NO-INTERACTION THEOREMS IN RELATIVISTIC PARTICLE DYNAMICS

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No-Interaction Theorems in Relativistic Action-at-a-Distance Mechanics are reviewed. The most suggestive proofs are quoted and some of them are simplified. Physical requirements which N-Particle Predictive Relativistic Dynamics should satisfy are formulated and the existing examples are discussed. Some methods of constructing examples of Predictive Relativistic Dynamics are proposed.

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1. *Introduction*

In relativistic mechanics we encounter a very interesting and yet unsolved problem: is it possible to construct a Lorentz invariant dynamics of N-point particles without field degrees of freedom? Such a dynamics would be the relativistic analogue of the non-relativistic Newtonian dynamics. The Lorentz transformation of time causes serious difficulties in constructing such a dynamics. There exist in literature theorems stating that straightforward generalization of nonrelativistic dynamics leads to a theory describing only noninteracting particles. These theorems are often called "No Interaction Theorems" (NIT).

The common opinion is that the relativistic dynamics must contain field degrees of freedom; one often says that there must exist a physical object which transfers interaction with the velocity of light. Therefore one tends to believe that NIT's are physically plausible. There does not exist, however, a theorem completely excluding dynamics without field, so called Action-at-a-Distance Mechanics — see the collection of papers by Kerner [1]. On the contrary, consistent models of this kind do exist; for instance the model of Wheeler and Feynman [2], which is a particular case of a more general model given by Van Dam

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and Wigner [3, 37]. In these models the Lorentz invariant equations of motion are integro--differential equations, as in the case of the Wheeler and Feynman model. Unfortunately, such models cannot be formulated in the Hamiltonian form.

The following question thus arises: can a system of N-particles in Action-at-a-Distance Theory be described by Newtonian-like ordinary differential equations? Some special examples of such dynamics do exist [5–7, 34–36]; Droz-Vincent [8] and Bel [9–11] developed a general mathematical formalism to deal with what they call the Predictive Relativistic Dynamics (PRD). In this dynamics the motions of particles are described by a $6N$--parameter family of trajectories which has the property that from a solution in one inertial frame we can obtain a solution in another inertial frame by changing properly the values of parameters. Thus the principle of relativity is satisfied, no inertial frame is distinguished, in each frame equations of motion have the same form. The requirements of the principle of relativity are, however, very strong, so if other physical requirements are added the dynamics may easily turn out to describe only free particles. Unfortunately, we do not have examples satisfying all these requirements. We discuss this problem in detail in the last section.

The aim of the first part of this paper is to give a review of the No-Interaction Theorems. We think that it may be valuable to have such a review because there exist in the literature many forms of these theorems, which use different assumptions and different methods of proofs. It happens that the best ones are not given in separate papers [6, 12] but constitute a small part in papers devoted to broader problems, so frequently they are not known. The best known NIT — formulated by Currie, Jordan and Sudarshan — has a very long proof and one can hardly see which assumptions are crucial in the theorem. We hope that this part of the paper will be useful for those readers who do not know the whole literature of the problem and want to have a general outlook.

The paper is organized as follows: in Section 2 we present the derivation of the Currie--Hill equations which guarantee the Lorentz invariance of Newtonian equations of motions. In Sections 3,4 and 5 we quote almost all NIT's and give the most suggestive proofs. Some of them we give in a version slightly simpler than the original one. In Section 6 we formulate the physical requirements which PRD should satisfy.

The whole paper is devoted to "instantaneous" form of Action-at-a-Distance Theory, in which all physical quantities are taken at one instant of time in each inertial frame. Obviously, in this case each inertial observer describes his own set of events, which complicates the relations between physical variables but does not exclude interaction. In the last section we give some information about noninstantaneous PRD.

2. *The Lorentz transformation of positions and velocities in instantaneous relativistic dynamics and Lorentz invariant Newtonian equations*

We shall try to describe an N-body relativistic system in the instantaneous formalism in which all variables in each inertial frame are taken at one instant.

Let the trajectory of the n-th particle in the inertial system S be described by the functions $x_i^n(t_n)$, $i = 1, 2, 3$, $n = 1, 2, \ldots, N$. The Loretnz transformations say that in the system moving with infinitesimal velocity $\varepsilon \ll 1$[1] directed along j axis, the trajectory is described by the functions $x_i'^n(t_n')$ determined by the equations:

$$x_i'^n(t_n') = x_i^n(t_n) - \varepsilon\delta_{ij}t_n, \tag{1}$$

$$t_n' = t_n - \varepsilon x_j^n. \tag{2}$$

We put (2) into (1) and make the Taylor expansion around $t_n' = t_n$. Thus we get

$$x_i'^n(t_n) - \varepsilon v_i'^n(t_n)x_j^n = x_i^n(t_n) - \varepsilon\delta_{ij}t_n,$$

where $\frac{dx_i'^n}{dt_n} = v_i'^n$. Since we neglect the terms quadratic in ε we can put in the last equation v_i^n instead $v_i'^n$. Thus by putting $t_n = t$ we get

$$x_i'^n(t) = x_i^n(t) + \varepsilon[v_i^n(t)x_j^n(t) - \delta_{ij}t]. \tag{3}$$

This is the desired relation between positions of all particles calculated at one instant in both frames. We illustrate it in Fig. 1.

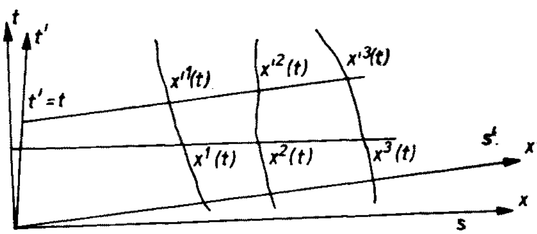

Fig. 1. Illustration of instantaneous description of particle trajectories in different inertial systems

Differentiating the equation (3) with respect to time we get the velocity transformation formula:

$$v_i'^n = v_i^n + \varepsilon(a_i^n x_j^n + v_i^n v_j^n - \delta_{ij}), \tag{4}$$

where accelerations a_i^n are treated as given functions of the particle positions and velocities i.e. they are treated as "forces". Similarly, the next differentiaton gives:

$$\frac{dv_i'^n}{dt} = \frac{dv_i^n}{dt} + \varepsilon(\dot{a}_i^n x_j^n + 2a_i^n v_j^n + a_j^n v_i^n), \tag{5}$$

where

$$\dot{a}_i^n = \frac{\partial a_i^n}{\partial x_s^m} v_s^m + \frac{\partial a_i^n}{\partial v_s^m} a_s^m. \tag{6}$$

The Newtonian equations of motion are

$$\frac{dv_i^n}{dt} = a_i^n(\vec{x}^1, \ldots, \vec{x}^N, \vec{v}^1, \ldots, \vec{v}^N). \tag{7}$$

[1] We use the system of units in which $c = 1$.

In order to guarantee translational and rotational invariance we require that a_i^n depend on the relative positions of particles and form a vector with respect to rotations[2].

If we assume that these equations constitute the fundamental law of the considered dynamics, then we must require that they have the same form in the moving inertial frame S' i.e. we must have

$$\frac{dv_k'^n}{dt} = a_i^n(\vec{x}'^1, \ldots, \vec{x}'^N, \vec{v}'^1, \ldots, \vec{v}'^N), \tag{8}$$

where a's are the same functions of their arguments as in (7).

Since (8) ought to describe the same motion, we have to be able to derive (7) from (8) by applying Eqs (3), (4), (5), (6) and making Taylor expansion of functions $a_i^n(x', v')$. This will occur only if the "forces" a_i^n satisfy the following system of $9N$ quasi-linear partial differential equations:

$$\sum_{m=1}^{N} \sum_{s=1}^{3} \left\{ v_s^m(x_j^n - x_j^m) \frac{\partial a_i^n}{\partial x_s^m} + [a_s^m(x_j^n - x_j^m) + \delta_{sj} - v_j^m v_s^m] \frac{\partial a_i^n}{\partial v_s^m} \right\}$$
$$+ 2v_j^n a_i^n + x_i^n a_j^n = 0, \quad n = 1, \ldots, N, \quad i = j = 1, 2, 3. \tag{9}$$

These equations are frequently called Currie–Hill equations [17, 18]. Bel proved [9] that these equations constitute the necessary and sufficient conditions which guarantee that the dynamics is Lorentz invariant with respect to the finite Lorentz transformations. Our consideration proves only that they are necessary conditions.

The problem of existence of physically meaningful solutions of these equations is still open and we postpone its discussion to Section 6.

Equations (9) constitute certainly a very strong condition on possible "forces" $\vec{a}^n$ and we shall see in subsequent sections that one has to be careful about adding further requirements.

3. Can the total kinematical momentum be a constant of motion in the relativistic dynamics?

We start our review with the simplest theorem given by Beard and Fong [15] and Van Dam and Wigner [4] in the case of non-instantaneous dynamics.

Let us make the following assumptions:

a) the total kinematical four-momentum $(\vec{P}, E)$ of the two-body system is a Lorentz four-vector and a constant of motion, i.e. we have:

$$\vec{P} = \vec{p}^1 + \vec{p}^2, \quad E = E^1 + E^2 + V, \quad \frac{dP}{dt} = 0, \quad \frac{dE}{dt} = 0;$$

[2] $\vec{a}^n$ must have the following form:

$$\vec{a}^n = \sum_{n,m} b_{nm}(\vec{x}^n - \vec{x}^m) + \sum_n c_n \vec{v}^n$$

where b_{nm}, c_n are functions of all possible scalars with respect to rotations.

b) the individual particle momenta (p^1, E^1), (p^2, E^2) are four-vectors.

The theorem states: $\frac{dp^1}{dt} = \frac{dp^2}{dt} = 0$, i.e. the particles do not interact.

Proof.

It is enough to use the fact that the transverse part of momentum is the same in all inertial frames[3]. Let us consider the system S which is moving with respect to the system S' with the velocity $\varepsilon \ll 1$ along j-axis. Thus we have

$$p_i'^n(t_n') = p_i^n(t), \tag{10}$$

$$t_n' = t - \varepsilon x_j^n, \tag{11}$$

for $j \neq i$, $j = 1, 2, 3$, $i = 1, 2, 3$, $n = 1,2$.

We put (11) into (10) and make the Taylor expansion around $t_n' = t$, thus we have

$$p_i'^n(t) = p_i^n(t) + \frac{dp_i^n}{dt} x_j^n \varepsilon, \quad j \neq i. \tag{12}$$

Since $(\vec{P}, E)$ is a four-vector and a constant of motion, we have $P_i = P_i'$ for each t and $i \neq j$

$$p_i^1(t) + p_i^2(t) = p_i^1(t) + p_i^2(t) + \left(\frac{dp_i^1}{dt} x_j^1 + \frac{dp_i^2}{dt} x_j^2\right) \varepsilon.$$

Because, from $d\vec{P}/dt = 0$,

$$\frac{dp_i^1}{dt} = -\frac{dp_i^2}{dt}$$

we have

$$\frac{dp_i^1}{dt}(x_j^1 - x_j^2) = 0 \quad \text{for} \quad j \neq i \tag{13}$$

which means that

$$\frac{dp_i^1}{dt} = \frac{dp_i^2}{dt} = 0.$$

The theorem obviously does not hold in the case of one dimensional motion (we have then $x_j^1 = x_j^2 = 0$ for $j \neq i$ and Eq. (13) is automatically satisfied). Beard and Fong have shown that in that case the only possible motion results from the potential $V = \text{const} \cdot |x^1 - x^2|$ which describes for instance the motion of two charged planes moving along the lines of the electrid field. We may also have the canonical formalism describing such motions as stated in Ref. [15] and [16, 38].

From this theorem the following conclusions follow: in the relativistic mechanics the total kinematical momentum must contain a"potential". Only nonrelativistic mechanics

[3] We give here a proof slightly simpler than the original one.

allows an asymmetry between energy and momentum of the system which can be simply the sum of individual particle momenta. We see also that conservation of the kinematical momentum may hold only asymptotically in collision processes, if we require that the "momentum potential" vanishes for large distances between particles. The value of the total momentum in bound motions depends on dynamics while in the nonrelativistic dynamics it depends only on the initial conditions.

4. *Can the relativistic N-body system be described by the canonical formalism?*

The assumptions of previous theorem were not made in the most famous NIT of Currie, Jordan and Sudarshan [13] but they made other very strong ones: they assumed that the Lorentz transformations written in the form (3), (4), translations and space rotations are canonical symmetry transformations with generators satisfyining the Poincaré group Poisson bracket relations. Furthermore, they assumed — and this appears to be the crucial assumption — that the particle positions are canonical variables. In the proof they used almost all Poincaré-group Poisson bracket relations and made several re-definitions of particle canonical momenta which led to the free particle form of the two body Hamiltonian. Leutwyler [14] generalized their result to the case of N-particles.

We make the following assumptions:

a) the particle positions are canonical variables, i.e.

$$[x_i^n, x_s^m] = 0 \tag{14}$$

for each $n, m = 1, 2, \ldots, N$, $s = 1, 2, 3$,

b) the Lorentz transformations of particle positions (3) are canonical transformations, i.e. there exist in the phase space three functions $K_j(\vec{x}^1, \ldots, \vec{x}^N, \vec{p}^1, \ldots, \vec{p}^N, t)$ such that

$$[x_i^n, K_j] = v_i^n x_j^n - \delta_{ij} t, \tag{15}$$

$$[v_i^n, K_j] = a_i^n x_j^n + v_i^n v_j^n - \delta_{ij}, \tag{16}$$

where

$$v_i^n = [x_i^n, H], \quad a_i^n = [v_i^n, H] \tag{17}$$

and $H = H(\vec{x}^1, \ldots, \vec{x}^N, \vec{p}^1, \ldots, \vec{p}^N)$ is the Hamiltonian of the system,

c) the mechanics is nondegenerate which means that the transition from x_i^n, v_i^n to phase space variables x_i^n, p_i^n is nonsingular i.e. $\det(\partial v_i^n/\partial p_s^m) \neq 0$ or $\det(\partial^2 H/\partial p_i^n \partial p_s^m) \neq 0$. This allows to make the transition from equations $\dot{v}_i^n = [v_i^n, H]$ to $3N$ Newtonian equations

$$\dot{v}_i^n = a_i^n(\vec{x}^1, \ldots, \vec{x}^N, \vec{v}^1, \ldots, \vec{v}^N).$$

Theorem: The acceleration of n-th particle can be only a function of its velocity;

$$a_i^n = a_i^n(\vec{x}^n, \vec{v}^n), \qquad \frac{\partial a_i^n}{\partial x_s^m} = \frac{\partial a_i^n}{\partial v_s^m} = 0 \quad \text{for } n \neq m$$

i.e. the particles do not interact.

Note: If we add the requirement that the accelarations satisfy the Currie–Hill equations (9) one can easily verify that a_i^n must be zero, so the "self-acceleration" can be easily eliminated by the postulate of Lorentz invariance of Newtonian equations of motion.

The proof[4] consists in multiple usage of the relations $[x_i^n, x_s^m] = 0$, Eq. (15), (16) and the Jacobi identity. Let us first prove the following lemma:

$$[x_i^n, a_s^m] = 0, \tag{18}$$

$$[v_i^n, a_s^m] = 0, \tag{19}$$

for $n \neq m \quad n, m = 1, 2, \ldots, N$.

One can prove the following series of relations:

$$[x_i^n, v_s^m] = [x_s^m, v_i^n], \tag{20a}$$

$$[x_i^n, v_s^n] = 0, \tag{20b}$$

$$[v_i^n, v_s^m] = [x_s^m, a_i^n] = -[x_i^n, a_s^m], \tag{20c}$$

$$[v_i^n, v_s^m] = 0, \tag{20d}$$

$$[v_i^n, a_s^m] = 0. \tag{20e}$$

Relations (20b)–(20e) are valid only for $n \neq m$. Below we give the proof of (20a), (20b) in detail, the next relations can be proved by using the same method and the previously shown formulas, the order (20a)–(20e) is important in the proof

$$[x_i^n, v_s^m] = [x_i^n, [x_s^m, H]] = -[H, [x_i^n, x_s^m]] - [x_s^m, [H, x_i^n]] = [x_s^m, v_i^n].$$

In order to show (20b) we rewrite equation (15) with interchanged indices m, s and n, j:

$$[x_s^m, K_j] = v_s^m x_j^m - \delta_{sj} t. \tag{21}$$

The Poisson bracket of x_i^n with both sides of Eq. (21) and the Poisson bracket of x_s^m with both sides of Eq. (15) give

$$[x_i^n, [x_s^m, K_j]] = [x_i^n, v_s^m] x_j^m + v_s^m [x_i^n, x_s^m], \tag{22}$$

and

$$[x_s^m, [x_i^n, K_j]] = [x_s^m, v_i^n] x_j^n + v_i^n [x_s^m, x_j^n]. \tag{23}$$

Subtracting Eq. (22) from Eq. (23), and using relations (14), (20a) and the Jacobi identity one gets

$$(x_j^m - x_j^n)\,[x_i^n, v_s^m] = 0$$

which means that $[x_i^n, v_s^m] = 0$ for $n \neq m$.

Relations (20e) follow directly from (20b) and Eq. (14). Formulas (20d), (20e) can be deduced by evaluating the appropriate Poisson brackets of positions and velocities with (15), (16) (and using all relations obtained earlier). Now, one can easily show that

[4] The proof is based on Ref. [6].

in the case of nondegenerate mechanics the relations (20b), (20c), (20d), (18) and (19) give the theorem. Namely, from Eq. (20b) we deduce that

$$\frac{\partial v_s^m}{\partial p_i^n} = 0 \quad \text{for } m \neq n \tag{24}$$

so

$$\det\left(\frac{\partial v_s^m}{\partial p_i^n}\right) = \det\left(\frac{\partial v_s^1}{\partial p_i^1}\right) \cdot \det\left(\frac{\partial v_s^2}{\partial p_i^2}\right) \dots \det\left(\frac{\partial v_s^N}{\partial p_i^N}\right) \neq 0. \tag{25}$$

Thus $\det\left(\frac{\partial v_s^m}{\partial p_i^m}\right) \neq 0$ for each $m = 1, 2, \dots, N$. From (18) and (24) follows that:

$$0 = \frac{\partial a_s^m(x, p)}{\partial p_i^n} = \sum_{r=1}^{3} \frac{\partial a_s^m(x, v)}{\partial v_r^n} \frac{\partial v_r^n}{\partial p_i^n} \quad \text{for } n \neq m.$$

Finally, with the help of (24) one obtains

$$\frac{\partial a_s^m}{\partial v_r^n} = 0 \quad \text{for } n \neq m. \tag{26}$$

Similarly, from (20), (23), (24) one gets

$$0 = [v_i^n, a_s^m] = \sum_{r=1}^{3} \sum_{q=1}^{N} \left[\frac{\partial v_i^n}{\partial x_r^q} \frac{\partial a_s^m(x, p)}{\partial p_r^q} - \frac{\partial v_i^n}{\partial p_r^q} \frac{\partial a_s^m(x, p)}{\partial x_r^q}\right]$$

$$= \sum_{r=1}^{3} \left[\frac{\partial v_i^n}{\partial x_r^m} \frac{\partial a_s^m(x, p)}{\partial p_r^m} - \frac{\partial v_i^n}{\partial p_r^n} \frac{\partial a_s^m(x, p)}{\partial x_r^n}\right]$$

$$= \sum_{t=1}^{3} \sum_{q=1}^{N} \frac{\partial a_s^m}{\partial v_t^q} [v_i^n, v_t^q] - \sum_{r=1}^{3} \frac{\partial v_i^n}{\partial p_r^n} \frac{\partial a_s^m(x, v)}{\partial x_r^n} = - \sum_{r=1}^{3} \frac{\partial v_i^n}{\partial p_r^n} \frac{\partial a_s^m}{\partial x_r^n} = 0, \quad \text{for } n \neq m.$$

Since the matrices $\partial v_i^n / \partial p_r^n$ are nonsingular for each $n \neq m$, we obtain

$$\frac{\partial a_s^m}{\partial x_r^n} = 0 \quad \text{for } n \neq m.$$

This proof is instructive as it suggests how to introduce an interaction into the canonical formalism. The point is that one has to drop the requirement (14) which implies that the positions of particles are canonical variables. This was for the first time proposed by Kerner [16, 38], further considerations of this problem can be found in Ref. [6, 7, 10, 11, 20, 26].

Other forms of this theorem can be found in the papers of Bel [10], written in the language of modern differential geometry and in the paper of Kerner [19], whose proof seems to be the most simple one.

Kerner considers motion of two particles in one dimension; the Lagrangian is $L(x, \dot{x}^1, \dot{x}^2)$, where $x = x^1 - x^2$ and $\dot{x}^1, \dot{x}^2$ are velocities (this is equivalent to the description of motion with the help of the Hamiltonian, if the mechanics is nondegenerate, i.e. the assumption (c) is satisfied). In this case the assumption (a) is automatically fulfilled; the Lorentz transformations (3) and (4) will be symmetry transformations if a function $G(x, \dot{x}^1, \dot{x}^2)$ exists such that

$$L(x', \dot{x}'^1, \dot{x}'^2) = L(x, \dot{x}^1, \dot{x}^2) + \frac{d}{dt} G(x, \dot{x}^1, \dot{x}^2), \tag{27}$$

where $x, \dot{x}^1, \dot{x}^2$ are given by Eqs. (3) and (4), which in this case have the form

$$x'^n = x^n + \varepsilon(x^n \dot{x}^n - t), \tag{28}$$

$$\dot{x}'^n = \dot{x}^n + \varepsilon(x^n \ddot{x}^n + \dot{x}^n \dot{x}^n - 1), \quad n = 1, 2. \tag{29}$$

This requirement is so strong that (25), (26), and (27) allow only the following form of the Lagrange function

$$L = \alpha \sqrt{1 - \dot{x}^1 \dot{x}^1} + \beta \sqrt{1 - \dot{x}^2 \dot{x}^2} + \gamma |x|, \tag{30}$$

where α, β, γ are constants.

This Lagrangian describes either free particles or the case of constant forces mentioned in Section 3. Proof of this result is given in the Appendix.

5. Can the Lorentz invariant systems be described by a Hamiltonian?

This section is devoted to the theorem first formulated by Jordan [22] and later by Bel [10]. The following idea led to the formulation of this theorem: the requirement (b) of the previous theorem which says that Lorentz transformations (3) and (4) are to be canonical seems to be too strong, it would be sufficient to derive from the Hamiltonian the Lorentz invariant Newtonian equations $dv_i^n/dt = a_i^n(\vec{x}^1, \dots \vec{x}^N, \vec{v}^1, \dots, \vec{v}^N)$ where $a_i^n = [v_i^n, H]$ and particle momenta are expressed by particle positions and velocities.

The assumptions of this new NIT which is proved for the case of two particles only, are the following:

(a) x_i^n are canonical variables, $n = 1, 2$,

(b) rotations, space and time translation are canonical symmetry transformations,

(c) the generators J_i, P_i and H (it follows from the assumption (b) that they are constants of motion) transform inertial frame like angular momentum and energy of a single particle, i.e. the whole system behaves like a single particle with angular momentum J_i, momentum P_i and energy H[5].

Thesis: The particles do not interact.

[5] This is only the physical meaning of assumption (c); details may be found in [10] and [22].

The proof is rather tedious, it requires many calculations and therefore we shall not repeat it here.

The general conclusion of this theorem may be stated as follows: if we try to describe the two-body system with the help of a Hamiltonian $H(x, p)$ or a Lagrangian $L(x, \dot{x})$, we are not allowed to identify the space-time translations generators with momentum and energy and the rotations generators with the angular momentum of the system. We shall see in Section 7 that despite these difficulties it may be convenient to use a Lagrangian to calculate the forces $\vec{a}^n$.

6. *Physical requirements which Predictive Relativistic Dynamics should satisfy*

In previous Sections we have seen that formalisms used in nonrelativistic mechanics are not able to describe relativistic systems. The following question arises: what is the minimal set of physical requirements that a PR Dwith the forces $\vec{a}^n$, fulfilling the Currie–Hill equations should satisfy?

In our opinion it would be hard to resign from the following requirements:

I. $a_i^n \xrightarrow{|\vec{x}^n| \to \infty} 0$ other $\vec{x}^m$ being fixed i.e. a_i^n should tend to zero when the n-th particle is far away from all other particles. It seems that the "forces" a_i^n should vanish at least as fast as $1/|\vec{x}^n - \vec{x}^m|^2$;

II. $a_i^n \xrightarrow{|\vec{v}^n| \to 1} 0$ rapidly enough to ensure the particles do not exceed the velocity of light,

III. The Newtonian equations of motion $d^2 x_i^n/dt^2 = a_i^n$ have the integrals of motion $H, \vec{P}, \vec{J}$ with the following properties:

(a) they should reduce to the free particle form of energy, momentum and angular momentum when all relative distances between particles tend to infinity,

(b) they transform under Poincaré group of transformations like energy, momentum and angular momentum of a single particle,

IV. The forces a_i^n should have appropriate symmetry properties with respect to interchange of variables of particles, i.e. they should satisfy the relations:

$$\begin{aligned} &a_i^n(\vec{x}^1, \dots, \vec{x}^n, \dots, \vec{x}^m, \dots, \vec{x}^N, \vec{v}^1, \dots, \vec{v}^n, \dots, \vec{v}^m, \dots, \vec{v}^N, m_1, \dots, m_N) \\ &= a_i(\vec{x}^1, \dots, \vec{x}^m, \dots, \vec{x}^n, \dots, \vec{x}^N, \vec{v}^1, \dots, \vec{v}^m, \dots, \vec{v}^n], \dots, \vec{v}^N, m_1, \dots, m_N) \end{aligned}$$

for $m > n$.

V. The dynamics should have a nontrivial non relativistic limit, i.e. for velocities small with respect to the velocity of light the constants of motion $H, \vec{P}, \vec{J}, \vec{K}$ should have the form (in the case of two particles)

$$H = \frac{m_1}{2} \dot{\vec{x}}^1 \dot{\vec{x}}^1 + \frac{m_2}{2} \dot{\vec{x}}^2 \dot{\vec{x}}^2 + V(|\vec{x}|),$$

$$\vec{P} = \vec{p}_1 + \vec{p}_2,$$

$$\vec{J} = \vec{x}^1 \times \vec{p}^1 + \vec{x}^2 \times \vec{p}^2,$$

$$\vec{K} = m_1 \vec{x}^1 + m_2 \vec{x}^2 - \vec{P} t \quad \text{and} \quad V \neq 0.$$

VI. The forces $\vec{a}^n$ should, be continuous functions of variables $\vec{v}^1, \ldots, \vec{v}^n, \vec{x}^1 - \vec{x}^2, \vec{x}^1 - \vec{x}^3, \ldots$ for all their physical values i.e. $0 \leqslant |\vec{v}^n| < 1$, $0 < |\vec{x}^n - \vec{x}^m| < \infty$ $n, m = 1, 2, \ldots, N$.

Unfortunately, an exact solution of equations (9) satisfying all these conditions does not exist. There exist two simple solutions of Eqs. (9) in the case of one dimensional motion. In that case Eqs (9) reduce to:

$$(1-v^1v^1)\frac{\partial a^1}{\partial v^1} + (1-v^2v^2+xa^2)\frac{\partial a^1}{\partial v^2} - v^2x\frac{\partial a^1}{\partial x} + 3v^1a^1 = 0,$$

$$(1-v^2v^2)\frac{\partial a^2}{\partial v^2} + (1-v^1v^1-xa^1)\frac{\partial a^2}{\partial v^1} - v^1x\frac{\partial a^2}{\partial x} + 3v^2a^2 = 0. \tag{31}$$

Their known solutions have the form (6), (7)

$$a^2 = -a^1 = \frac{(v^1-v^2)^2}{2x}, \tag{32}$$

and the other one

$$a^1 = (1-v^1v^1)\frac{v^2-v^1}{v^2x}, \quad a^2 = (1-v^2v^2)\frac{v^2-v^1}{v^1x}. \tag{33}$$

Both solutions do not satisfy the assumption I which is probably the main cause that they do not satisfy the assumption IIIa.

The solution (32) does not satisfy the condition II.

Hirondel [32] found a three dimensional counterpart of the example (33):

$$\vec{a}^1 = \frac{(1-\vec{v}^1\cdot\vec{v}^1)(\vec{v}^2-\vec{v}^1)}{\vec{x}\cdot\vec{v}^2}, \quad \vec{a}^2 = \frac{(1-\vec{v}^2\cdot\vec{v}^2)(\vec{v}^2-\vec{v}^1)}{\vec{x}\cdot\vec{v}^1}.$$

It does not satisfy the condition VI because of singularities for $\vec{x}$ perpendicular to velocities.

Bel [33] found a complicated solution of Currie–Hill equation (9) which probably does not satisfy the condition III, it seems to be difficult to find physical energy, momentum and angular momentum for the dynamics described by this solution.

There exist solutions of Eqs. (9) obtained with the help of perturbation methods [23, 18, 19] but the problem of convergence of the perturbation series for all values of variables $\vec{x}, \vec{v}^m$ $m = 1, 2, \ldots, N$ is not clear and seems to be difficult.

In our opinion the question of existence of nonzero "forces" $\vec{a}^n$ satisfying conditions I–VI is very important in PRD; having no exact example of a PRD one may suspect that it can describe free particles only. One can suspect also that giving up some of the conditions I–III, one will have problems with the definitions of physical energy, momentum and angular momentum of the whole system. The condition IIIb is probably not sufficient to define these quantities uniquely.

If we drop the condition VI we shall have, for given "forces" $\vec{a}^n$, a dynamics in which not all initial conditions are allowed, for instance

$$a^n \sim \sqrt{\frac{|\vec{x}^1-\vec{x}^2|}{l_0}-1},$$

where l_0 — some constant. Driver [24] suggests that such a case may easily occur.

7. Is it possible to construct physical examples of PRD?

The following problem remains to be solved in Predictive Relativistic Dynamics: do the No-Interaction Theorems result from the limitations imposed by canonical formalism or even the Newtonian-like dynamics cannot describe interacting relativistic particles? We hope that one can get an answer to this interesting question by trying to construct an algorithm which would allow us to construct physical examples of PRD. We propose two methods of constructing such examples.

The first method is based on the papers by Pauri and Prosperi [25, 26] who developed the idea of Bakamjian and Thomas [27].

Let us consider the system of two particles described by some fictitious external $\vec{Q}, \vec{P}$ and internal $\vec{\varrho}, \vec{\pi}$ canonical variables. If we do not require that there exists a canonical transformation between $\vec{Q}, \vec{P}, \vec{\varrho}, \vec{\pi}$ and $\vec{x}^1, \vec{x}^2, \vec{p}^1, \vec{p}^2$ variables, where $\vec{x}^1, \vec{x}^2$ are positions of particles and $\vec{p}^1, \vec{p}^2$ their kinematical momenta, we can construct a Poincaré invariant dynamics of this system simply with the help of the following generators:

$$\begin{aligned}
\vec{T} &= \vec{P}, \\
H &= \sqrt{\vec{P}^2+M^2(\vec{\varrho}, \vec{\pi})}, \\
\vec{J} &= \vec{Q}\times\vec{P}+\vec{\varrho}\times\vec{\pi}, \\
\vec{K} &= M\vec{Q}+\frac{\vec{P}\times(\vec{\varrho}\times\vec{\pi})}{M+H},
\end{aligned}$$

where $\vec{T}$ is the translation generator, $\vec{K}$ the generator of Lorentz transformations and

$$M = \sqrt{\vec{\pi}^2+m_1^2}+\sqrt{\vec{\pi}^2+m_2^2}+U(\vec{\varrho}, \vec{\pi})$$

is the invariant mass of the system and U an arbitrary "potential" satisfying the condition

$$U(\vec{\varrho}, \vec{\pi}) \xrightarrow{|\vec{\varrho}|\to\infty} 0.$$

Now, the most difficult problem is to construct the physical positions of particles $\vec{x}^n = \vec{x}^n(\vec{Q}, \vec{P}, \vec{\varrho}, \vec{\pi})$, $n = 1,2$, which would transform properly under a Poincaré group, namely they should satisfy the following relations:

$$[x_i^n, P_j] = \delta_{ij}, \tag{34}$$

$$[x_i^n, J_j] = \varepsilon_{ijk}x_k^n, \tag{35}$$

$$[x_i^n, K_j] = x_j^n[x_i^n, H]. \tag{36}$$

The first two relations are easily satisfied, the third constitutes a system of quasi linear partial differential equations, whose solutions should coincide in the limit $|\vec{\varrho}| \to \infty$ with known functions $\vec{x}^n(\vec{Q}, \vec{P}, \vec{\varrho}, \vec{\pi})$ for free particles [25, 26]. If, for given $U(\vec{\varrho}, \vec{\pi})$ symmetric with respect to the interchange of particles, it is possible to find the appropriately smooth functions $\vec{x}^n(\vec{Q}, \vec{P}, \vec{\varrho}, \vec{\pi})$ and $\vec{v}^n(\vec{Q}, \vec{P}, \vec{\varrho}, \vec{\pi}) = [\vec{x}^n, H]$ such that the mapping of $\vec{Q}, \vec{P}, \vec{\varrho}, \vec{\pi}$ into $\vec{x}^1, \vec{x}^2, \vec{v}^1, \vec{v}^2$ is reversible in the whole phase space and its image covers all physically possible values of variables $\vec{x}^1, \vec{x}^2, \vec{v}^1, \vec{v}^2$ (i.e. $0 < |x^n| < \infty$, $|\vec{v}^n| < 1$), then the conditions I–VI would be automatically satisfied. It seems that this way of seeking physical examples of PRD is simpler than looking for solutions of the Currie–Hill equations satisfying the conditions I–VI, since in the latter case, having forces a_i^n we must solve very complicated equations which guarentee the existence of energy, momentum and angular momentum [10, 11]. In Ref. [25] the authors found solutions of (34)–(36) with the help of expansions in powers $1/c^2$, which does not give a decisive answer to our question.

The second method of constructing physical examples of PRD may be based on the Lagrange formalism, in which according to the theorem presented in Section 5 the Lagrangian cannot give physical energy, momentum and angular momentum in the usual form.

For simplicity reason let us consider one dimensional motion. We know that the conditions (27)–(29) from Section 4 lead to the case of free particles. We relax the condition (27), assuming that

$$L(x', \dot{x}^{1'}, \dot{x}^{2'}) = L(x, \dot{x}^1, \dot{x}^2) + \frac{d}{dt} G + \Delta L(x, \dot{x}^1, \dot{x}^2). \tag{37}$$

Both $L(x', \dot{x}^{1'}, \dot{x}^{2'})$ and $L(x, \dot{x}^1, \dot{x}^2)$ should give the same motion, hence

$$\left(\frac{d}{dt}\frac{\partial}{\partial \dot{x}^m} - \frac{\partial}{\partial x^m}\right)\Delta L = \sum_{s=1}^{2} A_{ms}(x, \dot{x}^1, \dot{x}^2)\left(\frac{d}{dt}\frac{\partial}{\partial \dot{x}^s} - \frac{\partial}{\partial x^s}\right)L, \tag{38}$$

where $A_{ms}(x, \dot{x}^1, \dot{x}^2)$ are such arbitrary functions that the matrix $A_{ms}(x, \dot{x}^1, \dot{x}^2)$ is nonsingular.

If we find a Lagrangian satisfying equations (38) and such that the matrix $\frac{\partial^2 L}{\partial \dot{x}^n \partial \dot{x}^m}$ is nonsingular, we may calculate with the help of Euler–Lagrange equations the force $\vec{a}^n$ which would satisfy the Currie–Hill equations. One might hope that Eqs (38), being linear partial-differential equations, will be easier to handle than the Currie–Hill equations. The simplest solution of (38) is $L = \frac{x}{2}(\dot{x}^1\dot{x}^1 - \dot{x}^2\dot{x}^2)$ which leads just to the example (32). Further investigation of these two methods will be presented in a separate paper.

8. *Concluding remarks*

We have considered the instantaneous form of PRD only. There exist covariant versions of PRD in which the canonical variables are four-vectors constrained by some covariant relations [21, 28–31, 40] which appropriately reduce the $8N$ parameter family of trajectories (or even larger family [21]) to $6N$ parameter family. We shall mention only one very simple example of non-instantaneous dynamics based on the Fokker action principle, in which position four-vector lies on the upper part of the light come, whose vertex coincides with one particle [5, 34–36]. Thus is a two-body electrodynamic problem, in which one particle moves in the retarded field of the second particle, while the second particle moves in the advanced field of the first particle. Obviously, such a dynamics is not symmetric with respect to the interchange of particles and thus, if for this case instantaneous "forces" $\vec{a}^n$ exist, they do not satisfy the symmetry condition IV. Staruszkiewicz [5, 34–36] has shown that this dynamics admits the Hamiltonian form with the canonical positions of particles taken on the light cone; this example shows that No-Interaction Theorem holds only in the case of instantaneous form of PRD. Unfortunately, this example of PRD can describe only the two particle system, it is, in general, impossible to describe in this way the three-particle system. (When the second particle lies on the light cone of the first one and the third particle lies on the light cone of the second one, then the third particle does not have to lie on the light cone of the first one.)

In Section 6 we formulated the requirements which physical examples of relativistic N-body dynamics should satisfy. We do not think that each such example must describe real physical objects, but it may be physically interesting to know the whole class of solutions of the Currie–Hill equations satisfying our requirements. We do not think that all physically interesting examples of PRD must have their counterpart in the field theory.

It seems that the non-instantaneous versions may be helpful in solving the formulated questions, but the presence of nonholonomic constraints in the formalism complicates the problem.

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APPENDIX

Kerner's proof of the No-Interaction Theorem

We give here a proof of the statement, that the Lagrangian satisfying Eqs (27), (28) and (29) must have the form

$$L = \alpha\sqrt{1-\dot{x}^1\dot{x}^1}+\beta\sqrt{1-\dot{x}^2\dot{x}^2}+\gamma x.$$

Let us make the Taylor expansion in the left-hand side of Eq. (27) using Eq. (28) and (29). We obtain

$$\frac{\partial L}{\partial x}(\dot{x}^1\dot{x}^1-\dot{x}^2\dot{x}^2)+\sum_{n=1}^{2}\left[\frac{\partial L}{\partial \dot{x}^n}(\dot{x}^n\dot{x}^n-1)+\frac{\partial L}{\partial \dot{x}^n}x^n\ddot{x}^n\right]=\frac{\partial G}{\partial \dot{x}^n}\ddot{x}^n+\frac{\partial G}{\partial x^n}\dot{x}^n. \tag{39}$$

In this equation the terms with acceleration $\ddot{x}^n$ occur lineary, so we must have

$$\frac{\partial L}{\partial \dot{x}^1}x^1=\frac{\partial G}{\partial \dot{x}^1} \quad \text{and} \quad \frac{\partial L}{\partial \dot{x}^2}x^2=\frac{\partial G}{\partial \dot{x}^2}. \tag{40}$$

The function $G(x^1, x^2, \dot{x}^1, \dot{x}^2)$ exists only if

$$\frac{\partial^2 L}{\partial \dot{x}^1 \partial \dot{x}^2} x^1 = \frac{\partial^2 L}{\partial \dot{x}^1 \partial \dot{x}^2} x^2,$$

so the Lagrangian must satisfy the equation

$$\frac{\partial^2 L}{\partial \dot{x}^1 \partial \dot{x}^2} = 0, \tag{41}$$

which means that L has the form

$$L = A(\dot{x}^1, x) + D(\dot{x}^2, x), \tag{42}$$

where $A(\dot{x}^1, x)$, $D(\dot{x}^2, x)$ are arbitrary functions and from (32) we deduce that

$$G = A(\dot{x}^1, x)x^1 + D(\dot{x}^2, x)x^2 + h(x^1, x^2), \tag{43}$$

where $h(x^1, x^2)$ is an arbitrary function. Now we put (42) and (43) into Eq. (39). This gives the following equation

$$(\dot{x}^1\dot{x}^1 - 1)\frac{\partial A}{\partial \dot{x}^1} + (\dot{x}^2\dot{x}^2 - 1)\frac{\partial D}{\partial \dot{x}^2} + \dot{x}^2 x \frac{\partial A}{\partial x} + \dot{x}^1 x \frac{\partial D}{\partial x}$$

$$-\dot{x}^1 A - \dot{x}^2 D - \frac{\partial h}{\partial x^1}\dot{x}^1 - \frac{\partial h}{\partial x^2}\dot{x}^2 = 0. \tag{44}$$

Applying to this equation the operator $\dfrac{\partial^2}{\partial \dot{x}^1 \partial \dot{x}^2}$ we find

$$x\left(\frac{\partial^2 A}{\partial x \partial \dot{x}^1} + \frac{\partial^2 D}{\partial x \partial \dot{x}^2}\right) = 0. \tag{45}$$

From (45) it follows that

$$\frac{\partial^2 A}{\partial x \partial \dot{x}^1} = C(x), \qquad \frac{\partial^2 D}{\partial x \partial \dot{x}^2} = -C(x), \tag{46}$$

where $C(x)$ is arbitrary function. Thus A and B must have the form:

$$A(\dot{x}^1, x) = A(\dot{x}^1) + C(x)\dot{x}^1 + g(x),$$

$$D(\dot{x}^2, x) = D(\dot{x}^1) - C(x)\dot{x}^2 + f(x). \tag{47}$$

We do not loose generality by taking $C(x) = 0$, beacause the term $C(x)(\dot{x}^1 - \dot{x}^2)$ in the Lagrangian is a total derivative.

Finally, after putting (47) into (44) we have

$$(\dot{x}^1\dot{x}^1 - 1)\frac{dA(\dot{x}^1)}{d\dot{x}^1} + (\dot{x}^2\dot{x}^2 - 1)\frac{dD(\dot{x}^2)}{d\dot{x}^2} + \dot{x}^1\left[x\frac{df(x)}{dx} - g(x) - \frac{\partial h(x^1, x^2)}{\partial x^1} - A(\dot{x}^1)\right]$$

$$+\dot{x}^2\left[x\frac{dg(x)}{dx} - f(x) - \frac{\partial h}{\partial x^2} - D(\dot{x}^2)\right] = 0. \tag{48}$$

This equation is equivalent to the following set of equations:

$$x\frac{df}{dx}-g(x)-\frac{\partial h}{\partial x^1}=C_1,\qquad x\frac{dg}{dx}-f-\frac{\partial h}{\partial x^2}=C_2,$$

$$(\dot{x}^1\dot{x}^1-1)\frac{dA(\dot{x}^1)}{d\dot{x}^1}+\dot{x}^1[C_1-A(\dot{x}^1)]=C_3,$$

$$(\dot{x}^2\dot{x}^2-1)\frac{dD}{d\dot{x}^2}+\dot{x}^2[C_2-D(\dot{x}^2)]=C_4,$$

where C_1, C_2, C_3, C_4 are arbitrary constants satisfying the relation $C_3+C_4=0$.

It is an easy matter to check that this system of equations allows only the following form of the Lagrangian

$$L=\alpha\sqrt{1-\dot{x}^1\dot{x}^1}+\beta\sqrt{1-\dot{x}^2\dot{x}^2}+\gamma x,$$

or

$$L=\alpha\sqrt{1-\dot{x}^1\dot{x}^1}+\beta\sqrt{1-\dot{x}^2\dot{x}^2}+\gamma|x|,$$

if one requires the invariance with respect to space reflections; α, β, γ are constants and the terms which are a total derivative have been dropped.

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