THE SUPERENERGY TENSOR OF THE EINSTEIN–ROSEN GRAVITATIONAL WAVE

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In this paper the explicit form of the components of the gravitational superenergy tensor, ${}_gS$, for the Einstein–Rosen gravitational wave is given.

1. Introduction

The purpose of this paper is to show, based on an example of the cylindrical Einstein–Rosen gravitational wave, that the superenergy tensor of the gravitational field which was introduced in [1] better describes a gravitational wave than the canonical energy-momentum pseudotensor.

2. The Einstein–Rosen gravitational wave

The interval for the exact solution of the vacuum Einstein's equations which was given by Einstein and Rosen possesses, in "cylindrical" coordinates $x^0 = ct$, $x^1 = \varrho$, $x^2 = \varphi$, $x^3 = z$ the following form

$$ds^2 = e^{2(\gamma-\psi)}(c^2dt^2 - d\varrho^2) - \varrho^2 e^{-2\psi}d\varphi^2 - e^{2\psi}dz^2. \tag{1}$$

The functions γ and ψ depend only on the coordinates x^0 and x^1 and satisfy the following system of partial, differential equations

$$\begin{aligned} \psi_{,11} + \frac{1}{\varrho}\psi_{,1} - \psi_{,00} &= 0, \\ \gamma_{,1} &= \varrho[\psi_{,1}^2 + \psi_{,0}^2], \\ \gamma_{,0} &= 2\varrho\psi_{,0}\psi_{,1}. \end{aligned} \tag{2}$$

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We get equations (2) if we substitute the metric tensor determined by interval (1) into the vacuum Einstein equations.

The Einstein–Rosen solution for the vacuum Einstein equations is interpreted physically as the cylindrical gravitational wave. It is easy to check, e.g., [2, 3], that for the Einstein–Rosen gravitational wave, in cylindrical coordinates ct, ϱ, φ, z, globally vanish the components, ${}_{E}t_{0\cdot}^{\cdot 0}$, ${}_{E}t_{0\cdot}^{\cdot k}$ of the canonical energy-momentum pseudotensor ${}_{E}t$ of the gravitational field $\{^{\alpha}_{\beta\gamma}\}$.[1]

Therefore, the "energy density" and the "energy flux" vanish globally for such gravitational wave. Consequently Einstein's pseudotensor, ${}_{E}t$, has no value in describing the energy-momentum transfer by the Einstein–Rosen gravitational wave in cylindrical coordinates.

3. *The components, ${}_{g}S_{\mu\cdot}^{\cdot\nu}$, of the gravitational superenergy tensor, ${}_{g}S$, for the Einstein–Rosen gravitational wave*

Let us calculate the components, ${}_{g}S_{\mu\cdot}^{\cdot\nu}$, of the gravitational superenergy tensor, ${}_{g}S$, for the Einstein–Rosen gravitational wave. We have the following analytic expression for these components, see e.g., [1]

$$
\begin{aligned}
{}_{g}S_{\mu\cdot}^{\cdot\nu}(P; v^{\varrho}) = \frac{2k}{9}\,(2\overset{0}{v}{}^{\alpha}\overset{0}{v}{}^{\beta} - \overset{0}{g}{}^{\alpha\beta})\,[&\overset{0}{K}{}^{\nu\lambda\sigma\cdot}_{\cdot\cdot\cdot\alpha}\overset{0}{K}{}_{\mu\lambda\sigma\beta}^{\cdot\cdot\cdot\cdot} + \overset{0}{K}{}^{\nu\lambda\sigma\cdot}_{\cdot\cdot\cdot\beta}\overset{0}{K}{}_{\mu\lambda\sigma\alpha}^{\cdot\cdot\cdot\cdot} \\
-\tfrac{1}{2}\,\delta^{\nu}_{\mu}\underset{0}{K}{}^{\lambda\sigma\gamma\cdot}_{\cdot\cdot\cdot\alpha}\underset{0}{K}{}_{\lambda\sigma\gamma\beta}^{\cdot\cdot\cdot\cdot} + \underset{0}{K}{}^{\nu\lambda\sigma\cdot}_{\cdot\cdot\cdot\alpha}\underset{0}{K}{}_{\mu\sigma\lambda\beta}^{\cdot\cdot\cdot\cdot} &+ \underset{0}{K}{}^{\nu\lambda\sigma\cdot}_{\cdot\cdot\cdot\beta}\underset{0}{K}{}_{\mu\sigma\lambda\alpha}^{\cdot\cdot\cdot\cdot} - \tfrac{1}{2}\,\delta^{\nu}_{\mu}\underset{0}{K}{}^{\lambda\sigma\gamma\cdot}_{\cdot\cdot\cdot\alpha}\underset{0}{K}{}_{\lambda\gamma\sigma\beta}^{\cdot\cdot\cdot\cdot} \\
&-\tfrac{1}{2}\,\delta^{\nu}_{\mu}\overset{0}{K}{}^{\gamma\delta\varrho\cdot}_{\cdot\cdot\cdot\beta}(\overset{0}{K}{}_{\gamma\delta\varrho\alpha}^{\cdot\cdot\cdot\cdot} + \overset{0}{K}{}_{\gamma\varrho\delta\alpha}^{\cdot\cdot\cdot\cdot})].
\end{aligned}
\tag{3}
$$

Using (1)–(3) we obtain

$$
\begin{aligned}
{}_{g}S_{0\cdot}^{\cdot 0} = \frac{4k}{9}\,e^{-4(\gamma-\psi)}\Bigg\{&\left[\gamma_{,00} - \gamma_{,11} - \frac{1}{\varrho}\,\psi_{,1}\right]^2 \\
+[\psi_{,11} - \gamma_{,0}\psi_{,0} + 2\psi_{,1}^2 - \psi_{,0}^2 - \gamma_{,1}\psi_{,1}]^2 &+ \left[\gamma_{,0}\left(\psi_{,1} - \psi_{,0} - \frac{1}{2\varrho}\right) - 2\psi_{,0}^2 - \psi_{,00}\right]^2 \\
+3\left[\psi_{,0}^2 + \psi_{,00} + \frac{1}{\varrho}\,\gamma_{,1} - \gamma_{,0}\psi_{,0} - \gamma_{,1}\psi_{,1}\right]^2 &+ 3[\gamma_{,0}\psi_{,0} - \psi_{,0}^2 + \gamma_{,1}\psi_{,1} - 2\psi_{,1}^2 - \psi_{,11}]^2\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
{}_{g}S_{0\cdot}^{\cdot 1} = (-)\,\frac{16k}{9}\,e^{-4(\gamma-\psi)}&\left[\gamma_{,0}\left(\psi_{,1} - \frac{3}{2\varrho}\right) + \gamma_{,1}\psi_{,0} - \psi_{,10}\right] \\
\times&\left[2\gamma_{,0}\psi_{,0} - 2\psi_{,0}^2 + 2\gamma_{,1}\psi_{,1} - 2\psi_{,1}^2 - \psi_{,11} - \psi_{,00} - \frac{1}{\varrho}\,\gamma_{,1}\right],
\end{aligned}
$$

[1] Only the components ${}_{E}t_{2\cdot}^{\cdot 2}$ and ${}_{E}t_{3\cdot}^{\cdot 3}$ of ${}_{E}t$ are different from zero in this case.

$$ {}_{g}S_{1\cdot}^{\cdot 1} = (-)\frac{4k}{9}\left\{e^{-4(\gamma-\psi)}\left[\left(\gamma_{,00}-\gamma_{,11}-\frac{1}{\varrho}\psi_{,1}\right)^2+8\left(\gamma_{,0}\left(\psi_{,1}-\frac{3}{2\varrho}\right)\right.\right.\right. $$

$$ \left.\left.+\gamma_{,1}\psi_{,0}-\psi_{,10}\right)^2\right]-3e^{-4(\gamma+\psi)}\left(\psi_{,1}^2-\psi_{,0}^2-\frac{1}{\varrho}\psi_{,1}^2\right)^2\Bigg\}, $$

$$ {}_{g}S_{2\cdot}^{\cdot 2} = (-)\frac{2k}{9}e^{-4(\gamma-\psi)}\Big\{2[\psi_{,11}-\gamma_{,0}\psi_{,0}+2\psi_{,1}^2-\psi_{,0}^2-\gamma_{,1}\psi_{,1}]^2 $$

$$ +3\left[\gamma_{,0}\left(\psi_{,1}-\frac{3}{2\varrho}\right)+\gamma_{,1}\psi_{,0}-\psi_{,10}\right]^2-6[\gamma_{,0}\psi_{,0}-\psi_{,0}^2-\gamma_{,1}\psi_{,1}-2\psi_{,1}^2-\psi_{,11}]^2\Big\}, $$

$$ {}_{g}S_{3\cdot}^{\cdot 3} = (-)\frac{4k}{9}e^{-4(\gamma-\psi)}\left\{\left[\gamma_{,0}\left(\psi_{,1}-\psi_{,0}-\frac{1}{2\varrho}\right)-2\psi_{,0}^2-\psi_{,00}\right]^2\right. $$

$$ \left.+3\left[\gamma_{,0}\left(\psi_{,1}-\frac{3}{2\varrho}\right)+\gamma_{,1}\psi_{,0}-\psi_{,10}\right]^2-3\left[\psi_{,0}^2+\psi_{,00}+\frac{1}{\varrho}\gamma_{,1}-\gamma_{,0}\psi_{,0}-\psi_{,1}\gamma_{,1}\right]^2\right\}. \tag{4} $$

The rest of the components of the tensor ${}_{g}S$ vanishes in this case.

4. *Conclusions*

It is easy to see from formulae (4) that the components ${}_{g}S_{\mu\cdot}^{\cdot\nu}$ of the gravitational superenergy tensor do not vanish for the Einstein–Rosen wave in cylindrical coordinates and, therefore, do not vanish in every other coordinates. In particular the components ${}_{g}S_{0\cdot}^{\cdot 0}$ and ${}_{g}S_{0\cdot}^{\cdot 1}$ which define, respectively, the superenergy density and the superenergy flux density do not vanish. Therefore, we can satisfactorily define the superenergy density and the superenergy flux density of the Einstein–Rosen gravitational wave. In contradiction, we cannot do this for the energy density and for the energy flux density because the gravitational field does not possess an energy-momentum tensor. In summary, we can say that the gravitational superenergy tensor, ${}_{g}S$, is the better tool for the local description of the Einstein–Rosen gravitational wave than is the energy-momentum pseudotensor ${}_{E}t$.

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