UNIVERSAL SYMMETRY AND THE KLOTZ METRIC IN EINSTEIN'S NON-SYMMETRIC FIELD THEORY

BY L. J. GREGORY

Department of Applied Mathematics, University of Sydney*

(*Received September 13, 1979*)

In general relativity, a coordinate transformation is called a symmetry if the metric tensor is form invariant with respect to the transformation. Einstein's non-symmetric unified field theory has no a priori metric and the definition of symmetry is more difficult. Traditionally, invariance of a metric tensor was replaced by form invariance of the fundamental tensor of the non-symmetric theory but this definition is inadequate for the metrisation procedure used by Klotz. The concept of universal symmetry is introduced to overcome this problem and is used to find the general form of the Klotz metric for a spherically symmetric space.

1. *Introduction*

Klotz [3] has proposed that the symmetric metric tensor, $a_{\mu\nu}$, in Einstein's non-symmetric unified field theory (see Appendix) should be determined from the equations

$$a_{\mu\nu,\lambda}-\Gamma^{\sigma}_{(\mu\lambda)}a_{\sigma\nu}-\Gamma^{\sigma}_{(\nu\lambda)}a_{\mu\sigma} = 0, \tag{1}$$

where $\Gamma^{\lambda}_{(\mu\nu)}$ are the symmetric components of the affine connection. Round and square brackets denote symmetrisation and skew-symmetrisation respectively and a comma denotes partial differentiation. If such a symmetric tensor of rank four exists then the space is metrisable ([1]) with respect to the connection $\Gamma^{\lambda}_{(\mu\nu)}$.

The general resolution of equation (1) has not been contemplated (apart from examining the integrability conditions) but finding the solution for special symmetries of the space-time is a tractable problem.

The definition of symmetry in unified theories is more difficult than in general relativity. Until now, most researchers used the symmetric part of the fundamental tensor $g_{\mu\nu}$ as the metric tensor and defined symmetry in terms of form invariance (see Section 2) of $g_{\mu\nu}$ [3, 7, 8]. It immediately follows that the affine connection and Ricci tensor derived

* Address: Department of Applied Mathematics, University of Sydney, Sydney N.S.W. 2006, Australia.

from $g_{\mu\nu}$ are also form invariant but the solutions of equation (1) are not necessarily form invariant. Consequently, a stronger concept of symmetry is needed in the study of non--symmetric theories with no a priori metric.

2. *Universal symmetry*

A geometric object Ω is form invariant under a coordinate transformation $x^{\mu} \to x^{\mu'}$ if Ω' is the same function of its arguments $x^{\mu'}$ as Ω was of its arguments, that is $\Omega'(x^{\mu}) = \Omega(x^{\mu})$ for all x^{μ}.

Form invariance with respect to the infinitesimal transformation generated by $\xi^{\mu}(x)$

$$x^{\mu'} = x^{\mu}+\varepsilon\xi^{\mu}(x), \quad \varepsilon^2 \ll \varepsilon \tag{2}$$

is equivalent to the vanishing of the Lie derivative [13] with respect to ξ^{μ}

$$L_{\xi}(\Omega) = 0. \tag{3}$$

In general relativity, a coordinate transformation is said to be a symmetry of space-time if the metric tensor is form invariant under the transformation. For the special case of the (infinitesimal) transformation generated by ξ^{μ}, the vector field ξ^{μ} is called a motion of the space and the Riemannian metric $h_{\mu\nu}$ satisfies

$$L_{\xi}h_{\mu\nu} = 0. \tag{4}$$

Given a space (not necessarily Riemannian) with an affine connection $\Gamma^{\lambda}_{\mu\nu}$, the vector field ξ^{μ} is called an affine motion [9] if the connectione is invariant for the transformation generated by ξ^{μ}, that is if,

$$L_{\xi}\Gamma^{\lambda}_{\mu\nu} = 0. \tag{5}$$

Einstein's non-symmetric theory has an affine connection $\Gamma^{\lambda}_{\mu\nu}$ but no a priori metric. The connection can be expressed in terms of the fundamental tensor $g_{\mu\nu}$ and its first derivatives $g_{\mu\nu,\lambda}$ by solving the sixty-four algebraic field equations

$$g_{\mu\nu,\lambda}-\Gamma^{\alpha}_{\mu\lambda}g_{\alpha\nu}-\Gamma^{\alpha}_{\lambda\nu}g_{\mu\alpha} = 0. \tag{6}$$

Since $g_{\mu\nu}$ is a generalisation of the metric tensor in general relativity, form invariance of $g_{\mu\nu}$ is the simplest extension of symmetry to a non-symmetric theory. The properties of the Lie derivative and the solution of equations (6) [11] ensure that form invariance of $g_{\mu\nu}$ leads to form invariance of $\Gamma^{\lambda}_{\mu\nu}$ and the tensor constructed from $\Gamma^{\lambda}_{\mu\nu}$. This is enough to enable the field equations to be solved for special forms of $g_{\mu\nu}$ [7, 8, 12].

However, Einstein only used $g_{(\mu\nu)}$ as the metric tensor for simplicity in the early stages of the unified theory and once other metric tensors are considered, the definition of symmetry is more complicated. For example, form invariance of $g_{\mu\nu}$ and $\Gamma^{\lambda}_{\mu\nu}$ is not enough to ensure that the Klotz metric, defined as the solution of equation (1), is also form invariant.

Thus the traditional definition of symmetry in non-symmetric theories is not adequate and the concept of universal symmetry is introduced to rectify the situation.

The vector ξ^μ will be called a universal symmetry of the space and the space said to admit the universal symmetry ξ^μ if all the geometric objects defined in the space are form invariant with respect to the infinitesimal transformation generated by ξ^μ. For example, a space-time is universally spherically symmetric if it admits as universal symmetries the three vectors (in polar coordinates (r, θ, ψ, t))

$$\overset{1}{\xi}{}^\mu = (0, \sin\psi, \cot\theta\cos\psi, 0)$$

$$\overset{2}{\xi}{}^\mu = (0, -\cos\psi, \cot\theta\sin\psi, 0)$$

$$\overset{3}{\xi}{}^\mu = (0, 0, -1, 0). \tag{7}$$

It can be shown [10] that the (universally) spherically symmetric form of a tensor $C_{\mu\nu}$ is

$$C_{\mu\nu} = \begin{bmatrix} C_1 & 0 & 0 & C_4+C_5 \\ 0 & C_2 & C_6\sin\theta & 0 \\ 0 & -C_6\sin\theta & C_2\sin^2\theta & 0 \\ C_4-C_5 & 0 & 0 & C_3 \end{bmatrix}, \tag{8}$$

where the C_i's are functions of r and t.

In Section 3, a static, universally spherically symmetric space is considered. That is, a space with the three symmetries (7) plus the time symmetry

$$\overset{4}{\xi} = (0, 0, 0, 1). \tag{9}$$

Equation (3) for $\overset{4}{\xi}$ implies that all the geometric objects of the space are independent of time. The tensor $C_{\mu\nu}$ has the same form as in (7) but the C_i's are now functions of r only. The allowable group of coordinate transformations of r and t which preserve the static, universal spherical symmetry reduce to

$$r' = at+f(r),$$

$$t' = bt+g(r),$$

$$\frac{bdf}{dr} - \frac{adg}{dr} \neq 0, \qquad a, b \text{ constants}. \tag{10}$$

Given a tensor $C_{\mu\nu}$, it is always possible to choose coordinates r and t so that $C_4 = 0$, that is, the symmetric part of $C_{\mu\nu}$ is diagonal. Papapetrou [8] used this form for the fundamental tensor and the field equations were finally solved by Vanstone [12]. The fundamental tensor and affine connection used by Papapetrou are given in the Appendix.

3. *The Klotz metric*

Under the assumption that $a_{\mu\nu}$ was diagonal, Klotz [3] solved equations (1) for a spherically symmetric space and in a series of publications [4, 5, 6] extensively examined the physics of the solution. In this section, the equations will be examined for the most general tensor $a_{\mu\nu}$.

It has been shown that in a universally spherically symmetric space, a symmetric tensor can be reduced to a diagonal form by a suitable coordinate transformation, though in general, two distinct symmetric tensors cannot be simultaneously diagonalised. Consequently, if Papapetrou's fundamental tensor is used, the Klotz metric $a_{\mu\nu}$ will have the form

$$a_{\mu\nu} = \begin{bmatrix} a_1 & 0 & 0 & a_4 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_2 \sin^2\theta & 0 \\ a_4 & 0 & 0 & a_3 \end{bmatrix}$$

and the solution of (1) breaks into three distinct cases depending on whether $a_{14} = a_4$ is zero, a non-zero constant or nonconstant.

(*i*) $a_{14} = a_4 = 0$. Klotz [3] examined the case $a_{14} = 0$, equivalent to assuming that both the fundamental tensor and $a_{\mu\nu}$ are diagonal. The solution is

$$a_1 = \left(1 - \frac{\lambda k^2}{4r}\right)^{-1}, \quad a_2 = r^2, \quad a_3 = \frac{4}{k^2}\left(1 - \frac{\lambda k^2}{4r}\right)\left(1 + \frac{k^2}{r^4}\right),$$

or

$$a_1 = \frac{2}{\left(1 - \frac{r^2}{r_0^2}\right)\left(1 + c\sqrt{\frac{r_0^2}{r^2} - 1}\right)}, \quad a_2 = r^2, \quad a_3 = w\left(1 + c\sqrt{\frac{r_0^2}{r^2} - 1}\right),$$

whre λ, k, r_0, w, c are constants.

(*ii*) a_{14} a non-zero constant. When the general form of the connection $\Gamma^\lambda_{(\mu\nu)}$ (see Appendix) is substituted into the system of algebraic and differential equations (1), it can be shown that all the components of $a_{\mu\nu}$ are constants and the only non-zero components of $\Gamma^\lambda_{\mu\nu}$ are $\Gamma^2_{33} = -\sin\theta\cos\theta$, $\Gamma^3_{23} = \cot\theta$.

(*iii*) a_{14} not a constant. Repeating the procedure given in (*ii*), it follows that $a_{11} = a_0\alpha$, $a_{22} = b_0$, $a_{44} = c_0$, $a_{14} = d_0\sqrt{\alpha}$, a_0, b_0, c_0, d_0 constants, and the only non-zero components of $\Gamma^\lambda_{(\mu\nu)}$ are $\Gamma^1_{11} = \alpha'/2\alpha$, $\Gamma^2_{33} = -\sin\theta\cos\theta$, $\Gamma^3_{(23)} = \cot\theta$. If new coordinates are chosen so that $a_{\mu\nu}$ is diagonal, it has the form $a_{11} = a_0\alpha(r)$, $a_{22} = b_0$, $a_{33} = b_0\sin^2\theta$, $a_{44} = c_0$, a_0, b_0, c_0 constant.

For either of the two cases (*ii*) or (*iii*), substitution of the surviving components of $\Gamma^\lambda_{\mu\nu}$ into the field equations yields the condition $\sin^2\theta = 0$.

Hence cases (*ii*) and (*iii*) lead to physically unacceptable solutions and can be eliminated. The only remaining solutions are those found by Klotz [3] for a diagonal tensor, $a_{\mu\nu}$.

4. Discussion

The initial identification of $g_{(\mu\nu)}$ and $g_{[\mu\nu]}$ as the metric and electromagnetic tensors, respectively, made form invariance of $g_{\mu\nu}$ a natural definition of symmetry in the unified theory.

However, this identification of tensors was only tentative and recent work suggests that other choices are more appropriate [2, 3]. The inadequacies of simply considering form invariance of $g_{\mu\nu}$ was emphasized by the metrisation procedure of Klotz [3]: invariance of $g_{\mu\nu}$ and $\Gamma^{\lambda}_{\mu\nu}$ does not imply that the solution of equation (1) is also form invariant. Even for a Riemannian space with metric $h_{\mu\nu}$ and Christoffel symbol $\left\{ {\lambda \atop \mu\nu} \right\}$, it is possible to have an affine motion ξ^{μ} which is not a motion of the space. That is $h_{\mu\nu}$ and $\left\{ {\lambda \atop \mu\nu} \right\}$ satisfy

$$h_{\mu\nu,\lambda} - \left\{ {\alpha \atop \mu\lambda} \right\} h_{\alpha\mu} - \left\{ {\alpha \atop \nu\lambda} \right\} h_{\mu\alpha} = 0,$$

$$L_{\xi} \left\{ {\lambda \atop \mu\nu} \right\} = 0 \quad \text{and} \quad L_{\xi} h_{\mu\nu} \neq 0.$$

Universal symmetry generalises the definition of symmetry from form invariance of the metric tensor to form invariance of all the geometric objects of the space. At first this appears a much stronger notion but in a Riemannian space it follows from the properties of the Lie derivative that if the metric tensor is form invariant, the curvature tensor, Ricci tensor, Christoffel bracket etc. are also form invariant. Similarly, in the Einstein–Maxwell theory, form invariance of the metric and the electromagnetic tensor implies that all the tensors and connections used in the theory are form invariant.

Once the metric and electromagnetic tensors are identified in the non-symmetric theory, it is expected that the fundamental tensor and affine connection will be easily defined in terms of them and the property of universal symmetry will follow from form invariance of the two physical tensors.

APPENDIX

Einstein's weak field equations are

$$g_{\mu\nu,\lambda} - \Gamma^{\sigma}_{\mu\lambda} g_{\sigma\nu} - \Gamma^{\sigma}_{\lambda\nu} g_{\mu\sigma} = 0, \tag{11}$$

$$\Gamma_{\mu} = \tfrac{1}{2}(\Gamma^{\sigma}_{\mu\sigma} - \Gamma^{\sigma}_{\sigma\mu}) = 0, \tag{12}$$

$$R_{(\mu\nu)} = 0, \tag{13}$$

$$R_{[\mu\nu,\lambda]} = 0, \tag{14}$$

where

$$R_{\mu\nu} = -\Gamma^{\sigma}_{\mu\nu,\sigma} + \Gamma^{\sigma}_{\nu\sigma,\mu} + \Gamma^{\varrho}_{\mu\sigma}\Gamma^{\sigma}_{\varrho\nu} - \Gamma^{\varrho}_{\mu\nu}\Gamma^{\sigma}_{\varrho\sigma}$$

is the generalised Ricci tensor, $\Gamma^{\lambda}_{\mu\nu}$ the non-symmetric affine connection and $g_{\mu\nu}$ the non--symmetric fundamental tensor.

Papapetrou [8] considered a fundamental tensor of the form

$$g_{\mu\nu} = \begin{bmatrix} -\alpha & 0 & 0 & \omega \\ 0 & -\beta & f\sin\theta & 0 \\ 0 & -f\sin\theta & -\beta\sin^2\theta & 0 \\ -\omega & 0 & 0 & \sigma \end{bmatrix},$$

where α, β, σ, f, ω are functions of r and the coordinates are:

$$x^{\mu} = (r, \theta, \psi, t).$$

The sixty-four equations (11) can be solved algebraically to give the affine connection in terms of the fundamental tensor [11] and the solution is

$$\Gamma^1_{11} = \frac{\alpha'}{2\alpha}, \quad \Gamma^1_{22} = \Gamma^1_{33}\operatorname{cosec}^2\theta = \frac{1}{2\alpha}(fB' - \beta A'),$$

$$\Gamma^1_{44} = \frac{\sigma}{2\alpha}(\ln yU)', \quad \Gamma^2_{33} = -\sin\theta\cos\theta,$$

$$\Gamma^3_{(23)} = \cot\theta, \quad \Gamma^2_{(12)} = \Gamma^3_{(13)} = \tfrac{1}{2}A' = \tfrac{1}{2}\frac{\varrho'}{\varrho},$$

$$\Gamma^2_{(34)} = -\Gamma^3_{(24)}\sin^2\theta = \frac{\omega B'}{2\alpha}\sin\theta,$$

$$\Gamma^4_{(14)} = \tfrac{1}{2}y'/y \quad \Gamma^1_{[23]} = \frac{1}{2\alpha}(\beta B' + fA')\sin\theta,$$

$$\Gamma^2_{[13]} = -\Gamma^3_{[12]}\sin^2\theta = B'/2\sin\theta,$$

$$\Gamma^1_{[14]} = -2\Gamma^2_{[24]} = -2\Gamma^3_{[34]} = \frac{\omega\varrho'}{\alpha\beta} = \frac{\sigma}{2\omega}(\ln U)',$$

where

$$\varrho^2 = f^2 + \beta^2 = e^{2A}, \quad U = 1 - \frac{\omega^2}{\alpha\sigma}, \quad y = \sigma U, \quad \tan B = \beta/f.$$

The field equations (12), (13), (14) reduce to

$$U = \frac{\varrho^2}{k^2 + \varrho^2}, \quad k \text{ constant},$$

$$R_{11} = R_{22} = R_{44} = 0, \quad R_{[23]} = c\sin\theta.$$

REFERENCES

[1] S. Gołąb, *Tensor Calculus*, Warsaw 1974.
[2] L. J. Gregory, A. H. Klotz, *Acta Phys. Pol.* **B8**, 601 (1977).
[3] A. H. Klotz, *Acta Phys. Pol.* **B9**, 573 (1978).
[4] A. H. Klotz, *Acta Phys. Pol.* **B9**, 595 (1978).
[5] A. H. Klotz, *Acta Phys. Pol.* **B10**, 589 (1979).
[6] A. H. Klotz, *Lett. Nuovo Cimento* **25**, 191 (1979).
[7] A. H. Klotz, G. K. Russell, *Acta Phys. Pol.* **B2**, 623 (1971).
[8] A. Papapetrou, *Proc. Roy. I. A.* **52**, 69 (1948).
[9] J. A. Schouten, *Ricci Calculus*, Würzburg 1954.
[10] H. Takeno, *The Theory of Spherically Symmetric Space-Times*, Hiroshima 1966.
[11] M. A. Tonnelat, *Einstein's Unified Field Theory*, New York 1966.
[12] J. R. Vanstone, *Can. J. Math.* **14**, 568 (1962).
[13] K. Yano, *The Theory of Lie Derivatives and its Applications*, Groningen 1955.