THERE IS NO SLOW UNIFORM CONTRACTION OF A FLUID SPHERE OBEYING AN EQUATION OF STATE*

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It is shown that there is no slowly uniform contraction of a spherically symmetric fluid configuration in general relativity — as opposed to the Newtonian case — obeying an equation of state $p = p(\varrho)$ and having an energy-momentum tensor with compact support.

1. *Introduction*

In this paper we will look at spherically symmetric solutions of Einstein equations representing a finite fluid sphere obeying an equation of state $p = p(\varrho)$. McVittie [1] conjectured that in the non-static case, the equation of state should be of the form $p = p(\varrho, t, r)$. It has been shown actually in [2] that in the non-static case there is no solution of Einstein equations with an equation of state $p = p(\varrho)$.

In the static case the situation is a bit different. There we know already such solutions [3]. Let us look at this case more carefully. Imagine a fluid sphere of mass M (Schwarzschild constant) with an equation of state $p = p(\varrho)$. We now consider an infinitesimal symmetry-preserving uniform perturbation where the total mass M remains constant. We look at this perturbation in a dynamical sense. This means that the perturbed configurations should be linked by the dynamical (non-static) field equations. In other words we consider slowly uniform expansion of a spherically symmetric configuration. A requirement for an equation of state is that it should be also valid under such perturbations. Otherwise the intuitive concept of a (global) equation of state breaks down. This requirement is also tacitly made in all treatments of stability of gravitating systems. In the following we will show that spherically symmetric static finite fluid spheres do not allow — as opposed to the Newtonian case (Section 2) — any uniform symmetry preserving perturbations.

One can look at this result from two different points of view. From thermodynamical considerations of gravitating systems it is to be expected that the pressure p depends also

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explicitly on some non-local variable specifying the system, such as the radius of the system R as the only degree of freedom in the case of spherical symmetry. As the system is perturbed, R will vary, and therefore p as a function of ϱ alone will not have the same functional dependence as before. This could explain the negative result of the paper. But one could ask: how does the general theory of relativity know about the thermodynamical behavior of the system? This is the second aspect of the result. It seems that the mathematics of general relativity "knows" in advance the physics of the situation [4].

2. *The Newtonian case*

We consider the uniform expansion (contraction) of a fluid sphere in Newton's theory. An expansion (contraction) of a spherical distribution of matter is said to be uniform if the distance between any two points is altered in the same way as the radius of the configuration [5]. Let the radii of the initial and the final configuration of the fluid sphere be R_0 and R_1, and further let

$$R_1 = y \cdot R_0.$$

Then, if r_1 and r_0 are the distances of any specified element of matter from the center before and after the expansion

$$r_1 = yr_0$$

if the expansion has been carried out uniformly. Let ϱ_0, p_0 and ϱ_1, p_1 be the density and pressure at corresponding points. It is clear that

$$\varrho_1 = y^{-3}\varrho_0 \tag{3}$$

since the corresponding volume elements in the two configurations are in the ratio y^{-3}, while the mass enclosed in either is the same.

Consider now a gas sphere in gravitational equilibrium. Then

$$\frac{dp}{d\tau} = -\frac{GM(r)}{r^2}\varrho,$$

where r denotes the radius vector. Furthermore $M(r)$ is the mass enclosed inside a spherical surface of radius r. G is the gravitational constant. It is an easy task to infer from the above equation written down for the corresponding points r_0 and r_1 that

$$p_1 = y^{-4}p_0 \tag{4}$$

and with (3)

$$\frac{p_1}{p_0} = \left(\frac{\varrho_1}{\varrho_0}\right)^{4/3} \quad \text{or} \quad p_1 = \left(\frac{p_0}{\varrho_0^{4/3}}\right)\varrho_1^{4/3}. \tag{5}$$

Thus, if a gas sphere expands (contracts) uniformly through a sequence of equilibrium configurations, then the matter at every point undergoes a polytropic change belonging to the exponent $\gamma = 4/3$ or $n = 3$ (Ritter's Theorem [5]).

Now imagine that the equilibrium gas sphere obeys the following polytropic equation of state,

$$p = k \cdot \varrho^{\gamma'}.$$

Then, according to a theorem due to Emden, we obtain as a result of the uniform expansion another polytropic of index γ' with a polytropic temperature different from the polytropic temperature of the original fluid sphere:

$$p' = k' \varrho'^{\gamma'}.$$

If we denote by ϱ_0, p_0 and ϱ_1, p_1 the density and pressure at a point of the gas configuration before and after the expansion, we get

$$\frac{k'}{k} = \frac{p_1}{p_0} \cdot \frac{\varrho_0^{\gamma'}}{\varrho_1^{\gamma'}}.$$

If we substitute for p_1/p_0 from (5), we obtain

$$\frac{k'}{k} = \left(\frac{\varrho_1}{\varrho_0}\right)^{4/3} \cdot \left(\frac{\varrho_0}{\varrho_1}\right)^{\gamma'} = \left(\frac{\varrho_0}{\varrho_1}\right)^{\gamma' - 4/3}.$$

Therefore in the case of $\gamma' = 4/3$ we have

$$k' = k.$$

This means that the polytropic temperature remains constant during the expansion. It is also a known fact that in this case the entropy remains unchanged.

But we know that the Lane–Emden equations for $n = 3$ or $\gamma' = 4/3$ give a solution which corresponds to a finite sphere of gas configuration. Therefore the Newtonian theory allows a finite fluid sphere obeying an equation of state $p \equiv p(\varrho) = k\varrho^{4/3}$, which admits uniform perturbation with the same equation of state.

3. *The general relativistic case*

3.1. Specifying the metric for the case of uniform expansion

We start from the general metric for spherically-symmetric time-dependent solutions of the Einstein field equations in which the source of the gravitational field is a perfect fluid whose energy momentum tensor has a compact support. The metric can be written in the form

$$ds^2 = e^{2\alpha}dt^2 - e^{2\beta}dr^2 - e^{2\mu}d\Omega^2, \tag{6}$$

where α, β and μ are functions of r and t, and

$$d\Omega^2 = d\vartheta^2 + \sin^2 \vartheta d\varphi^2. \tag{7}$$

This metric should describe the region of space-time $0 \leqslant r \leqslant r_b$, $0 \leqslant \vartheta \leqslant \pi$, $0 \leqslant \varphi \leqslant 2\pi$, $-\infty \leqslant t \leqslant \infty$ filled with matter co-moving with the coordinates $(t, r, \vartheta, \varphi)$. Therefore the four-velocity of the fluid is given by [9]

$$u^\mu = e^{-\alpha}\delta_4^\mu$$

and satisfies

$$u^{\mu} u_{\mu} = 1.$$

The energy momentum tensor is that of an ideal fluid

$$T^{\mu\nu} = (p+\varrho) u^{\mu} u^{\nu} + p g^{\mu\nu}.$$

Now we formulate the condition for a uniform expansion. Let x_{μ} be the connecting vector of two worldlines of neighboring particles [6]. Then the relative position vector lying in the rest frame of an observer with the velocity u^{μ} is $x_{\perp\mu} = h^{\nu}_{\varrho}\, x_{\nu}$ where the projection tensor $h_{\mu\nu}$ is defined by $h_{\mu\nu} = g_{\mu\nu} - u_{\mu} u_{\nu}$. It can be split into a relative distance δl of neighboring particles and a direction n_{μ}; then $x_{\perp\mu} = n_{\mu} \cdot \delta l$, where $n_{\mu} n^{\mu} = -1$. The rate-of--change of relative distance is

$$\frac{(\delta l)^{\cdot}}{\delta l} = \sigma_{\mu\nu} n^{\mu} n^{\nu} + \tfrac{1}{3}\,\theta,$$

where the shear tensor $\sigma_{\mu\nu}$ and the expansion θ are defined by [6]

$$\sigma_{\mu\nu} = \tfrac{1}{2}\,(v_{\mu\nu} + v_{\nu\mu}) - \frac{\theta}{3}\, h_{\mu\nu},$$

$$\theta = v^{\mu}{}_{\mu},$$

and

$$v^{\mu}{}_{\nu} = h^{\mu}{}_{\gamma} h^{\sigma}{}_{\nu} u^{\gamma}{}_{;\sigma}.$$

For a uniform expansion the following conditions should be satisfied:

a) the rate-of-change of relative distance should be direction independent,

b) it should be independent of the position r.

The first condition means $\sigma_{\mu\nu} = 0$, and the second one is equivalent to the independence of θ from r. For the present metric, we have

$$\sigma \equiv (\tfrac{1}{2}\,\sigma_{\mu\nu}\sigma^{\mu\nu})^{1/2} = \sqrt{\tfrac{1}{3}}\, e^{-\alpha}(\beta_{,t} - \mu_{,t}),$$

$$\theta = e^{-\alpha}(\beta_{,t} + 2\mu_{,t}).$$

The condition $\sigma_{\mu\nu} = 0$ yields

$$\beta_{,t} = \mu_{,t} \tag{8}$$

and therefore

$$\theta = 3 e^{-\alpha} \mu_{,t}$$

or

$$\tfrac{1}{3}\,\theta = e^{-\alpha}\,\frac{R_{,t}}{R} \tag{9}$$

where we have set

$$\mu = \log R(r, t). \tag{10}$$

It will be shown below (see Eq. (19)) that, as a consequence of the field equations, θ does not depend on r.

We have therefore shown that the condition $\mu_{,t} = \beta_{,t}$ together with the requirement that the coordinate system in which Eq. (6) holds is a co-moving one and the Einstein field equations guarantee the uniform expansion of the fluid sphere (as was defined for the Newtonian case).

For the co-moving coordinate system $(t, r, \vartheta, \varphi)$, the variable r is the analogue of the Lagrangian coordinate of classical hydrodynamics. The function $R(r, t)$ is the Eulerian coordinate of that theory. That is $R(r, t)$ is the coordinate position at time t of the fluid particle which at $t = 0$ was at the coordinate position r, if we require that $R(r, 0) = r$. With this interpretation of $R(r, t)$ it follows that

$$U \equiv e^{-\alpha}R_{,t} = \tfrac{1}{3}\,\theta \cdot R \tag{11}$$

is the rate-of-change of R with respect to proper time relative to the observer at $t =$ const, $\vartheta =$ const, $\varphi =$ const.

Using the relation (8), the metric (6) can be written in the following form:

$$ds^2 = e^{2\alpha}dt^2 - \frac{R^2}{f^2}(dr^2 + f^2 d\Omega^2), \tag{12}$$

where

$$\log f(r) = \mu - \beta,$$

or

$$ds^2 = e^{2\alpha}dt^2 - R^2(d\bar{r}^2 + d\Omega^2), \tag{13}$$

where $d\bar{r} = dr/r$.

3.2. The field equations

To gain more insight into the nature of the problem under consideration, we first write the field equations for the metric (6). We have

$$-G_0^0 = e^{-2\alpha}(\mu_{,t}^2 + 2\mu_{,t}\beta_{,t}) - e^{-2\beta}(2\mu_{,rr} + 3\mu_{,r}^2 - 2\mu_{,r}\beta_{,r}) + e^{-2\mu}, \tag{14}$$

$$-G_1^1 = e^{-2\alpha}(2\mu_{,tt} + 3\mu_{,t}^2 - 2\mu_{,t}\alpha_{,t}) - e^{-2\beta}\mu_{,r}(\mu_{,r} + 2\alpha_{,r}) + e^{-2\mu}, \tag{15}$$

$$\begin{aligned} -G_2^2 = {} & e^{-2\alpha}(\beta_{,tt} + \mu_{,tt} + \mu_{,t}^2 + \beta_{,t}^2 - \beta_{,t}\alpha_{,t} + \mu_{,t}(\beta_{,t} - \alpha_{,t})) \\ & - e^{-2\beta}(\alpha_{rr} + \mu_{rr} + \mu_r^2 + \alpha_{,r}^2 - \alpha_{,r}\beta_{,r} + \mu_{,r}(\alpha_{,r} - \beta_{,r})), \end{aligned} \tag{16}$$

$$-G_4^1 = 2e^{-2\beta}(\mu_{,rt} - \mu_{,t}\alpha_{,r} - \mu_{,r}\beta_{,t} + \mu_{,t}\mu_{,r}). \tag{17}$$

Now the slowness of the expansion means that U given by (11) and the time derivatives are so small that their products (as well as second derivatives) may be neglected.

Next we notice that all time derivatives in (14)–(16) are negligible, and therefore Eqs. (14)–(16) are identical with those of a static sphere. At any value of t the configuration is therefore a static one. This sequence of static models is linked by (17) and (11) which determines U, and thus how the material moves. We further assume that all members of the sequence have the same mass M and therefore are matched to the same external Schwarzschild metric. Subject to this one restriction we are, however, quite free. In other words, if we take any continuous one-parameter family of static spheres, all of the same mass M, then suitable internal motions of the material as described by the equation $G_4^1 = \kappa T_4^1$ will generally deform the models into each other, provided the parameter of the family is treated as a function of time t only.

Now the field equation $G_4^1 = \kappa T_4^1$ plus the condition that the coordinate system be co-moving, that is the condition

$$T^{14} = T_{41} = 0$$

implies that

$$-G_4^1 = 2e^{-2\beta}(\mu_{,rt} - \mu_{,t}\alpha_{,r} - \mu_{,r}\beta_{,t} + \mu_t\mu_r) = 0.$$

It then follows from the condition (8)

$$\mu_{,rt} = \mu_{,t}\alpha_{,r}$$

or

$$e^{\alpha} = \frac{R_{,t}}{R} \cdot \frac{P}{P_{,t}}, \tag{18}$$

where R is defined by (10) and P is an arbitrary function of time. We therefore see that

$$\tfrac{1}{3}\theta = e^{-\alpha}\frac{R_{,t}}{R} = \frac{P_{,t}}{P} \tag{19}$$

is independent of r, as was required for a uniform expansion.

For our purposes it is more suitable to work with the field equations in a form first given by Misner and Sharp [7]. In a modified form they have been used in [8] and [2]. In what follows we use these modified field equations as they are given in [2] and look for corresponding equations in the case of slow motion. We will use the metric (13) writing it in the unbarred r coordinate.

We define m by

$$R^2 = \frac{R_{,r}^2}{1 + U^2 - \dfrac{2Gm}{R}}$$

or

$$m = \tfrac{1}{2} R\left[1 + R^2\left(\frac{P_{,t}}{P}\right)^2 - \left(\frac{R_{,r}}{R}\right)^2\right]. \tag{20}$$

The Einstein field equations then imply that

$$m_{,r} = 4\pi\varrho R^2 R_{,r}, \tag{21}$$

$$m_{,t} = -4\pi p R^2 R_{,t}. \tag{22}$$

These two equations are valid also for the case of slow expansion. Note that, e.g., $m(r, t)$ as defined by (20) is not valid for the slow motion case, because it depends on U^2. Nevertheless the derivatives of m as given by (21), (22) are first order in U and in the time derivatives.

The condition that the stresses be isotropic, that is the condition

$$T_1^1 = T_2^2 = T_3^3$$

with the restriction (8) leads to the following equation for R, valid also for the slow motion case:

$$\left(\frac{1}{R}\right)_{,rr} = \frac{1}{R} - \frac{B(r)}{R^2}, \tag{23}$$

where $B(r)$ is an arbitrary function of its argument.

We now come to the last equations we need to consider. The conservation laws $T^{\mu\nu}{}_{,\nu} = 0$ imply that

$$R\varrho_{,t} + 3R_{,t}(\varrho + p) = 0, \qquad p_{,r} + (\varrho + p)\alpha_{,r} = 0,$$

which are first order equations in the time derivative. As a consequence of the existence of an equation of state and Eq. (18) we obtain from the above equations

$$R = \frac{h(r)}{X(x)}, \tag{24}$$

where h and X are suitable functions of r and x as defined in [2], and

$$x = \frac{Q(r)}{P(t)},$$

where $Q(r)$ is a function of r alone. It can be shown that the thermodynamic quantities p and ϱ depend only on this variable x. Equation (24) is again valid also for the case of slow motion.

It has been shown in [2] that Eqs. (21)–(24), together with the assumption that the pressure vanishes at the boundary of the fluid, are not consistent. We therefore conclude that there is no symmetry preserving uniform infinitesimal perturbation of a static fluid sphere obeying an equation of state $p = p(\varrho)$, if the perturbed configurations are to be linked by the dynamical equations.

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[9] Throughout the paper we use units in which $G = C = 1$. Greek indices run from 0 to 3. A comma denotes partial differentiation and a semicolon denotes covariant differentiation.