THE EFFECTS OF TORSION FIELDS ON A SPINNING TOP

LIU YI-FEN* AND WU YONG-SHI

Institute of Theoretical Physics, Academia Sinica, Peking, China

(Received August 18, 1979)

In this paper we derive the equations of motion for a spinning particle in the presence of the metric-torsion field using a different method from that of Mathisson and Papapetrou. We also discuss the behaviour of a spinning body in a static spherically symmetric metric-torsion field and compare our results with that of Schiff. We also proved that there is no precession of gyroscop in absolutely parallelizable space. Finally, we discuss the top experiment which can test our theory.

1

Here we shall discuss the dynamical effects of the torsion field, in particular, on the motion of a spinning particle.

In Section 3 we shall derive the equation of motion for the moving spinning particle in a metric-torsion field and equations which govern the variation in time of the spin of the particle. We derive these equations starting from conservation laws of energy-momentum and angular momentum. The resultant equations are different from those of Mathisson [1], Papapetrou [2] and Schiff [3]. We note that our derivation of the equations of motion for a spinning particle requires only the invariance of Lagrangian density for matter, $\mathscr{L}_m$, under local Lorentz rotation and under arbitrary coordinate transformations. This derivation does not depend on the choice of the Lagrangian for metric and torsion fields. Therefore, our equations of motion for spin hold generally for all types of theories which extend general relativity (GR) by including torsion with the exception of the supergravity and extended supergravity theory.

To illustrate more concretely the effects of the torsion field, in Section 4 we shall calculate the time variation of the spin of a particle which moves in a circular orbit in a static spherically symmetric metric-torsion field, using as an example for the metric-trosion field the exact solution proposed in [4]. It was found that the Schiff's effect in GR is only

* At the Institute of Theoretical Physics, University of Warsaw, Hoża 69, 00-681 Warszawa, Poland.

a special case in our results. In general, in addition to the precession of spin, the magnitude of spin will vary periodically along the orbit. In our opinion it may be possible to detect the existence of the torsion field using this effect.

2

Previously [4], we proposed a program for the gauge theory of interactions of torsion and metric fields and found exact solutions for spherically symmetric static fields. Here we shall use these solutions to discuss the precession of a spinning particle in a torsion field. For convenience, let us briefly review some results of the previous paper which will be used below.

The lagrangian density for the metric-torsion and matter fields proposed in [3] is:

$$L = \int (\mathscr{L}_g + \mathscr{L}_0 + \mathscr{L}_m)\sqrt{-g}\,(dx)^4$$
$$= \int (\chi R + \eta F^a{}_{b\alpha\beta} F_a{}^{b\alpha\beta} + \mathscr{L}_m)\sqrt{-g}\,(dx)^4. \tag{1}$$

The equations of motion for the metric and torsion fields derived from it are:

$$\chi G_{\alpha\beta} + \left[2\eta F^a{}_{b\alpha\gamma} F_a{}^b{}_\beta{}^\gamma - \frac{\eta}{2} F^a{}_{b\gamma\varrho} F_a{}^{b\gamma\varrho} g_{\alpha\beta}\right] = T_{\alpha\beta}, \tag{2}$$

$$\eta F^a{}_{b\alpha}{}^\beta{}_{\|\beta} \equiv \eta\left[F^a{}_{b\alpha}{}^\beta{}_{,\beta} + B^a{}_{c\beta} F^c{}_{b\alpha}{}^\beta - B^c{}_{b\beta} F^a{}_{c\alpha}{}^\beta - \begin{Bmatrix}\gamma\\ \alpha\beta\end{Bmatrix} F^a{}_{b\gamma}{}^\beta + \begin{Bmatrix}\beta\\ \varrho\beta\end{Bmatrix} F^a{}_{b\alpha}{}^\varrho\right] = S^a{}_{b\alpha},$$

where χ and η are coupling constants. The definition of $F^a{}_{b\alpha\beta}$ and $B^a{}_{b\alpha}$ are given in [4].

$$T^{\alpha\beta} = \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g_{\alpha\beta}}(\sqrt{-g}\,\mathscr{L}_m)$$

is energy-momentum tensor and

$$S^a{}_{b\alpha} = \frac{g_{\alpha\beta}}{\sqrt{-g}}\frac{\delta}{\delta B^a{}_{b\beta}}(\sqrt{-g}\,\mathscr{L}_m)$$

according to the gauge theory, is the spin current.

In the case of spherically symmetric and static fields we found that the components of the metric tensor are:

$$g_{00} = \left(1 + \frac{h}{r}\right), \quad g_{11} = -\left(1 + \frac{h}{r}\right)^{-1}, \quad g_{22} = g_{33}/\sin^2\theta = -r^2, \tag{4}$$

and the other components are zero.
The components of the torsion are:

$$T_{01}{}^0 = \frac{\pm\left(L + \frac{1}{r}\right) - h/2r^2}{1 + h/r}, \quad T_{21}{}^2 = T_{31}{}^3 = \frac{1}{r} - \sqrt{\frac{L'}{1 + \frac{h}{r}}} \tag{5}$$

and the other independent components are zero.

For h we can take the same value as in GR. $h = -2KM/c^2$, while L satisfies the following equations:

$$\frac{r+h}{2r} L'' - L'L + \frac{h}{r} L' = 0. \tag{6}$$

If we write L as series

$$L = \sum_{n=1}^{\infty} \frac{a_n}{r^n} \tag{7}$$

then we have:

$$\begin{aligned} a_1 &= -1 \\ a_2 &= \text{arbitrary constant}, \\ a_3 &= a_2\left(a_2 + \frac{h}{2}\right), \\ a_4 &= a_2\left(a_2 + \frac{h}{2}\right)\left(\frac{3h}{5} + a_2\right), \\ &\dots \end{aligned} \tag{8}$$

3

From the invariance of $\mathscr{L}_g$, $\mathscr{L}_0$ and $\mathscr{L}_m$ under local Lorentz transformations and arbitrary coordinate transformation, the following two laws of conservations can be easily derived:

$$T^{\alpha\beta}{}_{;\beta} = \tfrac{1}{2} S_{ab}{}^{\beta} F^{ab\alpha}{}_{\beta},$$

$$S^{ab\alpha}{}_{\|\alpha} = 0, \tag{9}$$

where $T^{\alpha\beta}$ is the energy-momentum tensor, or the momentum current, and $S^{ab\alpha}$ is the spin current. For a single mass point they can be taken as:

$$T^{\alpha\beta} = \varrho U^\alpha U^\beta = P^\alpha U^\beta, \tag{10}$$

$$S^{ab\alpha} = S^{ab} U^\alpha, \tag{11}$$

where ϱ is the mass density of the particle, U^α is its velocity, $P^\alpha = \varrho U^\alpha$ is momentum and S^{ab} is its spin density.

From (9), (10) and (11) we obtain the equations of motion for a spinning particle which read as follows:

$$\varrho\left(\frac{d}{d\tau}U^{\alpha}+\begin{Bmatrix}\alpha\\ \beta\gamma\end{Bmatrix}U^{\beta}U^{\gamma}\right) = -\tfrac{1}{2}S_{ab}U^{\beta}F^{ab\alpha}{}_{\beta}, \tag{12}$$

$$\frac{d}{d\tau}S^{ab}+B^{a}{}_{c\beta}S^{cb}U^{\beta}+B^{b}{}_{d\alpha}S^{ad}U^{\alpha} \equiv \frac{D}{d\tau}S^{ab} = 0. \tag{13}$$

For a macroscopic body we can identify S^{ab} with its rotating angular momentum. Here rotation is about the proper centre of mass, while the proper centre of mass refers to the point which moves with the velocity, U^{α}, determined by Eq. (12).

Similar but not identical equations were obtained by Mathisson [1] and Papapetrou [2]. By a quite different method, they obtained the following equations of motion:

$$\frac{D}{d\tau}\left[mU^{\alpha}+\left(\frac{D}{d\tau}S^{\alpha\beta}\right)U_{\beta}\right] = -\tfrac{1}{2}S^{\varrho\lambda}R^{\alpha}{}_{\beta\varrho\lambda}U^{\beta}, \tag{14}$$

$$\frac{D}{d\tau}S^{\alpha\beta}+U^{\alpha}U_{\lambda}\frac{D}{d\tau}S^{\beta\lambda}+U_{\beta}U_{\lambda}\frac{D}{d\tau}S^{\lambda\alpha} = 0. \tag{15}$$

For a rotating body in a metric field, where

$$S^{\alpha\beta} = \int[(x^{\alpha}-X^{\alpha})T^{\beta 0}-(x^{\beta}-X^{\beta})T^{\alpha 0}]\sqrt{-g}\,(dx)^4, \tag{16}$$

here X^{α} are coordinates of the point which moves with velocity U^{α}.

Our equations (12) and (13) take into account the effects of the torsion, but Papapetrou's Eqs. (14) and (15) do not. Furthermore, there is a formal difference between our equations and those of Papapetrou. In fact, there are only three independent differential equations in (15), but there are six components of $S^{\alpha\beta}$. Therefore, Eqs. (15) do not determine $S^{\alpha\beta}$ completely. In contrast, our Eqs. (13) are six differential equations for six components of $S^{\alpha\beta}$. It is obvious that all the solutions of Eqs. (12) and (13) in torsion-free case satisfy Eqs. (14), (15). In this sense, our Eqs. (12) and (13) are compatible with Papapetrou's Eqs. (14) and (15).

Here we would like to discuss the formal difference mentioned above between Eqs. (13), (15) in more detail. Since Eqs. (15) are essentially three independent equations for six quantities of $S^{\alpha\beta}$, it is necessary to impose three additional conditions on $S^{\alpha\beta}$, in order that they may be determined completely. As to what conditions should be added, there are different opinions in the literature.

Papapetrou suggested that in the rest frame we should have

$$S^{\alpha 0} = 0. \tag{17}$$

Pirani [4] proposed the covariant version

$$S^{\alpha\beta}U_{\beta} = 0. \tag{18}$$

Tulczyjew [5] suggested another covariant auxiliary condition

$$S^{\alpha\beta}P_\beta \equiv S^{\alpha\beta}\left[mU_\beta + \left(\frac{D}{d\tau}S_{\beta\varrho}\right)U^\varrho\right] = 0. \tag{19}$$

These conditions had their origin in Special Relativity and are then generalized directly to GR.

In Newtonian Mechanics, the motion of a rigid body can be decomposed into a translation of some point of the rigid body and a rotation about this representative point. When discussing self-rotation, generally the centre of mass of the rigid body as the representative point is considered, and self-rotation refers to rotation about the centre of mass.

However, in Special Relativity the concept of the centre of mass is intimately related to the state of motion. Møller [6] introduced the concept of proper centre of mass in the rest system of reference with respect to the body. $S^{\alpha\beta}$ refers to rotation about the proper centre of mass, and $S^{0\beta}$ are the components of the centre of mass relative to the proper centre of mass, i.e., relative to the centre of rotation. In special relativity it follows that [6]

$$\vec{S} = \frac{\vec{V}\times\vec{J}}{\sqrt{c^2 - V^2}}, \tag{20}$$

where $\vec{S} = S^{io}$, $\vec{J}$ is the spin angular momentum in the vector form. If $\vec{V} = 0$, then

$$S^{io} = 0 \tag{21}$$

that is, the centre of mass coincides with the proper centre of mass for a rest body. Thus, it is generally accepted that in the absence of any field

$$S^{\alpha\beta}U_\beta = 0. \tag{22}$$

However, the problem becomes very complicated in GR because the metric fields have some effects on S^{io}. Generelly, $g_{\alpha\beta}$ are functions of spatial coordinates and time. A top with a finite size and shape, when it moves, occupies different spatial regions at different times. Therefore, according to Eq. (16) S^{io} will be affected by the change of $g_{\alpha\beta}(\vec{x}, t)$. In the same sense, $g_{\alpha\beta}$ "deforms" the body. Of course, the deformation depends on the shape and size of the body, so its effects will be very complicated. Even if $\vec{V} = 0$, this holds for a rotating unsymmetric body with a finite size, because when it is rotating, it occupies different spatial regions at different times. In addition, in general, $g_{\alpha\beta}(\vec{x}, t)$ varies with t and so does $S^{\alpha\beta}$. Therefore, Eqs. (21) and (22) do not generally hold in GR. Thus, direct generalization of those conditions from the Special to the General Theory of Relativity seems problematic.

To summarize, there is no need to add any auxiliary conditions such as Eqs. (17) and (18) to our Eqs. (12) and (13). Additional conditions are in general incompatible with equations (12) and (13).

4

Now let us solve equations (12) and (13) in fields given by Eqs. (4) and (5). First we choose a suitable tetrad as follows:

$$e_0^{(0)} = \sqrt{g_{00}} = \left(1+\frac{h}{r}\right)^{1/2},$$

$$e_1^{(1)} = \sqrt{-g_{11}} = \left(1+\frac{h}{r}\right)^{-1/2},$$

$$e_2^{(2)} = \sqrt{-g_{22}} = r,$$

$$e_3^{(3)} = \sqrt{-g_{33}} = r\sin\theta, \tag{23}$$

the other components of $e_\alpha^{(a)}$ are zero. Recall that

$$\Gamma_{\gamma\alpha}^{\beta} = \left\{ {\beta \atop \gamma\alpha} \right\} + \tfrac{1}{2}\left[-T^{\beta}{}_{\gamma\alpha} - T^{\beta}{}_{\alpha\gamma} + T_{\alpha\gamma}{}^{\beta}\right], \tag{24}$$

$$B^{a}{}_{b\alpha} = e_\beta^{(a)} e_{(b)}^{\gamma} \Gamma_{\gamma\alpha}^{\beta} + e_{(b),\alpha}^{\varrho} e_\varrho^{(a)}, \tag{25}$$

substituting (4) and (5) in these, we have

$$B^{0}{}_{10} = B^{1}{}_{00} = \pm\left(L+\frac{1}{r}\right),$$

$$B^{2}{}_{12} = B^{3}{}_{13} = -B^{1}{}_{33} = B^{1}{}_{22} = -r\sqrt{L'}, \tag{26}$$

the other independent components = 0
for the orbit plane taken to be the X–Y plane. Now, Eqs. (13) can be written in the simple form:

$$\frac{d}{d\tau}S^{01} = F_2 S^{03}, \quad \frac{d}{d\tau}S^{13} = F_1 S^{03}, \quad \frac{d}{d\tau}S^{03} = F_1 S^{13} - F_2 S^{01}, \tag{27}$$

$$\frac{d}{d\tau}S^{23} = F_2 S^{12}, \quad \frac{d}{d\tau}S^{02} = F_1 S^{12}, \quad \frac{d}{d\tau}S^{12} = F_1 S^{02} - F_2 S^{23}, \tag{28}$$

where

$$F_1 = \pm\left(L+\frac{1}{r}\right)\frac{v^0}{v^3}, \quad F_2 = r\sqrt{L'}. \tag{29}$$

Equations (27) (28) (12) give us four integrals

$$(S^{12})^2 + (S^{23})^2 - (S^{02})^2 = \text{constant},$$

$$(S^{03})^2 + (S^{01})^2 - (S^{13})^2 = \text{constant},$$

$$\left(1+\frac{h}{r}\right)v^0 \pm \left(L'+\frac{1}{r}\right)S^{01} = \text{constant},$$

$$r^2 v^3 + r\sqrt{L'}\,S^{13} = \text{constant},$$

which correspond to the usual integrals of energy and momentum.

A. Let us first discuss the particular case in which space is with absolute parallelism. In this case

$$a_2 = 0, \quad L = \frac{1}{r}, \quad F^a{}_{b\alpha\beta} = 0, \quad F_1 = 0, \quad F_2 = 1. \tag{30}$$

The general solution of (27), (28) is:

$$S^{13} = -B, \quad S^{02} = -E, \quad S^{01} = A\sin(\varphi+\delta_1),$$
$$S^{03} = A\cos(\varphi+\delta_1), \quad S^{23} = G\sin(\varphi+\delta_2), \quad S^{12} = G\cos(\varphi+\delta_2), \tag{31}$$

where A, B, G, E, δ_1 and δ_2 are constant.

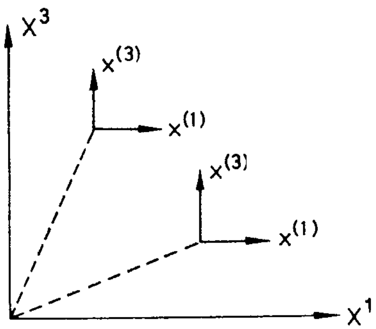

Fig. 1

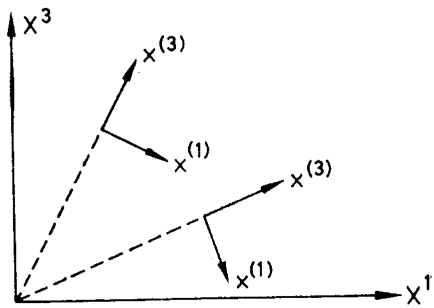

Fig. 2

Although the tetrad (23) was chosen to simplify the equations, it rotates with the orbit motion as shown in Fig. 1. We eliminate the effect of rotation of the tetrad by a local spatial rotation, as shown in Fig. 2.

$$\tilde{S}^{ab} = S^{cd} l^a_c l^b_d, \tag{32}$$

where

$$l^a_c = \begin{pmatrix} l^0_0 = 1 & 0 & 0 & 0 \\ 0 & l^1_1 = \cos\varphi & l'_3 = -\sin\varphi & 0 \\ 0 & l^3_1 = \sin\varphi & l^3_3 = \cos\varphi & 0 \\ 0 & 0 & 0 & l^2_2 = 1 \end{pmatrix}. \tag{33}$$

Hence:

$$\tilde{S}^{01} = A\sin\delta_1, \quad \tilde{S}^{02} = -E, \quad \tilde{S}^{03} = A\cos\delta_1,$$
$$\tilde{S}^{12} = G\cos\delta_2, \quad S^{13} = -B, \quad S^{23} = G\sin\delta_2. \tag{34}$$

From this result we conclude that in a space with absolute parallelism all the components of the spin of a top remain constant. This constant is independent of the orbit. We note that in this case the orbit is the same as the geodesics in the Schwarzschild field well known in GR.

B. Now let us go to the case in which the orbit of the spinning body is a circular one but fields are given by general solutions (4) and (5).

In principle, the orbit should be determined from equation (12). However, in most physical cases the right-hand side of (12) is negligible. For these cases the orbit is approximately the same as that obtained in GR for a particle moving in the Schwarzschild field. The circular orbit occurs as a particular case. In this case, it is obvious that

$$v^3/v^0 = \sqrt{-h/2r^3}\,. \tag{35}$$

Hence, according to Eq. (29) F_1 and F_2 are constants on the circular orbit, so that the general solution of Eqs. (27) and (28) can be found easily. If

$$|F_2| > |F_1| \tag{36}$$

we have

$$\begin{aligned} S^{03} &= A\cos(\sqrt{F_2^2-F_1^2}\,\varphi+\delta_1),\\ S^{13} &= \frac{AF_1}{\sqrt{F_2^2-F_1^2}}\sin(\sqrt{F_2^2-F_1^2}\,\varphi+\delta_1)-BF_2,\\ S^{01} &= \frac{AF_2}{\sqrt{F_2^2-F_1^2}}\sin(\sqrt{F_2^2-F_1^2}\,\varphi+\delta_1)-BF_1, \end{aligned} \tag{37}$$

$$\begin{aligned} S^{12} &= G\cos(\sqrt{F_2^2-F_1^2}\,\varphi+\delta_2),\\ S^{02} &= \frac{GF_1}{\sqrt{F_2^2-F_1^2}}\sin(\sqrt{F_2^2-F_1^2}\,\varphi+\delta_2)-EF_2,\\ S^{23} &= \frac{GF_2}{\sqrt{F_2^2-F_1^2}}\sin(\sqrt{F_2^2-F_1^2}\,\varphi-\delta_2)-EF_1. \end{aligned} \tag{38}$$

Making the local rotation of the tetrad as before using Eq. (32) we obtain:

$$\begin{aligned} J^x = \tilde{S}^{32} &= \left[\frac{-GF_2}{\sqrt{F_2^2-F_1^2}}\sin(\sqrt{F_2^2-F_1^2}\,\varphi+\delta_2)+EF_1\right]\cos\varphi\\ &\quad+G\cos(\sqrt{F_2^2-F_1^2}\,\varphi+\delta_2)\sin\varphi,\\ J^y = \tilde{S}^{21} &= \left[\frac{-GF_2}{\sqrt{F_2^2-F_1^2}}\sin(\sqrt{F_2^2-F_1^2}\,\varphi+\delta_2)+EF_1\right]\sin\varphi\\ &\quad-G\cos(\sqrt{F_2^2-F_1^2}\,\varphi+\delta_2)\cos\varphi, \end{aligned}$$

$$J^z = \tilde{S}^{13} = \frac{AF_1}{\sqrt{F_2^2-F_1^2}} \sin(\sqrt{F_2^2-F_1^2}\ \varphi+\delta_1)-BF_2, \tag{39}$$

$$\tilde{S}^{01} = \left[\frac{AF_2}{\sqrt{F_2^2-F_1^2}} \sin(\sqrt{F_2^2-F_1^2}\ \varphi+\delta_1)-BF_1\right]\cos\varphi - A\cos(\sqrt{F_2^2-F_1^2}\ \varphi+\delta_1)\sin\varphi,$$

$$\tilde{S}^{03} = \left[\frac{AF_2}{\sqrt{F_2^2-F_1^2}} \sin(\sqrt{F_2^2-F_1^2}\ \varphi+\delta_1)-BF_1\right]\sin\varphi + A\cos(\sqrt{F_2^2-F_1^2}\ \varphi+\delta_1)\cos\varphi,$$

$$\tilde{S}^{02} = \frac{GF_1}{\sqrt{F_2^2-F_1^2}} \sin(\sqrt{F_2^2-F_1^2}\ \varphi+\delta_2)-EF_2. \tag{40}$$

If the torsion field is very weak and $|a_2|$ very small, then $|F_2|^2 \gg |F_1^2|$, so that $F_2/\sqrt{F_2^2-F_1^2} \approx 1$. In this case, the above formulas can be written in a simple form:

$$J^x = G\sin[(1-\sqrt{F_2^2-F_1^2})\ \varphi+\delta_2]+EF_1\cos\varphi, \tag{41}$$

$$J^y = G\cos[(1-\sqrt{F_2^2-F_1^2})\ \varphi+\delta_2]+EF_1\sin\varphi, \tag{42}$$

$$J^z = \frac{AF_1}{\sqrt{F_2^2-F_1^2}} \sin[\sqrt{F_2^2-F_1^2}\ \varphi+\delta_1]-BF_2, \tag{43}$$

$$\tilde{S}^{01} = -A\sin[(1-\sqrt{F_2^2-F_1^2})\varphi+\delta_1]-BF_1\cos\varphi, \tag{44}$$

$$\tilde{S}^{03} = -A\cos[(1-\sqrt{F_2^2-F_1^2})\varphi+\delta_1]-BF_1\sin\varphi, \tag{45}$$

$$\tilde{S}^{02} = \frac{GF_1}{\sqrt{F_2^2-F_1^2}} \sin(\sqrt{F_2^2-F_1^2}\ \varphi+\delta_2)-EF_2. \tag{46}$$

In all the above formulas A, E, δ_1 and G, B, δ_2 are integration constants, and are determined by the initial conditions. It is interesting that when $a_2 = 0$, we obtain (34) and in (34) G, B, δ_2 are determined by the initial conditions of rotation rates of the body, and A, E, δ_1, are determined by the initial coordinates of the centre of mass, relative to the proper centre of mass.

If $A = E = 0$, Eqs. (41)–(46) can be written as:

$$J^x = G\sin[(1-\sqrt{F_2^2-F_1^2})\varphi+\delta_2], \tag{47}$$

$$J^y = G\cos[(1-\sqrt{F_2^2-F_1^2})\varphi+\delta_2], \tag{48}$$

$$J^2 = -BF_2, \tag{49}$$

$$\tilde{S}^{01} = -BF_2\cos\varphi, \tag{50}$$

$$\tilde{S}^{03} = -BF_1\sin\varphi, \tag{51}$$

$$S^{02} = \frac{GF_1}{\sqrt{F_2^2-F_1^2}} \sin(\sqrt{F_2^2-F_1^2}\ \varphi+\delta_2). \tag{52}$$

Eqs. (47)–(49) show the precession of rotation about the z-axis and the precession vector is:

$$\vec{\Omega} = [1-\sqrt{F_2^2-F_1^2}]\,\frac{\vec{r}\times\vec{V}}{r^2}\,. \tag{53}$$

It is interesting that if $a_2 = -\dfrac{h}{2}$, then all components of torsion vanish, and we come back to GR. In this case

$$a_2 = -\frac{h}{2}\,,\qquad h = -\frac{2KM}{c^2}\,,\qquad \frac{v^3}{v^0} = \sqrt{\frac{-h}{2r^3}}\,. \tag{54}$$

so that

$$(1-\sqrt{F_2^2-F_1^2}) = -\frac{3KM}{2c^2r}\,. \tag{55}$$

This is just the usual Schiff's result [3].

Incidentally, if we use the Pirani condition (18) we can also obtain Eqs. (47)–(52). But in general A and E are not exactly zero. Hence, we should use Eqs. (41)–(46) to describe the effect of rotation of a realistic top. However, of course, A and E are small, and so is F_1. So their effect on rotation is small.

C. If $|F_2| < |F_1|$. The general solutions of Eqs. (27) and (28) can be found easily. The physically meaningful one is given

$$S^{13} = \frac{-F_1A}{\sqrt{F_1^2-F_2^2}}\exp-\sqrt{F_1^2-F_2^2}\,\varphi-BF_2,$$

$$S^{12} = G\exp-\sqrt{F_1^2-F_2^2}\,\varphi,$$

$$S^{23} = \frac{GF_2}{\sqrt{F_1^2-F_2^2}}\exp-\sqrt{F_1^2-F_2^2}\,\varphi-EF_1, \tag{56}$$

$$S^{01} = \frac{-AF_1}{\sqrt{F_1^2-F_2^2}}\exp-\sqrt{F_1^2-F_2^2}\,\varphi-BF_1,$$

$$S^{02} = \frac{GF_1}{\sqrt{F_1^2-F_2^2}}\exp-\sqrt{F_1^2-F_2^2}\,\varphi-EF_2,$$

$$S^{03} = A\exp-\sqrt{F_1^2-F_2^2}\,\varphi. \tag{57}$$

Making the local rotation of the tetrad as before using Eqs. (32) we obtain:

$$J^x = \tilde{S}^{32} = -\left[\frac{GF_2}{\sqrt{F_1^2-F_2^2}}\exp-\sqrt{F_1^2-F_2^2}\,\varphi-EF_1\right]\cos\varphi+G(\exp-\sqrt{F_1^2-F_2^2}\,\varphi)\sin\varphi,$$

$$J^y = \tilde{S}^{21} = -\left[\frac{GF_2}{\sqrt{F_1^2-F_2^2}}\exp-\sqrt{F_1^2-F_2^2}\,\varphi-EF_1\right]\sin\varphi-G(\exp-\sqrt{F_1^2-F_2^2}\cos\varphi,$$

$$J^z = \tilde{S}^{13} = \frac{F_1A}{\sqrt{F_1^2-F_2^2}}\exp-\sqrt{F_1^2-F_2^2}\,\varphi-BF_2. \tag{58}$$

If $t \to \infty$, $\varphi \to \infty$, so we obtain

$$J^x = EF_1 \cos\varphi, \quad J^y = EF_1 \sin\varphi, \quad J^z = -BF_2. \tag{59}$$

Eqs. (59) show the precession of the rotation axis about the z-axis and the period of precession is the same as the period of orbital motion.

5

To test the Schiff effect, the giroscope experiments were proposed some time ago. In our opinion, the same experiments can also test our theory. Our theoretical predictions are given by Eqs. (41)–(46). Because we have taken into account the effect of torsion fields, the value of precession of the top's angular momentum will be different from that predicted by Schiff. The magnitude of the top's angular momentum will vary periodically. The period of variation is approximately the same as that of the orbit because $\sqrt{F_2^2-F_1^2} \approx 1$. The extrems in variation do not necessarily occur at the perihelion and aphelion. Furthermore, the axis of rotation would shift periodically. Therefore, we can observe the shift of the north pole.

It is too bad that we were unable to determine numerically the effects of torsion fields. For there is an integration constant, a_2, which should be determined numerically. It is well known that the integration constant, h, can be determined from the Newtonian approximation to be $h = \frac{-2KM}{c^2}$, where K is determined experimentally. However, there is no experimental measurement of the torsion fields up to now. The measurement of a_2 is still a challenge for experimentalists.

One of us (L.Y.F.) thanks Professor A. Trautman and members of his group for helpful discussions.

REFERENCES

[1] M. Mathisson, *Acta Phys. Pol.* **6**, 163 (1937).
[2] A. Papapetrou, *Proc. R. Soc.* (London) **209**, 249 (1951).
[3] L. O. Schiff, *Proc. Natl. Acad. Sci.* **46**, 871 (1960).
[4] Liu Yu-Fen, Tsou Chen-Lung, *Acta Phys. Pol.* **B10**, 893 (1979).
[5] F. A. E. Pirani, *Acta Phys. Pol.* **15**, 389 (1956).
[6] W. Tulczyjew, *Acta Phys. Pol.* **18**, 393 (1959).
[7] C. Møller, *The Theory of Relativity*, 1955 p. 170.