seems very unlikely that the previous era of our Cosmos, before the actual period of expansion, should share both properties, that is, the geometry to be of Friedmann type and the matter to behave like a quadratic Stokesian fluid.

## *4. Conclusion*

In this paper we have presented the basic idea to treat the viscosity of the galactic matter as a Stokesian nonlinear fluid. This could be a good description of the behaviour of the energy of the cosmic fluid in a highly compressed early epoch in which eventual large-scale anisotropy could be so important as to induce a nonlinear response.

In the preliminary analysis which we have presented here, the basic features of solutions of Einstein's equations under such a nonlinear Stokesian regime reveal a lot of new results. Among these, one can quote the absence of a vacuum stage and consequently the need to study the behaviour of matter coupled to geometry near the singularity; the very sensitive dependence of the stability of the Friedmann solution on the values of the quadratic coefficients of viscosity; and, finally, the non-symmetric behaviour of the Cosmos under time-inversion.

Finally, let us remark that, as it has been pointed out by some authors, e.g. Grishchuk [11], viscosity effects may be a consequence of the influence on the geometry of newly created particles by the non-stationary Cosmos. The relation of our present model and the created particle mechanism should be a matter for future investigation.

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[1] One could say that cosmologists must look for more general models which could give a justification for the present highly isotropic stage in which we live and possibly clarify our ideas about the behaviour of the Cosmos near the global singularity present in Friedmann models.

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