OBSERVABILITY IN GENERAL RELATIVITY

By D. H. Weinstein

Department of Biophysical Sciences, University of Houston*

(Received June 5, 1979)

The observability of physical quantities in general relativity is discussed by means of "operational observations". Application is made to planetary motion.

The question of the observability of physical quantities in general relativity is not necessarily unambiguous. Thus, for example, Einstein's [1] formulation of Mach's principle predicted three observable effects:

1. An increase in inertial mass of a test particle when brought closer to a neighbouring mass.
2. A force experienced by a test particle when a neighbouring mass is accelerated.
3. A "Coriolis field" generated in the interior of a rotating hollow body.

The first effect has been considered theoretically by Brans [2] and Dicke [3]. They concluded that it was not observable, and ascribed its Machian result to the choice of coordinates. Experimental measurements of sufficient sensitivity have also been carried out with null results [4, 5].

It is our purpose to describe the measurement of physical quantities by means of what may be termed "operational observations". These, along with the equivalence principle, were initially introduced and applied by Einstein [6]. We shall consider additional physical quantities in this way and stress the distinction between local and distant observers. This will enable us to comment further on the three Machian effects mentioned above. In addition, we shall examine planetary motion from this point of view.

1. Some results of operational observations

We now introduce some of the pertinent results of the Einstein approach. Consider a gravitational source of mass M and observers S_0 and S_p. Let S_0 be situated far distant from M. It is convenient to make the gravitational potential at S_0 zero. Assume S_p is situated at a distance r_p from M.

* Address: Department of Biophysical Sciences, University of Houston, Houston, Texas 77004, USA.

Let S_p generate a definite amount of energy E_p by the use of a given means. This energy, E_p is transmitted from S_p to S_o in the form of radiation. S_o then compares the energy received, E_o with the energy produced at S_o by the identical means used by S_p. This comparison results in

$$E_o = E_p\left(1 - \frac{GM}{c^2 r_p}\right). \tag{1}$$

Similarly, S_p measures by means of his local clock the frequency ν_p of monochromatic radiation which is transmitted to S_o. Then S_o measures the frequency of the received radiation ν_o by means of an identical clock at S_o. The result is

$$\nu_o = \nu_p\left(1 - \frac{GM}{c^2 r_p}\right). \tag{2}$$

Finally, S_o takes note of the time required by S_p to send light along and back a given length. Thus, S_o calculates a velocity of light, c_o, taking equation (2) into account. If c_p is the local velocity of light as determined by S_p, then

$$c_o = c_p\left(1 - \frac{GM}{c^2 r_p}\right). \tag{3}$$

It is implicit here that the travel path of the light is transverse to the radial direction from the source M. Also implicit is that transverse length [7] is unaffected by position in the gravitational field. Hence

$$r_o = r_p. \tag{4}$$

However, radial length [7, 8] is affected by position in the field, with the result that

$$\bar{r}_o = \bar{r}_p\left(1 - \frac{GM}{c^2 r_p}\right). \tag{5}$$

Here the bar indicates the radial direction. The corresponding radial velocity [7, 8] of light is then given by

$$\bar{c}_o = \bar{c}_p\left(1 - 2\frac{GM}{c^2 r_p}\right).$$

It should be clear, however, that these variations of the velocity of light do not constitute a violation of relativity. The local velocity of light c_p as measured by S_p, remains constant regardless of his position in the field.

2. Application to inertial mass

We now examine the measurement of inertial mass. For this purpose, let S_p give two identical masses, m_p, equal and opposite velocities. Let these suffer a completely inelastic collision. The heat radiation thus generated is transmitted to S_o. Then S_o receives this heat radiation, E_o, and compares with the heat radiation produced at S_o by the identical scheme used at S_p. We can then write

$$\frac{m_o v_o^2}{2} \Big/ \frac{m_p v_p^2}{2} = E_o/E_p. \tag{7}$$

That is to say

$$\frac{m_o}{m_p} = \left(\frac{r_p}{t_p}\right)^2 \left(\frac{t_o}{r_o}\right)^2 \frac{E_o}{E_p}. \tag{8}$$

If the motion is along the transverse direction, then we must use equations (1), (2) and (4). This leads to the relation for the transverse mass

$$m_o = m_p \left(1 + \frac{GM}{c^2 r_p}\right). \tag{9}$$

If the motion is along the radial direction, then we must use equations (1), (2) and (5). This leads to the relation for the radial mass [8]

$$\overline{m}_o = \overline{m}_p \left(1 + 3\frac{GM}{c^2 r_p}\right). \tag{10}$$

Note, however, that

$$\overline{m}_o \bar{v}_o^2 / \overline{m}_p \bar{v}_p^2 = m_o v_o^2 / m_p v_p^2. \tag{11}$$

Hence the collision heat radiation is independent of the orientation of the colliding masses.

3. *Discussion of the three Machian effects*

It thus appears from equations (9) and (10) that the first Machian effect is confirmed and, in addition, anisotropy of inertial mass is present. In fact, it was this possibility that led Hughes et al. [4], and Drever [5] to conduct experiments which took advantage of the non-spherical distribution of mass in our galaxy as suggested by Coccini and Salpeter [9]. As already mentioned, the experimental results were null.

However, we must emphasize that while there is indeed a change in inertial mass as a result of the changed gravitational field, this is only how it appears to S_o when far removed from S_p. If S_o were to approach S_p, this change in inertial mass would diminish until finally, when they are together, no change would be apparent to S_o. That is to say, in agreement with Brans [2] and Dicke [3], it is not possible to locally observe any change in inertial mass because of any change in the gravitational field.

While the above is based on the equivalence principle in addition to other considerations, one can also argue the non-observability of the first effect directly from the equivalence principle. Thus Nightingale and Ray [10] have pointed out that an observer in free fall toward a source would find that a test particle in his frame would have inertial properties independent of the strength of the accelerating source.

Similarly, one can equally well argue that in such a system we may regard the source as being accelerated toward the test particle. Again the observer would find that the behavior of the test particle would be independent of the source strength and hence its acceleration. That is to say, we must conclude that the second Machian effect is not observable either.

The third Machian effect is not directly susceptible to such arguments but it seems not unlikely that it too will not be observable in view of the above.

4. *Application to the gravitational constant*

We turn now to the influence of a gravitational field on the radial component of G itself. Consider two identical masses m separated by a fixed distance d and lined up radially in S_p. If they are released and allowed to collide (inelastically) because of their mutual gravitational attraction, we must have

$$\frac{\bar{G}_o \bar{m}_o^2}{\bar{d}_o} \Big/ \frac{\bar{G}_p \bar{m}_p^2}{\bar{d}_p} = E_o/E_p. \tag{12}$$

Using equations (1), (5) and (10) we obtain

$$\bar{G}_0 = \bar{G}_p\left(1-8\frac{GM}{c^2 r_p}\right). \tag{13}$$

We note here that we can apply the same arguments that were used to indicate the non-observability of the first Machian effect to the question of the observability of local changes in G. We must conclude, contrary to the well known conjecture of Dirac [11], that such changes cannot be observed.

We omit discussion of the corresponding transverse component of G since it is not required in what follows.

5. *Planetary motion*

It might be of interest to discuss other physical quantities such as charge, Planck's constant, the fine structure constant, etc., but instead we shall consider planetary motion in the light of the above development. Such motion can be put in perspicuous form by subjecting the standard Schwarzschild metric to the transformation [12]

$$r \to r\left(1+\frac{GM}{2c^2 r}\right)^2. \tag{14}$$

The result is the isotropic Schwarzschild metric

$$ds^2 = \left[\frac{1-\dfrac{GM}{2c^2 r}}{1+\dfrac{GM}{2c^2 r}}\right] dt^2 - \frac{1}{c^2}\left[1+\frac{GM}{2c^2 r}\right]^4 [dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)]. \tag{15}$$

Here the corresponding equation of planetary motion [12] is

$$\frac{d^2u}{d\varphi^2}+u=\frac{GM}{H^2}\left(1+\frac{6GMu}{c^2}\right). \tag{16}$$

It yields the usual value for the advance of perihelion per revolution

$$\delta=6\pi\left(\frac{GM}{Hc}\right)^2. \tag{17}$$

We can now examine the planetary motion as viewed by S_p when situated on the planet itself. If S_p had no means of making measurements relative to another observer such as S_o, he might expect that his orbit would follow the classical equation of motion

$$\frac{d^2u}{d\varphi^2}+u=\frac{GM}{H^2}. \tag{18}$$

For the purpose of what follows, we shall assume that S_p is in a nearly circular orbit.

From the point of view of S_o, equation (5) indicates that unit radial distance in S_p is decreased when r_p is decreased. At the same time, it appears to S_p that unit radial distance on S_o is increased as r_p is decreased. This is indicated by the inversion of equation (5)

$$\bar{r}_p=\bar{r}_0\left(1+\frac{GM}{c^2r_p}\right). \tag{19}$$

In view of the position of S_o, S_p concludes that his local measurement of radial distance requires correction in order to properly plot his position in space. That is

$$\bar{r}_p\rightarrow\bar{r}_p\left(1+\frac{GM}{c^2r_p}\right). \tag{20}$$

Note that this correction (20) is identical to the isotropic transformation (14) when equation (14) is expanded to the first order in GM/c^2r.

From equation (2) we see that S_p must also correct his clock rate by setting

$$\nu_p\rightarrow\nu_p\left(1+\frac{GM}{c^2r_p}\right). \tag{21}$$

If now S_p applies correction (20) to the left hand side of the equation of motion (18), he will find that it is unchanged for two reasons. The first is the differential identity

$$d\left[r\left(1+\frac{GM}{c^2r}\right)\right]=dr. \tag{22}$$

The second is that φ is in effect a transverse quantity.

However, S_p must correct G_p and H_p. For G_p we obtain equation (13)

$$\bar{G}_p\rightarrow\bar{G}_p\left(1+8\,\frac{GM}{c^2r_p}\right). \tag{23}$$

For $H = r^2 \frac{d\varphi}{dt}$, only the time requires correction and so inserting correction (21) we get

$$\frac{GM}{H^2} \to \frac{GM}{H^2} \frac{\left(1+8\frac{GM}{c^2 r_p}\right)}{\left(1+\frac{GM}{c^2 r_p}\right)^2} \to \frac{GM}{H^2}\left(1+6\frac{GM}{c^2}u\right). \tag{24}$$

Thus we see that when S_p completes his corrections to equation (18), he is able to properly plot his position in space. This corresponds to equation (16) for our limiting case of nearly circular motion.

We note that throughout this discussion we have restricted ourselves to the first order in $GM/c^2 r$. This contradicts Eddington [13], who indicated that the second order of this term was required in order to obtain the motion of perihelion. However, Synge [14] has investigated this matter in some detail. His conclusion was that the motion of perihelion was a first order effect and that it only appeared to be second order.

We therefore conclude that operational observations may be useful in some applications of general relativity. The fundamental role played by radiation in the process of making measurements should be evident.

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