ANALYTIC SOLUTION OF THE QCD EVOLUTION EQUATION FOR THE NON-SINGLET STRUCTURE FUNCTIONS

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The solution of the leading log QCD evolution equation for the non-singlet evolution function is given in the form of a convergent series. The convergence is rapid for small values of x. An asymptotic expansion in powers of $(1-x)$ is also obtained. Its first few terms reproduce within about one per cent *all* the moments of the evolution function in the kinematical range of present and near future interest. Using simultaneously the two expansions it is easy to calculate structure functions in all the region $0 < x \leqslant 1$ with an accuracy of the order of one per cent.

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1. *Introduction*

The Q^2 evolution of nucleon structure functions is being both determined experimentally and calculated theoretically with increasing precision. In particular, the comparison with experiment of the predicted corrections to the dominant leading log (LL) behaviour of the structure functions has become an important testing ground for QCD. For recent reviews cf. e.g. Refs. [1] and [2]. In order to study the corrections, it is necessary to know well the leading LL term. One possibility, advocated for instance in Ref. [2], is to consider moments of the structure functions only. For the Q^2 evolution of moments the LL approximation yields simple analytic formulae. Experimentally, however, one measures directly not the moments, but the structure functions — often in a kinematical region too small to calculate moments without a significant loss in accuracy. Therefore, approximate formulae (cf. e.g. [3]) and numerical tables [4] have been provided to calculate the Q^2 evolution

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of structure functions. Several numerical methods for such calculations have also been published [4–8].

In this paper we give an analytic solution in form of a convergent series for the LL evolution of the non-singlet structure functions. We also introduce an asymptotic expansion, which should be useful for phenomenology, because of its simplicity and of the precision with which it reproduces *all* the moments of the structure functions in a Q^2 range sufficient to cover the present and near future experiments.

In the following section we introduce our notation and recall some formulae needed for further work. In Section 3 the exact solution is given and discussed. Section 4 is devoted to the asymptotic expansion. Our conclusions are summarized in Section 5.

2. *Integral representation for the non-singlet evolution function*

The LL evolution with Q^2 of an arbitrary non-singlet structure function f_v can be expressed in terms of a universal evolution function E_v (cf. e.g. Ref. [4]) according to the formula

$$f_v(x, Q^2) = \int_0^1 dy \int_0^1 dz\delta(x-yz)E_v(z, t)f_v(y, Q_0^2). \tag{2.1}$$

The cascade length t, which appears in this formula, is defined by

$$t = \frac{6}{33-2f}\log(\log(Q^2/\Lambda^2)/\log(Q_0^2/\Lambda^2)), \tag{2.2}$$

where f is the number of flavours and Λ is the characteristic scale of QCD. As seen from relation (2.1), the evolution function may be interpreted as a structure function, which would have evolved at Q^2, if at Q_0^2 it had been $\delta(1-z)$. Therefore, it satisfies the LL evolution equation with particularly simple initial conditions. Here and in the following structure functions normalized to the number of partons (and not to their energy) are used so that

$$\int_0^1 E_v(x, t)dx = 1. \tag{2.3}$$

The LL evolution equation for the Mellin transform of the evolution function is easily solved and one finds the famous result for moments

$$E_v(s) = \int_0^1 x^{s-1}E_v(x, t) = \exp[A(3/4+1/(2s(s+1))-\psi(s+1)-\gamma)], \tag{2.4}$$

where $A = 8t/3$, ψ is the logarithmic derivative of the Γ-function and γ is Euler's constant. Inverting relation (2.4) one finds

$$E_v(x, t) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dse^{ys}E_v(s), \tag{2.5}$$

where $y = \log\frac{1}{x}$ and the integration path leaves all the singular points $s = 0, -1, -2, \ldots$

on its left-hand side. Integral (2.5) can be evaluated in the $x \to 1$ and $x \to 0$ limits. One finds

$$E_v(x, t) = \frac{e^{A(3/4-\gamma)}}{\Gamma(A)} (1-x)^{A-1}(1+O(1-x)) \tag{2.6}$$

and

$$E_v(x, t) = e^{A/4} \sqrt{\frac{A}{2y}}\, I_1(\sqrt{2Ay}) \left(1+O\left(\sqrt{\frac{A}{y}}\right)\right), \tag{2.7}$$

where I_1 is the modified Bessel function. Formulae analogous to (2.5), (2.6) and (2.7), curves obtained by a numerical integration of integral (2.5) and references to earlier work on the subject may be found in Ref. [9].

3. Convergent series expansion for the non-singlet evolution function

In order to find the evolution function, we evaluate integral (2.5). Since y is positive and asymptotically for large $|s|$, $|\arg s| < \pi$: $\psi(s+1) \sim \log s$, the integration contour can be deformed so that it starts at $-\infty$ just below the real axis, goes parallel to the real axis,

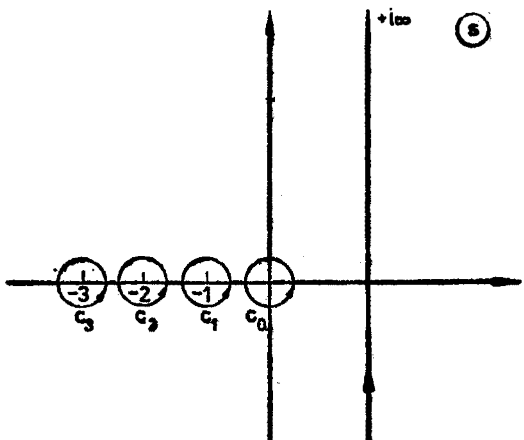

Fig. 1. Deformation of the integration contour leading from Eq. (2.5) to Eq. (3.1)

up to $\mathrm{Re}(s) = 0$, circles counterclockwise around $s = 0$ and just above the real axis returns to $-\infty$. The contributions from the integrals along the real axis between the singularities at $s = 0, -1, \ldots$ cancel, and one is left with a sum of contributions: one from a small circle around each of the singularities. Thus

$$E_v(x, t) = \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_{C_k} ds\, e^{ys} E_v(s), \quad 0 < y < \infty \tag{3.1}$$

where c_k denotes a small circle going anticlockwise around $s = -k$ (cf. Fig. 1). Using the identity

$$\psi(s+1) = \psi(s+k+1) - \sum_{j=1}^{k} 1/(s+j) \tag{3.2}$$

and the fact that $\psi(1+s)$ is holomorphic for $|s| < 1$ it is easy to show that for k large enough

$$\left|\frac{1}{2\pi i}\int_{c_k} ds e^{ys} E_\nu(s)\right| = O(k^A x^k). \tag{3.3}$$

Thus series (3.1) converges for all $|x| < 1$. For each c_k the integral can be written in the form

$$\frac{1}{2\pi i}\int_{c_k} ds e^{ys} E_\nu(s) = x^k \frac{1}{2\pi i}\int_{c_0} du e^{yu + a_k/u} \sum_{n=0}^{\infty} g_{kn}(A) u^n, \tag{3.4}$$

where $a_0 = a_1 = A/2$ and $a_k = A$ for all $k \geqslant 2$. The series in the integrand is convergent for $|u| < 1$. Integrating term by term according to the formula

$$\frac{1}{2\pi i}\int_{c_0} du e^{yu + a/u} u^n = (a/y)^{(n+1)/2} I_{n+1}(2\sqrt{ay}), \tag{3.5}$$

where I_ν are modified Bessel functions, one obtains a convergent series for integral (3.4) and substituting it into relation (3.1) a convergent series for $E_\nu(x)$:

$$E_\nu(x, t) = \sum_{k=0}^{\infty} x^k \left[\sum_{n=0}^{\infty} g_{kn}(A) (a_k/y)^{(n+1)/2} I_{n+1}(2\sqrt{a_k y})\right]. \tag{3.6}$$

Note that an asymptotic expansion in inverse powers of $\sqrt{y}$, as obtained by direct generalization of formula (2.7), includes only the $k = 0$ contribution, because all the others are "exponentially small" in y. Since, however, they are exponentially small in a logarithm, they are quite important for practical calculations. The formulae for the coefficients $g_{kn}(A)$ are given in Appendix B.

For $t \to 0$ all the terms in expansion (3.6), valid for $y \neq 0$, tend to zero and the sum rule (2.3) implies $E_\nu(x) = \delta(1-x)$, which is correct. It is of interest, however, to see what happens for larger t. In Fig. 2 the results of a numerical calculation for $t = 0.45$ are shown and compared with results obtained in Ref. [4], where a Monte Carlo program was used. It is seen that effects of the $k \neq 0$ singularities show up at the one per cent level for $x \approx 0.01$, increase with increasing x and exceed 10 per cent at $x \gtrsim 0.15$. For $x \lesssim 0.6$ the effect of the $k = 3$ singularity is below 1 per cent. A comparable correction comes from putting $g_{2n} = 0$ for $n > 6$. For $k < 2$ we put $g_{kn} = 0$ for $n > 4$, but these series converge better and the corresponding error is smaller. To summarize: we think that our error at $x = 0.6$

does not exceed 2 per cent and decreases with decreasing x. For x close to one the convergence is poor and for this reason we studied an extension of approximation (2.6) as discussed in the following section.

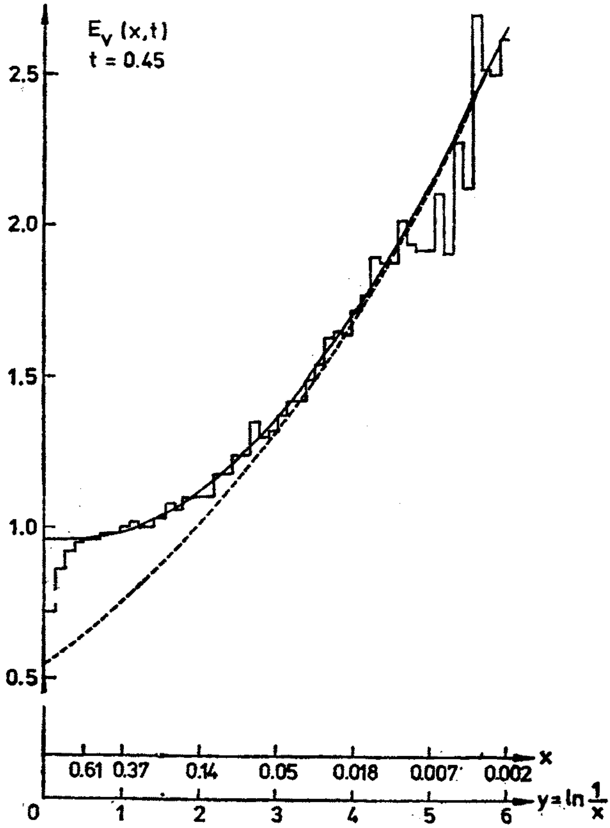

Fig. 2. Evolution function $E_v\left(y = \log \frac{1}{x}\right)$ for $t = 0.45$. The continuous curve and the dashed curve were obtained from formula (3.6) including $k = 0, 1, 2$ in the first case and only $k = 0$ in the second. The histogram is the corresponding Monte Carlo result from Ref. [4]

4. *Asymptotic series for the non-singlet evolution function*

An asymptotic expansion of the non-singlet evolution function in powers of $(1-x)$:

$$E_v(x, t) = \frac{e^{A(3/4-\gamma)}}{\Gamma(A)}(1-x)^{A-1}\left(1+\sum_{n=1}^{\infty} A_n(1-x)^n\right) \tag{4.1}$$

can be obtained as follows (cf. e.g. Ref. [4]). One rewrites expression (2.5) in the form

$$E_v(x, t) = e^{A(3/4-\gamma)} \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i}\left(\frac{1}{s}\right)^{M} \exp\left[A\left(\sum_{n=1}^{\infty} \frac{B_{2n}}{2ns^{2n}} - \frac{1}{2(s+1)}\right)\right], \tag{4.2}$$

where the asymptotic expansion

$$\psi(s+1) = \log s + \frac{1}{2s} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2ns^{2n}}, \tag{4.3}$$

has been substituted. Since the Bernoulli numbers B_{2n} satisfy the inequality

$$(-1)^{n+1}B_{2n} > 2(2n)!/(2\pi)^{2n}, \tag{4.4}$$

expansion (4.3) diverges for all finite s. Expanding formally in powers of s^{-1} the second exponential in the integrand of integral (4.2) and integrating term by term according to the formula

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{ys} s^{-\alpha} ds = y^{-1+\alpha}/\Gamma(\alpha) \tag{4.5}$$

one finds an expansion of the evolution function $E_\nu(x)$ in powers of $y = \log \frac{1}{x}$. For x small y becomes large and the series is useless. A significant improvement is obtained by expanding each y in powers of $(1-x)$. Rearranging the resulting expansion of $E_\nu(x)$ one obtains a series of the form (4.1). We show in Appendix A that this series is asymptotically convergent. An explicit calculation gives for the first few terms:

$$A_1 = A/2-1, \tag{4.6}$$

$$A_2 = (3A^2-7A+1+11/(A+1))/24, \tag{4.7}$$

$$A_3 = \tfrac{1}{48}(A^3-A^2-3A+11-11/(A+1)), \tag{4.8}$$

$$A_4 = \frac{1}{11520}\left(30A^4+60A^3-170A^2+2404A+1674-\frac{503}{A+1}+\frac{896}{A+2}-\frac{1053}{A+3}\right), \tag{4.9}$$

$$A_5 = \frac{1}{23040}\left(6A^5+40A^4+30A^3-16A^2-1602A+632+\frac{1097}{A+1}+\frac{1053}{A+3}\right). \tag{4.10}$$

Let us denote by N the highest n included in expansion (4.1). The curve obtained using expansion (4.1) with $N = 5$ is shown in Fig. 3. From estimates of higher terms and comparison with the Monte Carlo results from Ref. [4] we conclude that for $x \geqslant 0.6$ the curve is correct within two per cent. A standard method of estimating the goodness for an approximation to a non-singlet structure function is to calculate its moments and to compare them with the exact values, given in our case by formula (2.4) (cf. e.g. [3], [4]). We calculated the first twenty moments for $t = 0.15$ and for $t = 0.45$. For $s = 2, 3, \ldots$ the agreement with the exact results improves, as expected, with increasing s and/or decreasing t. The largest error was found for $s = 2$ and $t = 0.45$ (as usual the $s = 1$ moment (3.2) is not included), where we calculate 0.444, while the result exact to three digits is 0.449. Since it is also seen from formula (4.1) that for $t \to 0$ the approximation yields the exact result $\delta(1-x)$, it follows that the simple form (4.1) with $N = 5$ and the coefficients (4.6)–(4.10) reproduces *all* the moments with an error not exceeding 1 per cent.

Approximation (4.1) does not satisfy the sum rule (2.3). The deviation of the integral from one increases with t. It is -0.008 at $t = 0.15$ and -0.07 at $t = 0.45$. The reason is clear from Fig. 3. For $x \to 0$ expansion (4.1) has a smooth behaviour, while the exact structure function tends to infinity there. The discrepancy may be removed without spoiling the predictions for moments by adding to the expansion a term $a\delta(x)$, where a is a t-depen-

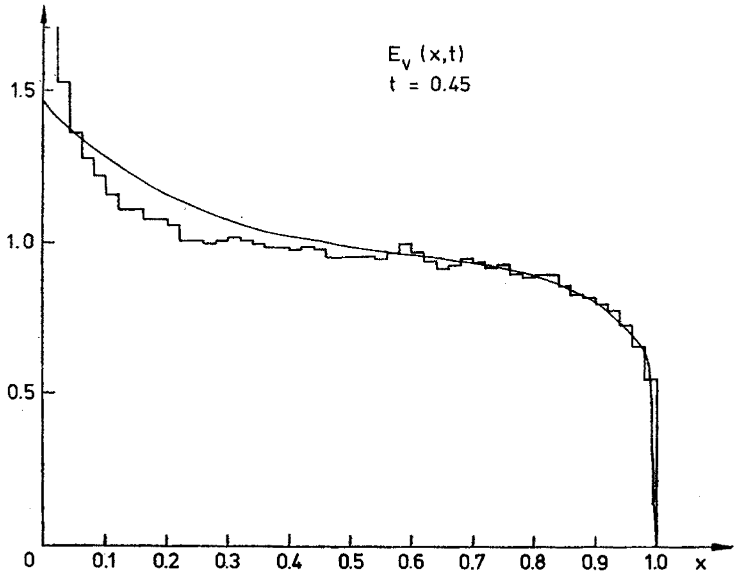

Fig. 3. Evolution function $E_V(x)$ for $t = 0.45$ calculated from formula (4.1) including $n \leqslant 5$ and using the coefficients (4.6)–(4.10). The histogram is the corresponding Monte Carlo result from Ref. [4]

dent constant determined from the sum rule (2.3). Our calculation of moments shows also that at low s a slightly better agreement with the exact values is obtained taking for $t = 0.15$ $N = 3$ and for $t = 0.45$ $N = 4$ instead of the $N = 5$ used in the main calculation. The reason is that extending the series (4.1) improves the curve $E_V(x)$ in the intermediate x region more than in the low x region. Thus the cancellation of errors, which is partly responsible for the excelent prediction of the low s moments, is spoiled. This, as well as a direct inspection of Fig. 3, shows that very good moments do not necessarily mean a very good approximation to the structure function.

5. *Discussion*

We have derived two expansions for the non-singlet evolution function. This is equivalent to two expansion for any non-singlet structure function because of identity (2.1), but it is easier to discuss the evolution function. Expansion (3.6) converges for all $x < 1$ and its convergence improves with decreasing x. Expansion (4.1) is only asymptotic, but gives a good estimate of the evolution function for 1-x small. Both expansions yield correctly $E_v(x) = \delta(1-x)$ for t tending to zero and deteriorate with increasing t. We checked, however, that even for $t = 0.45$, which is more than what is necessary for any present day or near future experiments, the two expansions, when used simultaneously, yield easily the evolution function in all the range $0 < x \leqslant 1$ with an error not exceeding two per cent. The asymptotic expansion reproduces almost perfectly all the moments of the evolution function. Since it is also very simple and easy to convolute with other functions, it may be useful for phenomenological analyses of the data. For practical applications, however, it should be supplemented with a small $\delta(x)$ term to correct the poorly reproduced very small x region and ensure the correct normalization (2.3).

APPENDIX A

Asymptotic convergence of series (4.1)

Let us deform the integration contour in integral (2.5) so that it goes from $-i\infty$ to $-i/y$ along the imaginary axis, then to $+i/y$ along the semicircle $|s| = 1/y$ in the $\mathrm{Re}\, s \geqslant 0$ half plane and finally from $+i/y$ to $+i\infty$ (cf. Fig. 4). This contour will be denoted by C.

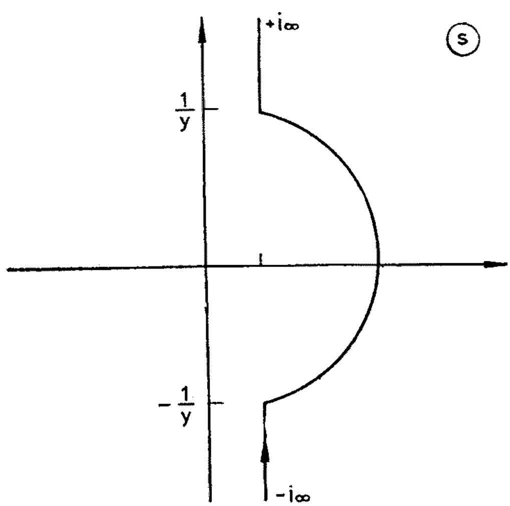

Fig. 4. Integration contour used in Appendix A to prove the asymptotic convergence of the asymptotic series (4.1)

Let us denote further by $\varphi_n(s)$ the difference between the second exponent in the integrand of integral (4.2) and its formal expansion in powers of s^{-1} up to and including s^{-n}. Since the expansions of $\psi(s)$, $(s+1)^{-1}$ and $\exp s^{-1}$ in powers of s^{-1} are all asymptotically convergent for $s \to \infty$, there exists a constant $C_n > 0$ such that

$$|\varphi_n(s)| < C_n |s^{-n-1}| \tag{A1}$$

for sufficiently large $|s|$. Thus for sufficiently small y this inequality is satisfied all along the integration path C. The difference between $E_\nu(x)$ and the sum of the first $n+2$ terms of its expansion in powers of y obtained as described in the text is therefore

$$\Delta_{n+1}(y) = \left| e^{A(3/4-\gamma)} \frac{1}{2\pi i} \int_C ds \left(\frac{1}{s}\right)^A e^{ys} \varphi_{n+1}(s) \right|$$

$$< e^{A(3/4-\gamma)} C_{n+1} \frac{1}{2\pi} \int_C |ds|\, |s^{-A-n-2}|\, |e^{ys}|. \tag{A2}$$

The modulus e^{ys} equals 1 along the imaginary axis and is less than e^{-1} along the semicircle. Thus

$$\Delta_{n+1}(y) < C_{n+1} e^{A(3/4-\gamma)} \left(2 \int_{\frac{1}{y}}^{\infty} \frac{ds}{2\pi} s^{-A-n-2} + \frac{y^{A+n+1}}{2e} \right) = O(y^{A+n+1}). \tag{A3}$$

Consequently,

$$\Delta_n(y) < \Delta_{n+1}(y) + b_{n+1} y^{A+n+1} = O(y^{n+A+1}), \tag{A4}$$

where b_{n+1} is the modulus of the coefficient of y^{n+A+1} in the formal expansion of $E_\nu(x)$ in powers of y, and the series in powers of y is asymptotically convergent. Since for $|1-x| < 1$ the function $y = \log \frac{1}{x}$ is holomorphic in $(1-x)$, substituting for each y its expansion in powers of $1-x$ and rearranging the series, which yields expansion (4.1), we obtain again an asymptotically convergent expansion q.e.d.

APPENDIX B

The general formula for the functions $g_{kn}(A)$, needed for expansion (3.6), will be given here. It follows from Eq. (3.4) that g's are defined by the expansion

$$e^{Af_k(u)} = \sum_{n=0}^{\infty} g_{kn}(A) u^n, \qquad k = 0, 1, \dots, \tag{B1}$$

where $f_k(u)$ are the regular (at $u = 0$) parts of the exponent (2.4) after substitution $s = -k + u$. Expanding $f_k(u)$ we get

$$f_k(u) = \sum_{r=0}^{\infty} f_{kr} u^r, \tag{B2}$$

where the coefficients f_{kr} are given in Table I. It is straightforward to express $g_{kn}(A)$ by the f_{kr}. The general formula is

$$g_{kn}(A) = e^{Af_{k0}} \sum_{m=1}^{n} W_m^{kn} \frac{A^m}{m!}, \qquad n \geqslant 1$$

$$g_{k0}(A) = e^{Af_{k0}}. \tag{B3}$$

TABLE I

Coefficients f_{kn} of the expansion of the regular parts of the exponent (2.4) in the variable $u = +k+s$. $\zeta(n)$ is the Riemann zeta function. For $k = 2$ the sum $\sum_{j=2}^{k-1}$ is defined to be zero

k \ r	0	$\geqslant 1$
0	$\frac{1}{4}$	$\frac{(-1)^{r+1}}{2} + (-1)^r\zeta(r+1)$
1	$\frac{1}{4}$	$-\frac{1}{2} + (-1)^r\zeta(r+1)$
$\geqslant 2$	$\frac{3}{4} - \frac{2k-1}{2k(k-1)} - \sum_{j=2}^{k-1}\frac{1}{k-j}$	$\frac{-1}{2}\left(\frac{1}{k^{r+1}} + \frac{1}{(k-1)^{r+1}} + 2\sum_{j=2}^{k-1}\frac{1}{(k-j)^{r+1}}\right) + (-1)^r\zeta(r+1)$

TABLE II

Coefficients W_m^{kn} for the first twelve polynomials $g_{kn}(A)$ as defined in Eqs (3.4) and (B1–B4)

k	n \ m	1	2	3	4	5	6
0	1	−1.145					
	2	0.702	1.311				
	3	−0.582	−1.608	−1.501			
	4	0.537	1.826	2.761	1.718		
1	1	−2.145					
	2	0.702	4.601				
	3	−1.582	−3.012	−9.868			
	4	0.537	7.281	9.690	21.17		
2	1	−2.270					
	2	0.640	5.153				
	3	−1.614	−2.904	−11.70			
	4	0.521	7.734	9.886	26.55		
	5	−1.525	−4.431	−27.73	−29.92	−60.27	
	6	0.504	10.19	22.52	88.14	84.89	136.8

The coefficients W_m^{kn} of the polynomial in (B3) are given by

$$W_m^{kn} = \sum_{\substack{i_1+i_2+\ldots+i_m=n \\ i_1\geqslant 1,\ldots,i_m\geqslant 1}} f_{ki_1}f_{ki_2}\cdots f_{ki_m}. \tag{B4}$$

It is seen from Eq. (B3) that up to an overall factor $e^{Af_{k0}}$, $g_{kn}(A)$ is an n-th order polynomial in A. In Table II we give coefficients W_m^{kn} of the first 12 polynomials (for $k \leqslant 2$,

$n \leqslant 6$). Higher order terms do not change the result for $E_v(x)$ by more than 2 per cent for $x \leqslant 0.6$.

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