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ISOTROPIC HYPERSURFACES IN GENERAL RELATIVITY ADMITTING GROUPS OF MOTIONS

By G. Dautcourt

Zentralinstitut für Astrophysik der AdW der DDR, DDR 1502 Potsdam-Babelsberg

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Isotropic hypersurfaces admitting inner groups of motions are classified. Normal forms for the inner metric are given, and its geometric properties are characterized by means of differential invariants. The problem of embedding a given null hypersurface into empty spacetime is studied locally.

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1. *Introduction*

In a previous paper [1] the local geometry of a general light-like hypersurface was studied with an intrinsic spin coefficient technique. The present paper discusses light-like hypersurfaces which admit inner groups of motions. Special null hypersurfaces with intrinsic symmetries have occured as "horizons" in black hole physics. This may justify a separate discussion, which otherwise would be of academic interest only.

A null hypersurface represents an isotropic 3-space with an inner metric γ_{ik} (i and k from 1 to 3) of matrix rank 2. It admits an inner isometry group if the Lie derivative of the inner metric vanishes with regard to a Killing field $\xi^i(x^i)$. As well known (see, e.g., [2–4]), the Lie derivative is already defined on a differentiable manifold and therefore independent of any affinity given additionally. It may be applied to degenerate metrics such as that of a null hypersurface, where an affinity is not uniquely determined [1]:

$$\underset{(\xi^i)}{L}\gamma_{ik} \equiv \frac{\partial \gamma_{ik}}{\partial x^l}\,\xi^l+\gamma_{kl}\frac{\partial \xi^l}{\partial x^i}+\gamma_{il}\frac{\partial \xi^l}{\partial x^k}\,. \tag{1}$$

Using the class of affinities introduced in [1], the expression for the Lie derivative may also be written

$$\underset{(\xi^i)}{L}\gamma_{ik} = \xi^l\nabla_l\gamma_{ik}+\gamma_{kl}\nabla_i\xi^l+\gamma_{il}\nabla_k\xi^l \tag{2}$$

(∇_l is the covariant derivative associated with the affinity). With $\xi_k = \gamma_{kl}\xi^l$ and $\nabla_i\gamma_{kl} \equiv \gamma_k h_{li} + \gamma_l h_{ki}$, the expression is transformed into a third form

$$\underset{(\xi^i)}{L}\gamma_{ik} = \nabla_k\xi_i + \nabla_i\xi_k - 2\xi^l\gamma_l h_{ik}, \tag{3}$$

with the Lie derivative of γ_{ik} with regard to ε^l given by

$$h_{ik} = \tfrac{1}{2}(\partial_k\gamma_{il} + \partial_i\gamma_{kl} - \partial_l\gamma_{ik})\varepsilon^l, \tag{4}$$

with ∂_k denoting the ordinary derivative. ε^k is the generator direction, defined by $\gamma_{ik}\varepsilon^k = 0$. γ_k is a covariant vector field satisfying $\gamma_k\varepsilon^k = 1$. Thus the intrinsic Killing equation of a null hypersurface can be written in the covariant form

$$\nabla_k\xi_i + \nabla_i\xi_k = 2h_{ik}\xi^l\gamma_l, \tag{5}$$

which differs from the usual covariant Killing equation by an additional term on the rhs.

There are two classes of solutions of (5). In the first case the trajectories of ε^k are generators (light-like geodesics) of the null hypersurface. Here $\xi^l\gamma_l \neq 0$ and $\xi_i \equiv \gamma_{ik}\xi^k = 0$, and from (3) follows $h_{ik} = 0$. As shown in [1], this is equivalent to the vanishing of the Ricci coefficients shear σ and rotation ϱ of the generator congruence. In [1] we have denoted this type of null hypersurfaces as "planar", to distinguish it from "plane" null hypersurfaces (where $\gamma_{ik} = \text{const}$). Thus, a null hypersurface admits a motion with light-like trajectories if and only if it is planar. Notice this Lie group represents an infinite Lie group G_∞ rather than a one-dimensional group, since with $X = \xi^k\partial_k$ every $X' = f(x^i)\xi^k\partial_k$ is also a generator, $f(x^i) \neq 0$ being an otherwise arbitrary function of position.

The other possible type of solutions of (3) has space-like trajectories (any direction in regular points on a null hypersurface is either light-like, hence coincident with the generator direction, or space-like). They may be classified by means of the well-known Bianchi classification of real Lie groups G_2, G_3 and G_4 according to nonisomorphic structures (see Table II for G_3). In Section 2 normal forms for the different types are derived. Their geometrical properties may be studied using the appropriate differential invariants given there. Section 3 discusses local embedding into four-dimensional Einstein space-times and into flat space-times. In a subsequent paper spacetimes are considered, which satisfy the Einstein vacuum field equations and admit the isometry groups — discussed here for a single null hypersurface — on a family of null hypersurfaces.

In order to facilitate the comparison with the subsequent paper and with the work by Petrov [5], we employ the tensor formulation, although the use of forms might appear more elegant.

2. *Classification of inner metrics*

2.1. Groups G_1 with a space-like generator

Coordinate transformations may be taken to transform the inner metric into normal forms. We may always reach

$$\gamma_{ik} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & E & F \\ 0 & F & G \end{pmatrix} \tag{6}$$

with E, F and G as functions on the null hypersurface, and this form is preserved under the transformations

$$x'_1 = x'_1(x_1, x_A), \qquad x'_A = x'_A(x_B). \tag{7}$$

The Killing equation splits into

$$\xi^B{}_{,1} = 0, \qquad \xi^1 \gamma_{AB,1} + \gamma_{AB,C}\xi^C + \gamma_{AC}\xi^C{}_{,B} + \gamma_{BC}\xi^C{}_{,A} = 0. \tag{8}$$

Let us first assume space-like generators and consider a G_1. We may choose coordinates such that $\underset{1}{\xi}^i = \delta^i_3$, say. In this coordinate system the metric is restricted by $\gamma_{AB,3} = 0$. The coordinate transformations preserving this as well as the chosen normal form for $\underset{1}{\xi}^i$ constitute a subgroup of (7):

$$x'_1 = f(x_1, x_2), \qquad x'_2 = g(x_2), \qquad x'_3 = x_3 + h(x_2) \tag{9}$$

with $f_{,1} g_{,2} \neq 0$.

2.2. Groups G_2 with space-like generators

Taking $\underset{1}{\xi}^i = \delta^i_3$ as the first Killing field of an Abelian G_2, the condition $[X_1, X_2] = 0$ or explicitly

$$\underset{1}{\xi}^i \underset{2}{\xi}^k{}_{,i} - \underset{2}{\xi}^i \underset{1}{\xi}^k{}_{,i} = 0 \tag{10}$$

requires $\underset{2}{\xi}^k = \underset{2}{\xi}^k(x_1, x_2)$. The Killing equation leads (writing $\underset{2}{\xi}^1 = \alpha(x_1, x_2)$, $\underset{2}{\xi}^2 = \beta(x_2)$, $\underset{2}{\xi}^3 = \gamma(x_2)$) to

$$\begin{aligned} \alpha E_{,1} + \beta E_{,2} + 2E\beta_{,2} + 2F\gamma_{,2} &= 0, \\ \alpha F_{,1} + \beta F_{,2} + \beta_{,2} F + \gamma_{,2} G &= 0, \\ \alpha G_{,1} + \beta G_{,2} &= 0. \end{aligned} \tag{11}$$

The functions α, β, γ transform under (9) as

$$\begin{aligned} \alpha' &= f_{,1}\alpha + f_{,2}\beta, \\ \beta' &= \beta g_{,2}, \\ \gamma' &= \gamma + \beta h_{,2}. \end{aligned} \tag{12}$$

Two cases must be distinguished. Suppose first that $\beta \neq 0$. Then by means of (12), $\beta = 1$, $\gamma = 0$, $\alpha = 0$ can be reached, that is, $\underset{2}{\xi}^i = \delta^i_2$. This gives the first normal form for a metric admitting an G_2 with space-like trajectories as shown in Table I. If, however, $\beta = 0$,

TABLE I

Isotropic hypersurfaces admitting as isometry group G_2 with space-like generators

No	Normal forms for metric tensor components	Normal form for generators	Coordinate transformations leaving normal forms invariant	Remarks
1	$E = a(x_1)$ $F = b(x_1)$ $G = c(x_1)$	$X_1 = \partial_3$ $X_2 = \partial_2$	$x_1' = f(x_1), f_{,1} \neq 0$ $x_2' = x_2 + \text{const}$ $x_3' = x_3 + \text{const}$	Abelian group G_2, $[X_1, X_2] = 0$
2	$E = k(x_2) + x_1^2/l(x_2)$ $F = x_1$ $G = l(x_2)$	$X_1 = \partial_3$ $X_2 = \partial_1 + \gamma\partial_3$ where $\gamma = \gamma(x_2), \gamma_{,2} = -1/l$	$x_1' = x_1 - h_{,2} l$ $x_2' = x_2 + \text{const}$ $x_3' = x_3 + h(x_2)$	Abelian group G_2, $[X_1, X_2] = 0$
3	$E = k(x_1)$ $F = m(x_1)e^{-x_2}$ $G = l(x_1)e^{-2x_2}$	$X_1 = \partial_3$ $X_2 = \partial_2 + x_3\partial_3$	$x_1' = f(x_1), f_{,1} \neq 0$ $x_2' = x_2 + g_0$ $x_3' = x_3 + h_0 e^{x_2}$ $g_0, h_0 = \text{const}$	non-Abelian group G_2, $[X_1, X_2] = X_1$
4	$E = 1$ $F = e_{-x_1}$ $G = k(x_2)e^{-2x_1}$	$X_1 = \partial_3$ $X_2 = \partial_1 + x_3\partial_3$	$x_1' = x_1$ $x_2' = x_2 + \text{const}$ $x_3' = x_3$	non-Abelian group G_2, $[X_1, X_2] = X_2$, admits a G_3 (see text)

we must suppose, that $\alpha \neq 0$ (otherwise, $\underset{2}{\xi}^i$ would be equivalent to $\underset{1}{\xi}^i$, and no G_2 would exist). (12) allows us to assume $\alpha = 1$, and (11) can be written:

$$E_{,1} + 2F\gamma_{,2} = 0,$$
$$F_{,1} + G\gamma_{,2} = 0,$$
$$G_{,1} = 0. \tag{13}$$

Integrating (13) and again using (12), on obtains the second normal form given in Table I.

Thus far we have assumed an Abelian G_2. For a non-Abelian G_2, we may again take $\underset{1}{\xi}^i = \delta_3^i$ as the first Killing vector. The second one is obtained from the relation $[X_1, X_2] = X_1$:

$$\underset{2}{\xi}^i = (\alpha[x_1, x_2], \beta[x_2], x_3 + \gamma[x_2]), \tag{14}$$

where α, β and γ now transform under (9) as

$$\alpha' = \alpha f_{,1} + \beta f_{,2},$$
$$\beta' = \beta g_{,2},$$
$$\gamma' = \gamma - h + \beta h_{,2}. \tag{15}$$

If $\beta \neq 0$, we may obtain $\alpha = 0$, $\beta = 1$, $\gamma = 0$. Integration yields metric 3 in Table I.

If, however, $\beta = 0$, we must assume $\alpha \neq 0$ (otherwise, the resulting metric tensor would not have rank 2). This allows us to put $\alpha = 1$ as well as $\gamma = 0$ because of (15). Integration of (8) yields metric 4 in Table I, if the remaining freedom of coordinate transformations

$$x_1' = x_1 + f(x_2),$$

$$x_2' = g(x_2),$$

$$x_3' = x_3, \tag{16}$$

is used to eliminate two of the three integration functions of x_2 in the metric.

Apart from metric 4 in Table I, the other metrics do not admit in general a third independent Killing field. The third Killing field of metric 4 is given by

$$\underset{3}{\xi}^i = (2x_3 + e^{x_1}\beta(x_2), 0, x_3^2), \tag{17}$$

and the commutator relations are

$$[X_1, X_2] = X_1, \quad [X_1, X_3] = 2X_2, \quad [X_2, X_3] = X_3. \tag{18}$$

Metric 4 therefore admits a three-parameter Lie group of Bianchi type VIII (cf. the normal forms for generators given by Petrov [5], which differ from that of Table II here in the case of type VIII metrics).

TABLE II

Commutator relations for G_3. Notice that Petrov [5] has used different operators X_i' for the Bianchi type VIII. They are connected with the ones given above by $X_1' = -X_2+X_3$, $X_2' = X_1$, $X_3' = -X_2-X_3$. We prefer the choice given here since the groups VIII and IX can be treated similarly

I. $[X_1, X_2] = 0$, $[X_1, X_3] = 0$, $[X_2, X_3] = 0$.
II. $[X_1, X_2] = 0$, $[X_2, X_3] = X_1$, $[X_1, X_3] = 0$.
III. $[X_1, X_2] = 0$, $[X_2, X_3] = 0$, $[X_1, X_3] = X_1$.
IV. $[X_1, X_2] = 0$, $[X_2, X_3] = X_1+X_2$, $[X_1, X_3] = X_1$.
V. $[X_1, X_2] = 0$, $[X_2, X_3] = X_2$, $[X_1, X_3] = X_1$.
VI. $[X_1, X_2] = 0$, $[X_2, X_3] = qX_2$, $[X_1, X_3] = -X_1 (q \neq 0,1)$.
VII. $[X_1, X_2] = 0$, $[X_2, X_3] = -X_1+qX_2$, $[X_1, X_3] = X_2 (q^2 < 4)$.
VIII. $[X_1, X_2] = X_3$, $[X_2, X_3] = X_1$, $[X_3, X_1] = -X_2$.
IX. $[X_1, X_2] = X_3$, $[X_2, X_3] = X_1$, $[X_3, X_1] = X_2$.

2.3. Groups G_3 with space-like generators

In the case of three-parameter Lie groups normal forms in Table I must be restricted by further commutator conditions imposed by the third Killing vector (Table II). The final normal forms can be obtained in the same manner as discussed above. The calculations are lengthy but rather trivial, thus we omit all details and give only the results in Table III. Some comments should be made. For the Bianchi groups I through VII we can confine the discussion to those cases in Table I, which admit an Abelian G_2 as subgroup. The Bianchi groups VIII and IX have been treated separately, since we employ different group

TABLE III

Isotropic hypersurfaces admitting an isometry group G_3 with space-like generators

No	Bianchi type	Normal form for the metric tensor components	Normal form for generators	Coordinate transformations leaving the normal forms invariant	Class	Rotation coefficients	Invariants
1	II	$E = a+bx_1^2-2cx_1$, $F = c-bx_1$, $G = b$ (here and subsequently, a, b, c are constants with $ab-c^2 > 0$)	$X_1 = \partial_3, X_2 = \partial_2$, $X_3 = \partial_1+x_2\partial_3$	$x_1' = x_1+\text{const}$, $x_2' = x_2$, $x_3' = x_3$	3	$\varrho = 0$, $\sigma = -ib/(2[ab-c^2]^{1/2})$, $\nu = -i\sigma, \tau = 0$	$j = 0$, $I_1 = -2$, $N = 0$
2	III	$E = a\exp(-2x_1)$, $F = c\exp(-x_1)$, $G = b$	$X_1 = \partial_2, X_2 = \partial_3$, $X_3 = \partial_1+x_2\partial_2$	$x_1' = x_1+\text{const}$, $x_2' = x_2$, $x_3' = x_3+\text{const}$	2	$\varrho = 1/2, \sigma = -1/2 -ic/(2[ab-c^2]^{1/2})$, $\nu = c/(2[ab-c^2]^{1/2})$, $\tau = 0$	$j = (1-c^2/ab)^{1/2}$ $I_1 = 2c/[ab]^{1/2}$, $I_2 = c, L = M = 0$
3	III	$E = a/x_2^2+x_1^2/(x_2^2b)$, $F = x_1$, $G = bx_2^2$	$X_1 = \partial_3$, $X_2 = \partial_1+\frac{1}{bx_2}\partial_3$, $X_3 = -x_2\partial_2+x_3\partial_3$	$x_1' = x_1+bd$, $x_2' = x_2$, $x_3' = x_3+d/x_2$, $d = \text{const}$	3	$\varrho = 0, \sigma = i/(2[ab]^{1/2})$, $\nu = -1/(2[ab]^{1/2})$, $\tau = i/[2a]^{1/2}$	$j = 0$, $I_1 = -2$, $N = 0$
4	IV	$E = e^{-2x_1}[a-2cx_1 +bx_1^2]$, $F = e^{-2x_1}[c-bx_1]$, $G = e^{-2x_1}b$	$X_1 = \partial_3, X_2 = \partial_2$, $X_3 = \partial_1+[x_2+x_3]\partial_3 +x_2\partial_2$	$x_1' = x_1+\text{const}$, $x_2' = x_2$, $x_3' = x_3$	1	$\varrho = 1, \sigma = -ib/(2[ab -c^2]^{1/2})$, $\nu = b/(2[ab-c^2]^{1/2})$, $\tau = 0$	$j = 2[ab-c^2]^{1/2}/b$, $I_1 = -2,\ I_2 = 0$, $J = 0$
5	IV	$E = a/x_2^2+x_1^2/(2x_2)$, $F = x_1$, $G = 2x_2$	$X_1 = \partial_3$, $X_2 = -\partial_1+\gamma\partial_3$, $X_3 = x_1\partial_1-2x_2\partial_2 +x_3\partial_3$, where $\gamma = \gamma(x_2)$ with $\gamma_{,2} = -1/(2x_2)$	$x_1' = x_1+cx_2^{-1/2}$, $x_2' = x_2$, $x_3' = x_3+cx_2^{-1/2}$	3	$\varrho = 0, \sigma = i(x_2/(8a))^{1/2}$, $\nu = (x_2/(8a))^{1/2}$, $\tau = i/(2(2a)^{1/2})$,	$j = 0, I_1 = -2$, $N = 0$

6	V	$E = ae^{-2x_1}$, $F = ce^{-2x_1}$, $G = be^{-2x_1}$	$X_1 = \partial_3, X_2 = \partial_2$, $X_3 = \partial_1 + x_2\partial_2 + x_3\partial_3$	$x_1' = x_1 + \text{const}$, $x_2' = x_2$, $x_3' = x_3$	5	$\varrho = 1, \sigma = 0, \nu = 0$, $\tau = 0$	no invariants
7	VI	$E = ae^{-2x_1q}$,	$X_1 = \partial_3, X_2 = \partial_2$,	$x_1' = x_1 + \text{const}$	1	$\varrho = (q+1)/2$, $\sigma = \frac{(1-q)}{2}(ic/(ab - c^2)^{1/2} + 1)$,	$j = \frac{1+q}{1-q}(1-c^2/ab)^{1/2}$,
		$F = ce^{-(q+1)x_1}$,	$X_3 = \partial_1 + x_3\partial_3 + qx_2\partial_2$	$x_2' = x_2$		$\nu = \frac{c}{2}(q-1)(ab-c^2)^{-1/2}$,	$I_1 = -2c(ab)^{-1/2}$,
		$G = be^{-2x_1}$	$(q \neq 0, 1)$	$x_3' = x_3$		$\tau = 0$	$I_2 = 0, J = 0$
8	VI	$E = \frac{a}{(1+q)^2x_2^2} + \frac{x_1^2}{k(x_2)}$	$X_1 = \partial_3$,	$x_1' = x_1 - kh_{,2}$	3	$\varrho = 0$, $\sigma = \frac{i}{2(ab)^{1/2}}[(1+q)x_2]^{q/(1+q)}$	$j = 0$,
		$F = x_1$, $G = k(x_2)$, with $k(x_2) = b[(1+q)x_2]^{2/(1+q)}$	$X_2 = \partial_1 + \gamma\partial_3$, $X_3 = qx_1\partial_1 - (1+q)x_2\partial_2 + x_3\partial_3$	$x_2' = x_2$, $x_3' = x_3 + h$ with $h = h_0x_2^{-1/(1+q)}$		$\nu = i\sigma$, $\tau = i(2a)^{-1/2}$	$I_1 = -2$, $N = 0$
9	VI	$E = a + x_1^2e^{-2x_2c}/b^2$,	$X_1 = \partial_3$,	$x_1' = x_1 - h_{,2}be^{cx_2}$	3	$\varrho = 0$,	$j = 0$,
		$F = x_1$,	$X_2 = \partial_1 + \frac{e^{-cx_2}}{ab}\partial_3$,	$x_2' = x_2 + \text{const}$		$\sigma = ie^{-cx_2/2}/(2(ab)^{1/2})$,	$I_1 = -2$,
		$G = be^{cx_2}$,	$X_3 = -x_1\partial_1 - \frac{2}{c}\partial_2 + x_3\partial_3$	$x_3 = x_3 + h$		$\nu = i\sigma$,	$N = 0$
		$(q = -1)$		with $h = h_0e^{-cx_2/2}$		$\tau = ic(2a)^{-1/2}$	
10	VII	$E = k(x_1)$, $F = 0$, $G = k(x_1)$	$X_1 = \partial_3, X_2 = \partial_2$, $X_3 = -x_3\partial_2 + x_2\partial_3$, $(q = 0)$	$x_1' = f(x_1)$, $x_2' = x_2$, $x_3' = x_3$	5 resp. 6	$\varrho = -k_{,1}/(2k)$, $\sigma = 0$, $\nu = \tau = 0$	if $k_{,1} \neq 0$, no invariants, if $k_{,1} = 0$, $K = 0$

TABLE III (continued)

No	Bianchi type	Normal form for the metric tensor components	Normal form for generators	Coordinate transformations leaving the normal forms invariant	Class	Rotation coefficients	Invariants
11	VII	$E = a+x_1^2b^2\cos^{-2}bx_2$, $F = x_1$, $G = \frac{1}{b}\cos^2 bx_2,\ q=0$	$X_1 = \partial_3$, $X_2 = \partial_1-\mathrm{tg}(bx_2)\partial_3$, $X_3 = [\mathrm{tg}(bx_2)x_1 + x_3]\partial_1 - \frac{1}{b}\partial_2 - \mathrm{tg}(bx_2)x_3\partial_3$	$x_1' = x_1 - h_0\sin bx_2$, $x_2' = x_2$, $x_3' = x_3+h_0/\cos bx_2$	3	$\varrho = 0$, $\sigma = i(b/a)^{1/2}/(2\cos bx_2)$, $\nu = i\sigma$, $\tau = -ib\,\mathrm{tg}\,x_2/(2a)^{1/2}$	$j = 0,\ I_1 = -2$, $N = 0$
12	VII	$E = a(c-qx_2)^{-2} + x_1^2(1-q^2/4)/(\cos^2 y(qx_2-c))$, $F = x_1$, $G = (qx_2-c)\cos^2 y/(1-q^2/4)$, $q \neq 0$	$X_1 = \partial_3$, $X_2 = \partial_1+\gamma\partial_3$, $X_3 = [(q-\gamma)x_1+x_3]\partial_1 + (c-qx_2)\partial_2 + \gamma x_3\partial_3$, with $\gamma = q/2+\sqrt{4-q^2}\,\mathrm{tg}\,y/2$, $y = b-\frac{\sqrt{4-q^2}}{2q}\ln(qx_2-c)$,	$x_1' = x_1 - h_{,2}\frac{(qx_2-c)\cos^2 y}{1-q^2/4}$, $x_2' = x_2+\mathrm{const}$, $x_3' = x_3+h$, $h = \frac{h_0\sqrt{4-q^2}}{2\cos y\sqrt{qx_2-c}}\, e^{qb(4-q^2)^{-1/2}}$	3	$\varrho = 0$, $\sigma = i((qx_2-c)(1-q^2/4)/a)^{1/2}/(2\cos y)$, $\nu = i\sigma$, $\tau = -i(q+\mathrm{tg}\,y(4-q^2)^{1/2})/(2(2a)^{1/2})$	$j = 0,\ I_1 = -2$, $N = 0$
13	VII	$E = e^{-qx_1}$, $k = a_1+a_2\cos px_1 + \frac{qa_2-2a_3}{p}\sin px_1$,	$X_1 = \partial_3,\ X_2 = \partial_2$,	$x_1' = x_1+\mathrm{const}$,	2	$\varrho = q/2$, $\sigma = q/2-l/m+i(km-l^2)^{-1/2}(k-m-2l^2/m+ql)/2$,	$j = \frac{q}{p}(p^2a_1^2/(4(a_2^2+a_3^2-qa_2a_3))-1)^{1/2}$,

		$F = e^{-qx_1}$ $+a_3 \cos px_1$ $+ \dfrac{2a_2-qa_3}{p} \sin px_1,$ $G = e^{-qx_1}m = a_1+(a_3q$ $-a_2)\cos px_1$ $+ \dfrac{(a_3p^2+qa_2-2a_3}{p}$ $\sin px_1,\ p^2+q^2 = 4.$	$X_3 = \partial_1 - x_3\partial_2$ $+(x_2+qx_3)\partial_3$	$x_2' = x_2,$ $x_3' = x_3$		$\nu = (km-l^2)^{-1/2}(2l^2/m$ $-ql+m-k)/2,$ $\tau = 0$	$I_2 = 0,$ $I_1 = a_1p/(a_2^2+a_3^2$ $-qa_2a_3)^{1/2},$ $L = M = 0$
14 15 16	VIII	$E = k+lx_3^2+2mx_3,$ $F = lx_3+m,$ $G = l,$ with $\gamma = kl-m^2$ Particular cases are: $\varepsilon = 0$: $k = a,$ $l = b+cx_1+ax_1^2/4,$ $m = -c-ax_1/2,$ $\varepsilon = 1$: $k = 2a+4be^{2x_1}$ $+4ce^{-2x_1},$ $l = -a/2+be^{2x_1}$ $+ce^{-2x_1},$ $m = -2be^{2x_1}+2ce^{-2x_1},$ $\varepsilon = -1$: $k = -2a-4b\cos 2x_1$ $-4c \sin 2x_1,$ $l = -a/2+b\cos 2x_1$ $+c\sin 2x_1,$ $m = 2b\sin 2x_1$ $-2c\cos 2x_1$	$X_1 = \partial_2,$ $X_{2,3} = e^{x_2}\partial_1$ $+ \dfrac{x_3}{2} e^{x_2}\partial_2$ $+([\varepsilon - \dfrac{x_3^2}{4}]e^{x_2}$ $\pm e^{-x_2})\partial_3$	$x_1' = x_1+\text{const},$ $x_2' = x_2,$ $x_3' = x_3$	4	$\varrho = 0,$ $\sigma = m/(2l)+i(m^2/l$ $-2\varepsilon l-k/2)/(2\gamma^{1/2}),$ $\nu = (k/2+2\varepsilon l$ $-m^2/l)/(2\gamma^{1/2}),$ $\tau = -i(l/(2\gamma)^{1/2}$	$\|\sigma\| I_1 = 2\nu-\gamma^{1/2}/l$

TABLE III (continued)

No	Bianchi type	Normal form for the metric tensor components	Normal form for generators	Coordinate transformations leaving the normal forms invariant	Class	Rotation coefficients	Invariants
17	VIII	$E = k(x_1)(1+x_3^2/4)$, $F = k(x_1)x_3/4$, $G = k(x_1)/4$	$X_1 = \partial_2$, $X_{2,3} = \frac{x_3}{2} e^{x_2}\partial_2 + [-e^{x_2}(1+x_3^2/4) \pm e^{-x_2}]\partial_3$	$x_1' = h(x_1)$, $x_2' = x_2$, $x_3' = x_3$	5 resp. 6	$\varrho = -k_{,1}/(2k)$, $\sigma = 0,\ \nu = 0$, $\tau = -i/(2k)^{1/2}$	if $k_{,1} \neq 0$, no invariants, if $k_{,1} = 0$, $K = -1/k < 0$
18	VIII	$E = e^{2x_1}$, $F = k(x_3)e^{x_1}$, $G = 1$, $(k < 1)$	$X_1 = \partial_2$, $X_{2,3} = (\pm e^{-x_2} - \frac{1}{4}e^{x_2})\partial_1 + (\frac{1}{4}e^{x_2} \pm e^{-x_2})\partial_2$	$x_1' = x_1$, $x_2' = x_2$, $x_3' = x_3 + \text{const}$	1	$\varrho = -1/2$, $\sigma = 1/2 + ik(1-k^2)^{-1/2}/2$, $\nu = -k(1-k^2)^{-1/2}/2$, $\tau = kk_{,3}/(2^{1/2}(1-k^2)) - ik_{,3}/(2^{1/2}(1-k^2)^{1/2})$	$j = -(1-k^2)^{-1/2}$, $I_1 = -2k$, $J = 0$
19	IX	$E = (2c-l)\cos^2 x_3$, $F = -m\cos x_3$, $G = l$, where $l = c - a\sin 2x_1 + b\cos 2x_1$, $m = a\cos 2x_1 + b\sin 2x_1$, $\delta = (c^2-a^2-b^2)^{1/2}$	$X_1 = \partial_2$, $X_2 = \frac{\cos x_2}{\cos x_3}\partial_1 - \text{tg}\, x_3 \cos x_2 \partial_2 + \sin x_2 \partial_3$, $X_3 = -\frac{\sin x_2}{\cos x_3}\partial_1 + \text{tg}\, x_3 \sin x_2 \partial_2 + \cos x_2 \partial_3$	$x_1' = x_1 + \text{const}$, $x_2' = x_2 + j\pi$, $x_3' = x_3(-1)^k + k\pi$ (j, k integer, $j+k$ even)	4	$\varrho = 0$, $\sigma = m/l - i(\delta^2 + cl)/(\delta l)$, $\nu = (\delta^2+cl)/(\delta l)$, $\tau = \text{tg}\, x_3 \cdot (1 - im(2lc - l^2 - m^2)^{-1/2})(2l)^{-1/2}$	$I_1 = 2c(a^2+b^2)^{-1/2}$

generators (cf. Table II). The results in Table III give a running number, the Bianchi type, normal forms of the metric tensor, the corresponding normal forms for the three generators, coordinate transformations which leave invariant either normal forms, a class number corresponding to a classification in terms of differential invariants (see Table VI), the Ricci rotation coefficients for a suitable threeleg, and the differential invariants up to second order (for the latter three entries see also the Appendix). The geometrical meaning of the vanishing of certain invariants is to some extent described in [1].

We have not included in Table III a metric of Bianchi type I. A third commuting Killing vector added to those of metric 1 of Table I is essentially a light-like Killing field, which is excluded in this section. On the other hand, metric 2 of Table I admits in fact an infinite number of commuting space-like Killing fields of the type $\xi^i = (-l(x_2)\,\alpha_{,2}, 0, \alpha)$, where $\alpha = \alpha(x_2)$ is an arbitrary function (the Killing fields listed in Table I are particular cases). Rotation coefficients and invariants are given by

$$\varrho = 0, \qquad \sigma = \frac{i}{2\sqrt{kl}}, \qquad \nu = i\sigma, \qquad \tau = \frac{il_{,2}}{2l\sqrt{2k}}, \qquad j = I_1+2 = N = 0.$$

The same metric admits for special choice of h and l — apart from the Killing vectors in Table I — a single third space-like Killing vector, corresponding to Bianchi types different from I. These particular metrics are also listed in Table III.

Most of the metrics listed in Table III admit groups G_3 which are simply transitive on the whole null hypersurface. Exceptions are the metrics listed as No 10, 17 and 18:

No 10, type VII_0 ($q = 0$):

The two-dimensional transitivity surfaces are given by $x_1 = $ const and represent Euklidean planes.

No 17, type VIII:

The two-dimensional transitivity surfaces are again given by $x_1 = $ const and represent pseudospheres.

No 18, type VIII:

The two-dimensional transitivity surfaces are the space-like surfaces

$$x_2 e^{x_1} + \int \frac{dx_3}{k(x_3)} = \text{const.}$$

2.4. Groups G_4 with space-like generators

Since every G_4 contains a G_3 as subgroup, we may take Table III as the starting point for a discussion of metrics admitting a G_4 with space-like generators, using the classification of G_4 groups provided by Petrov [5]. The result is Table IV.

2.5. Groups with a light-like generator

For an isotropic Killing field we have $\underset{1}{\xi^i} = \delta^i_1$ in the coordinate system (6). The Killing equation (8) then requires $\gamma_{AB,1} = 0$. Any further Killing field must be spacelike and has to satisfy

$$\gamma_{AB,C}\xi^C + \xi^C{}_{,A}\gamma_{CB} + \xi^C{}_{,B}\gamma_{CA} = 0. \tag{19}$$

TABLE IV

Isotropic hypersurfaces admitting an isometry group G_4 with space-like generators

	Metric	Generators	Admitted coordinate transformations	Rotation coefficients	Invariants
G_4 VII	$E = e^{2x_1}$, $F = ae^{x_1}$, $G = 1$	$X_1 = \partial_2$, $X_{2,3} = (\pm e^{-x_2} - \frac{1}{4}e^{x_2})\partial_1 + (\frac{1}{4}e^{x_2} \pm e^{-x_2})\partial_2$, $X_4 = \partial_3$	$x_1' = x_1$, $x_2' = x_2$, $x_3' = x_3 + \text{const}$	$\varrho = -\frac{1}{2}$, $\sigma = \frac{1}{2} + \frac{ik}{2\sqrt{1-k^2}}$, $\nu = \frac{-k}{2\sqrt{1-k^2}}$, $\tau = 0$	$j = -(1-k^2)^{-1/2}$, $I_1 = -2k$, $I_2 = 0$, $J = 0$
G_4 V	$E = ae^{-2x_1}$, $F = 0$, $G = ae^{-2x_1}$	$X_1 = \partial_3$, $X_2 = \partial_2$, $X_3 = \partial_1 + x_2\partial_2 + x_3\partial_3$, $X_4 = x_3\partial_2 - x_2\partial_3$	$x_1' = x_1 + \text{const}$, $x_2' = x_2$, $x_2' = x_3$	$\varrho = 1$, $\sigma = 0$, $\nu = 0$, $\tau = 0$	no invariants

TABLE V

Metrics of isotropic hypersurfaces admitting — apart from an isotropic Killing field — a G_3 with space-like generators

No	Metric	Generators	Rotation coefficients/invariants
1	$E = 1$, $F = 0$, $G = 1$	$X_0 = \varphi(x^i)\partial_1$, $X_1 = \partial_3$, $x_2 = \partial_2$, $X_3 = x_3\partial_2 - x_2\partial_3$	$\varrho = \sigma = 0$, $K = 0$
2	$E = a^2/x_3^2$, $F = 0$, $G = E$	$X_0 = \varphi(x^i)\partial_1$, $X_1 = \partial_3$, $X_2 = x_2\partial_2 + x_3\partial_3$, $X_3 = 2x_2x_3\partial_2 + [x_3^2 - x_3^2]\partial_3$	$\varrho = \sigma = 0$, $K = -1/a^2 = \text{const}$
3	$E = a^2$, $F = 0$, $G = a^2 \sin^2 x_2$	$X_0 = \varphi(x^i)\partial_1$, $X_1 = \partial_3$, $X_2 = \sin x_3\partial_2 + \text{ctg}\, x_2 \cos x_3\partial_3$, $X_3 = \cos x_3\partial_2 - \text{ctg}\, x_2 \sin x_3\partial_3$	$\varrho = \sigma = 0$, $K = 1/a^2 = \text{const}$

Equation (19) is the Killing equation of a two-dimensional surface with a positive-definite metric γ_{AB}. Here standard results apply. If a G_1 exists with a spacelike generator additionally to the isotropic Killing field, we may choose the coordinates so that $\xi^C = \delta^C_3$. From (19) one obtains the rotation surface

$$ds^2 = dx_2^2 + f(x_2)dx_3^2. \tag{20}$$

If the surface admits a G_2, it also admits a G_3 and has constant Gaussian curvature, that is, the Gaussian curvature K is independent of x_1 as well as of the two space-like directions corresponding to the coordinates x_2, x_3. The three possible types ($K > 0$, $K = 0$ and $K < 0$) are listed in Table V. (Notice that (19) determines only the two components ξ^A, and an arbitrary component ξ^1 can be added. This reflects the existence of an infinite Lie group G_∞, the total group may be denoted by $G_1 \times G_\infty$ or $G_3 \times G_\infty$, respectively.) In the case that the null hypersurface admits an isotropic Killing vector, but no G_2 or G_3 with space--like generators, the Gaussian curvature K exists as an invariant of the inner geometry (it is, in fact, the only second-order invariant). More specifically, K is an invariant function on the generator congruence, that is, K does not change along a given generator, but it may change across the generators. Thus K depends in general on the location at the slices $x_1 =$ const. An important example is the Kerr–Newman horizon with the inner metric

$$ds^2 = \frac{a^2}{l^2} \sin^2 \theta dt^2 - \frac{2ab^2 \sin^2 \theta}{l^2} dt d\theta + l^2 d\theta^2 + \frac{\sin^2 \theta b^4}{l^4} d\theta^2, \tag{21}$$

with

$$l^2 = r^2 + a^2 \cos^2 \theta,$$

$$r = M + (M^2 - Q^2 - a^2)^{1/2},$$

$$b^2 = a^2 + r^2,$$

which admits—apart from the isotropic Killing field ∂_t—a G_1 with the space-like generator ∂_ϕ. With the Boyer-Lindquist [6] coordinates (21), the calculation of K cannot be performed according to the simple rule given in the Appendix. A direct calculation gives, however, apart from $\varrho = \sigma = 0$,

$$K = \frac{(a^2 + r^2)(r^2 - 3a^2 \cos^2 \theta)}{(r^2 + a^2 \cos^2 \theta)^3}. \tag{22}$$

As noted, the Gaussian curvature K represents the only local second-order invariant of the inner geometry of the Kerr–Newman horizon.

3. *Embedding*

One could expect that the requirement of local embedding of a given three-dimensional null space into four-dimensional spacetime restrits the null space. Actually, this is in general not true: There is no *local* restriction of the *inner* geometry. Embedding into empty four dimensional spacetime means, however, that an affine parameter must be chosen in a suitable way. Within the "inner" geometry, all parameters

$$v' = v'(v, x_2, x_3) \tag{23}$$

with $\partial v'/\partial v \neq 0$ are on an equal footing. The "affine geometry" (defined in [1]) takes a certain subclass of these parameters, related by the linear transformation $v' = av + b$ ($a \neq 0$, a and b constant along the ray), and calls them "affine". In general, this subclass can be chosen in such a way, that the embedding requirements for empty spacetime are satisfied.

(Note that a null hypersurface embedded in a spacetime has always a well-determined affine geometry.) To see this notice that the relation

$$\omega \equiv D\varrho - \varrho^2 - \sigma\bar{\sigma} = \tfrac{1}{2} R_{\mu\nu} p^\mu p^\nu \tag{24}$$

for the rotation coefficients divergence ϱ and shear σ is the *only* restriction for these coefficients in a given spacetime. Thus, in an Einstein spacetime, ω must be zero. However, under a transformation (23), ω changes as

$$\omega' = \lambda^2 \omega + \varrho\lambda D\varrho, \tag{25}$$

where $\lambda = \partial v/\partial v'$.

We may therefore always select by means of (25) that subclass of parameters, for which ω' becomes zero (or equal to any prescribed value in a non-empty spacetime), provided only $\varrho \neq 0$ in the considered domain of the nullsurface. Thus — apart from metrics with $\varrho = 0$ — the majority of metrics listed in Table III may be considered as subspaces of a in general not uniquely determined vacuum field. The non-uniqueness results from the fact that to solve the characteristic initial value problem ([7]–[9]) starting from the given hypersurface as initial surface, we need in general a second initial hypersurface. This second hypersurface (on which also initial data must be given additionally and in general independently) may also be taken as null and intersects the given one in a two-dimensional space-like section. We may lift the non-uniqueness, for instance, by demanding that not only a single null hypersurface, but a whole family of null hypersurfaces admits the same symmetries as the given one. These spacetimes are discussed in a subsequent paper.

Returning to the single hypersurface discussed here it is easily seen that an affine parameter v_{aff}, belonging to embedding into vacuum, is given in terms of any parameter v by

$$v_{\mathrm{aff}} = \int_{v_0}^{v} \varrho dv \exp\left(-\int_{v_0}^{v} \varrho[1+|\sigma|^2/\varrho^2]dv\right). \tag{26}$$

v_{aff} is not uniquely determined by the inner geometry, because v_0 is arbitrary. The freedom left is just given by the linear transformation $v_{\mathrm{aff}} \to a v_{\mathrm{aff}} + b$, with a and b constant along the ray. (26) represents an (integral) invariant of the inner geometry. Thus, if one starts with any other parameter $v' = v'(v, w^A)$, one arrives at the same class of affine parameters. (26) holds as long as ϱ does not vanish in a three-dimensional region. If ϱ vanishes, σ must be zero also, in order to admit embedding into Einstein spacetime. The null spaces admitting a G_3 or G_4 belong to one of the following three types (constant means independence of v):

(a) $\varrho = \text{const} \neq 0, \quad |\sigma| = \text{const},$

(b) $\varrho = -\frac{1}{2}\dfrac{d}{dv} \ln f, \quad |\sigma| = 0,$

(c) $\varrho = 0, \quad |\sigma| = \text{const}.$

Calculating the transformed divergence ϱ_{aff} as a function of the affine parameter, the first two cases (a), (b) — those which can be embedded without restrictions — both give

$$\varrho_{\text{aff}} = 1/(1-[1+j^{-2}]\, v_{\text{aff}}) \tag{27}$$

($j^{-1} = 0$ for class (b)). Hence ϱ_{aff} encounters a singularity for a finite value v^* of v_{aff}, whereas — if one uses the original triad — ϱ need not to have a singularity: The scale transformation $v \to v_{\text{aff}}$ becomes singular at a certain value of v_{aff}. Calculating also σ_{aff}, one sees that in general this behaviour of ϱ at the singularity is different from that necessary at a regular vertex (where $|\sigma| \to 0$).

$$|\sigma_{\text{aff}}| = (v^*[1-v^*])^{1/2}/(v^*-v_{\text{aff}}). \tag{28}$$

Thus the singularities correspond to caustics. This illustrates a well-known properties of null hypersurfaces in Einstein spaces: Caustics will necessarily develop, if the null hypersurface does not admit a group of motions with light-like generators (embeddable case (c)). The problem, whether the caustics can be embedded without true local singularities in the four-dimensional spacetime, remains open.

We also mention shortly the problem of embedding into flat spacetime. The conditions for embedding into empty spacetime are necessary conditions also here. But apart from (25), we have to consider the transformation (23) also acting on the Penrose function

$$\Psi \equiv D\sigma - 2\varrho\sigma + 2i\nu\sigma, \tag{29}$$

which gives

$$\Psi' = \lambda^2\Psi + \lambda D\lambda\sigma. \tag{30}$$

Thus a necessary local condition for embedding into Minkowski space-time is $\varrho = \sigma = 0$ for a suitable class of "affine" parameters v'. From (25) and (30), this is equivalent to

$$\varrho\Psi = \sigma\omega. \tag{31}$$

Notice that (31) is an equation invariant with respect to (23). If both ϱ and $|\sigma|$ are different from zero, (31) may also be written

$$I_1 = 0, \qquad I_2 = 1/j - j. \tag{32}$$

(31) respective (32) is only a necessary condition for embedding into flat spacetime, because we cannot rely here on the characteristic initial value problem as in the previous case. The requirement that spacetime is flat everywhere restricts the initial data on the characteristic hypersurfaces beyond condition (31). For instance, if $\varrho = 0$, we need — apart from $\sigma = 0$ — also $K = 0$.

APPENDIX A

Differential invariants and rotation coefficients of isotropic hypersurfaces

The geometrical properties of lightlike hypersurfaces can be expressed in terms of its differential invariants, as described in [1]. The differential invariants can be found from the Ricci rotation coefficients ϱ (divergence), σ (shear) and τ, ν, χ and φ of the genera-

TABLE VI

Classification of isotropic hypersurfaces in terms of inner differential invariants up to second order. (See the Appendix for definition of the invariants)

Shear	Divergence		Type number	First order invariants	Second order invariants
$\lvert\sigma\rvert \neq 0$	$\varrho \neq 0$	$I_2^2+[I_1^2-4(1-j^2)]^2 \neq 0$	1	j	I, J
		$I_2 = 0, I_1^2 = 4(1-j^2)$	2	j	L, M
	$\varrho = 0$	$\lvert I_1\rvert = 2$	3	$j = 0$	N
		$\lvert I_1\rvert \neq 2$	4	$j = 0$	$I_1 = \mathrm{Re} I$
$\lvert\sigma\rvert = 0$	$\varrho \neq 0$		5	no invariants	no invariants
	$\varrho = 0$		6	no invariants	K

tor congruence. The invariants are uniquely determined by the metric γ_{ik}, whereas the rotation coefficients are not. A possible choice for the rotation coefficients in terms of γ_{AB} (in the coordinate system with $\gamma_{1A} = 0$) is listed below. A classification of null hypersurfaces according to differential invariants is given in Table VI. For details we refer to [1]. With $\gamma = |\gamma_{AB}|$ we have

$$\varrho = -\gamma'/(4\gamma)$$

$$\sigma = \gamma'/(4\gamma)-\gamma'_{33}/(2\gamma_{33})+\frac{i}{2\sqrt{\gamma}}(\gamma'_{23}-\gamma_{23}\gamma'_{33}/\gamma_{33}),$$

$$\nu = (\gamma'_{33}\gamma_{23}-\gamma_{33}\gamma'_{23})/(2\sqrt{\gamma}\,\gamma_{33}),$$

$$\tau = \frac{1}{2\sqrt{2\gamma_{33}}}(\gamma_{33,3}/\gamma_{33}-\gamma_{,3}/\gamma)+\frac{i}{2\sqrt{2\gamma\gamma_{33}}}(\gamma_{33,2}+\gamma_{23}\gamma_{33,3}/\gamma_{33}-2\gamma_{23,3}),$$

$$\chi = \varphi = 0.$$

(The prime denotes the derivative with respect to x^1.) Furthermore, the differential operators are given by

$$D = \frac{\partial}{\partial x_1},$$

$$\delta = \frac{i\sqrt{\gamma_{33}}}{\sqrt{2\gamma}}\,\partial_2+\frac{(\sqrt{\gamma}-i\gamma_{23})}{\sqrt{2\gamma\gamma_{33}}}\,\partial_3.$$

The differential invariants in Table VI are defined in terms of the rotation coefficients by (notice $e^{is} = \sigma/|\sigma|$ and the abbreviations $p = 2j-iI_1+I_2$, $q = 2j-iI_1-I_2$)

$$j = \varrho/|\sigma|,$$

$$I = I_1+iI_2 = \frac{i}{|\sigma|}\left(\frac{D\varrho}{\varrho}-\frac{D\sigma}{\sigma}\right)+\frac{2\nu}{|\sigma|},$$

$$J = e^{is/2}\left[(q\bar{q}-4)\left(\frac{\delta\sigma}{\sigma}-\frac{\delta\varrho}{\varrho}\right)+(p\bar{q}-4)\left(\frac{\delta\bar{\sigma}}{\sigma}-\frac{\delta\varrho}{\varrho}\right)+2\bar{q}(p-q)\bar{\tau}\right]$$
$$+e^{-is/2}\left[2(p-q)\left(\frac{\bar{\delta}\sigma}{\sigma}-\frac{\bar{\delta}\varrho}{\varrho}+2\tau\right)\right],$$
$$K = -\delta\tau-\bar{\delta}\bar{\tau}+2\tau\bar{\tau},$$
$$L = e^{is/2}\left[\frac{\delta\sigma}{\sigma}-\frac{\delta\varrho}{\varrho}-2\bar{\tau}\right]+e^{-is/2}\frac{p}{2}\left[\frac{\bar{\delta}\bar{\sigma}}{\bar{\sigma}}-\frac{\bar{\delta}\varrho}{\varrho}-2\tau\right],$$
$$M = -2e^{is/2}\delta j/j,$$
$$N = e^{is/2}\left[\frac{\delta\sigma}{\sigma}-\frac{\delta\bar{\sigma}}{\bar{\sigma}}-4\bar{\tau}\right]+\frac{iI_1}{2}e^{-is/2}\left[\frac{\bar{\delta}\sigma}{\sigma}-\frac{\bar{\delta}\bar{\sigma}}{\bar{\sigma}}+4\tau\right].$$

Some remarks should be added. K, L, M and N are invariants only if certain conditions are satisfied for the rotation coefficients. These conditions can be taken from Table VI. E.g., N is an invariant if and only if $|\sigma| \neq 0$, $\varrho = 0$ and $|I_1| = 2$. The listed invariants are also invariant if an inversion takes place (i.e., a coordinate transformation with $\det\left|\frac{\partial x^i}{\partial x^k}\right| < 0$), except of I_1, which changes sign. Notice further that all invariants are dimensionless quantities, with the exception of K, which has dimension (length)$^{-2}$.

APPENDIX B

Application of REDUCE

To facilitate the calculations, the algebraic formula manipulating computer system REDUCE 2 (Hearn 1973) was used. A procedure "NULLSURFACE" was written in REDUCE, with the metric tensor components γ_{22}, γ_{23}, γ_{33} as input. NULLSURFACE calculates the rotation coefficients, classifies isotropic hypersurfaces in terms of inner differential invariants according to Table VI and calculates the differential invariants. It was employed partly to check and partly to obtain the results in Table III. The calculations were carried out on the ES 1040 computer of the Leipzig University Computer Center. I am much indebted to Klaus-Peter Jann for processing the program.

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