Vol. B11 (1980) ACTA PHYSICA POLONICA No 5

ON THE MEANING OF THE METRIC HYPOTHESIS IN THE NONSYMMETRIC UNIFIED FIELD THEORY

By A. H. Klotz and B. T. McInnes

Department of Applied Mathematics, University of Sydney*

(Received October 4, 1979)

The meaning and geometrical significance of the method whereby a metric can be identified in the nonsymmetric unified field theory is investigated especially in relation to the setting up of a spinor analysis. It is shown that there is a close relation between the latter and the metric hypothesis adopted. A possible connection between the nonsymmetric theory and quantum electrodynamics is then obtained.

1. *Introduction*

The aim of the unified field theory is to provide a comprehensive geometrical description of the macroscopic gravitation and electromagnetism. Recent investigations into the structure of the nonsymmetric version of the theory (Klotz 1978 a, b, c; 1979 a, b, c) show that this is achieved if the theory itself is based on what may be called a weak principle of geometrisation; in addition, of course, to Einstein's principle of hermitian symmetry (Einstein 1945; Einstein and Straus 1946).

In General Relativity one assumes a priori that a model of the world is given by a Riemannian V_4. The field equations then serve only to determine the particular metric corresponding to a given physical situation. More correctly, they determine the gravitational field arising therein. We can call this a strong principle of geometrisation. Its counterpart in a unified field theory would be to hypothesise some well defined, but non-Riemannian geometry and to try to write physics into it. However, this is impossible in the absence of the concept of equivalence, replaced by the much weaker hermitian symmetry as the only means of restricting the choice of possible field equations. All that can be assumed to start with about geometry is that it possesses an affine structure used to define invariant differentiation and to construct a generalised curvature tensor.

This does not mean that the nonsymmetric theory is purely affine. Affine connection $\Gamma^{\lambda}_{\mu\nu}$ is an exclusively geometrical concept, except for its contracted, skew symmetric part

$$\Gamma_{\lambda} = \Gamma^{\sigma}_{\underset{\vee}{\lambda\sigma}} = \tfrac{1}{2}\,(\Gamma^{\sigma}_{\lambda\sigma}-\Gamma^{\sigma}_{\sigma\lambda}). \tag{1}$$

* Address: Department of Applied Mathematics, University of Sydney, Sydney, N.S.W. 2006, Australia.

Physical fields are represented by the fundamental tensor $g_{\mu\nu}$ which, in a four dimensional space has sixteen components. And geometry and physics do not determine each other in the same, direct sense as in General Relativity. Indeed, they are both determined but only after the field equations are found (and the principle of hermitian symmetry gives them uniquely: Klotz and Russell 1973) and solved under some simplifying conditions (of geometrical symmetry). Under these circumstances it is impossible to predict (and it is wrong to postulate a priori) what is to be the metric tensor

$$a_{\mu\nu} = a_{\nu\mu}, \qquad \det(a_{\mu\nu}) < 0,$$

say, of the geometro-physical space-time manifold. Like the connection between physics and geometry, $a_{\mu\nu}$ must be somehow determined after we known the solutions of the field equations. By a metric tensor we mean something more than just an entity used for raising and lowering of tensor indices. Any nonsingular, symmetric tensor could serve this purpose. Above all, the metric tensor determines the primitive, invariant distance measurement in the manifold and hence also the light-tracks therein.

A macroscopic field theory seems meaningless without such a tensor being defined in a nontrivial manner.

In the work referred to above (Klotz 1978a) a metric hypothesis has been assumed according to which $a_{\mu\nu}$ should be determined by the equation

$$\left\{ {\lambda \atop \mu\nu} \right\}_a = \tilde{\Gamma}^{\lambda}_{\underline{\mu\nu}}, \tag{2}$$

or

$$a_{\mu\nu;\lambda}(\tilde{\Gamma}^{\alpha}_{\underline{\beta\gamma}}) = a_{\mu\nu,\lambda} - a_{\sigma\nu}\tilde{\Gamma}^{\sigma}_{\underline{\mu\lambda}} - a_{\mu\sigma}\tilde{\Gamma}^{\sigma}_{\underline{\nu\lambda}} = 0,$$

where $\tilde{\Gamma}^{\lambda}_{\underline{\mu\nu}}$ is the symmetric part of the affine connection of the weak field equations which in Einstein's notation read

$$g_{\underset{+}{\mu}\underset{-}{\nu};\lambda}(\tilde{\Gamma}) = 0, \qquad \tilde{\Gamma}_{\mu} = 0, \qquad \tilde{R}_{\underline{\mu\nu}} = 0, \qquad \tilde{R}_{\mu\nu,\lambda} = 0. \tag{3}$$

The "twiddle" over an entity denotes that it should be constructed from $\tilde{\Gamma}^{\lambda}_{\mu\nu}$ which itself is related to the original, abstract connection $\Gamma^{\lambda}_{\mu\nu}$ by Schrödinger's equation

$$\tilde{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} + \tfrac{2}{3}\,\delta^{\lambda}_{\mu}\Gamma_{\nu}. \tag{4}$$

The aim of this article is to explore the geometrical and physical meaning of the hypothesis (2). It was introduced previously as an "inspired guess" in an attempt to find as general a relation as possible which would reduce to the general relativistic value

$$a_{\mu\nu} = g_{\underline{\mu\nu}}, \tag{5}$$

when Riemannian symmetry is reimposed. The far-reaching consequences as far as the field theory is concerned which result from (2), make the present investigation particularly imperative.

It turns out that the metric hypothesis, instead of being seemingly arbitrary, singles out a geometry of special significance not only in the realm of macrophysics (as shown in the work already cited) but also in setting up a covariant form of quantum electrodynamics.

Apart from the somewhat formal considerations of the next section, the remainder of this article will be concerned with constructing a spinor analysis in the non-Riemannian context of the nonsymmetric theory. The interaction between macroscopic field structure and quantum phenomena arises through the concept of "minimal coupling", that is replacement of ordinary partial derivatives by covariant operators. This may be justified if the latter involve only a gravitational correction (Christoffel brackets). The procedure, in the case of the generalised field theory, however is likely to make the Dirac equations depend on the tensor

$$\tilde{N}^{\lambda}_{\mu\nu} = \tilde{\Gamma}^{\lambda}_{\mu\nu} - \left\{ {\lambda \atop \mu\nu} \right\}, \tag{6}$$

which arises because of the presence of the electromagnetic field especially if we accept General Relativity as an accurate description of gravitation. A non-negligible effect may remain even in a flat-space approximation. This could lead to a difficulty since the theory described by the Dirac equations with an electromagnetic field term is empirically extremely well tested.

We shall find that the metric hypothesis (2) allows us to resolve the above problem albeit not entirely iniquely. Since we interpret Γ_{μ} as proportional to an electromagnetic vector potential (Gregory and Klotz 1977), we shall find it convenient to adopt a constant spinor for raising and lowering of spinor indices, (van der Waerden 1929) instead of the more elaborate van der Waerden-Infeld theory (eg. Bade and Jehle 1953 — we shall adopt throughout the notation of this review except when explicitely stated).

Throughout this article, Greek indices will take on the space-time values 0, 1, 2, 3 and capital Latin indices, the two component spinor values I, II, with the usual summation convention over either.

2. *Geometrical consequences of the metric hypothesis*

Let us define the covariant differential operators

$$\nabla^{+}{}_{\alpha} h_{\mu} = \partial_{\alpha} h_{\mu} - \Gamma^{\lambda}_{\mu\alpha} h_{\lambda}, \quad \nabla^{+}{}_{\alpha} h^{\mu} = \partial_{\alpha} h^{\mu} + \Gamma^{\mu}_{\lambda\alpha} h^{\lambda}, \tag{7}$$

and

$$\nabla^{-}{}_{\alpha} h_{\mu} = \partial_{\alpha} h_{\mu} - \Gamma^{\lambda}_{\alpha\mu} h_{\lambda}, \quad \nabla^{-}{}_{\alpha} h^{\mu} = \partial_{\alpha} h^{\mu} + \Gamma^{\mu}_{\alpha\lambda} h^{\lambda}. \tag{8}$$

(that is $h_{\underset{+}{\mu};\alpha}$, $h_{\underset{-}{\mu};\alpha}$ etc. in Einstein's notation).

Actually we shall make little use of ∇^{-}_{α} or of the operators $\tilde{\nabla}^{+}_{\alpha}$, $\tilde{\nabla}^{-}_{\alpha}$ which we could define similarly using Schrödinger's $\tilde{\Gamma}^{\lambda}_{\mu\nu}$ instatead of the "geometrical" connection $\Gamma^{\lambda}_{\mu\nu}$. All we need to remember is that, for example,

$$\nabla^{+}_{\alpha} h_{\mu\nu}$$

will stand for $h_{\underset{+}{\mu}\underset{+}{\nu};\alpha}$, and that (providing $g(g-2) \neq 0$, $g = \det g_{\mu\nu}$) $\tilde{\Gamma}^{\lambda}_{\mu\nu}$ is completely specified by the equations

$$\tilde{\nabla}^{+}_{\alpha} g_{\mu\nu} + 2\tilde{\Gamma}^{\lambda}_{\underset{\vee}{\nu\alpha}} g_{\mu\lambda} = 0,$$

or equivalently by

$$\tilde{\nabla}^{-}_{\alpha} g_{\mu\nu} + 2\tilde{\Gamma}^{\lambda}_{\underset{\vee}{\mu\alpha}} g_{\lambda\nu} = 0.$$

In general $\nabla^{+}_{\alpha} a_{\mu\nu}$ cannot vanish and we define the tensors

$$W_{\alpha\mu\nu} = W_{\alpha\nu\mu}, \qquad \tilde{W}_{\alpha\mu\nu} = \tilde{W}_{\alpha\nu\mu}, \tag{9}$$

by

$$\nabla^{+}_{\alpha} a_{\mu\nu} = W_{\alpha\mu\nu}, \qquad \tilde{\nabla}^{+}_{\alpha} a_{\mu\nu} = \tilde{W}_{\alpha\mu\nu}, \tag{10}$$

respectively.

We shall also find it convenient later to use the Riemannian operator D_α with Christoffel brackets (or their spinor analogues) as the connection.

With the necessarily nonsingular metric tensor $a_{\mu\nu}$ used for raising or lowering of tensor indices, we obtain from

$$a_{\mu\alpha} a^{\nu\alpha} = \delta^{\nu}_{\mu}, \tag{11}$$

the equations

$$\nabla^{+}_{\alpha} a^{\mu\nu} = -W_{\alpha}{}^{\mu\nu}, \qquad \tilde{\nabla}^{+}_{\alpha} a^{\mu\nu} = -\tilde{W}_{\alpha}{}^{\mu\nu}. \tag{12}$$

After these preliminary definition let us return to the basic ideas of the nonsymmetric theory. Regardless of its interpretation and of the field equations (3), the following three essentially geometrical concepts are relevant: the shortest (or rather extremal) distance, the path of no acceleration and geodesic curve.

The first two are defined respectively by

$$\delta \int ds = 0, \qquad \text{with} \qquad ds^2 = a_{\mu\nu} dx^\mu dx^\nu, \tag{13}$$

$$\frac{du^\alpha}{ds} + \Gamma^{\alpha}_{\underline{\mu\nu}} u^\mu u^\nu = 0, \qquad \text{with} \qquad u^\mu = \frac{dx^\mu}{ds}. \tag{14}$$

Similarly, for a geodesic, we have

$$h^\alpha - h^\beta \delta x^\gamma \Gamma^{\alpha}_{\beta\gamma} = (1+\delta L)\left(h^\alpha + \delta s\, \frac{dh^\alpha}{ds}\right) = h^\alpha + h^\alpha \delta L + \delta s\, \frac{dh^\alpha}{ds}$$

to the first order of smallness, the length L of the vector h^μ being defined by

$$L^2 = a_{\mu\nu} h^\mu h^\nu.$$

If now h^μ is the unit vector $u^\mu = \dfrac{\delta x^\mu}{\delta s}$ and s is the arc parameter along a space-time curve (to which u^μ is tangent) then L is 1, but

$$\delta L^2 = 2L\delta L = 2\delta L = u^\mu u^\nu \delta x^\sigma W_{\sigma\mu\nu},$$

so that

$$\frac{du^\alpha}{ds}+u^\beta u^\gamma \underline{\Gamma}^\alpha_{\beta\gamma}+\tfrac{1}{2}\, u^\alpha u^\mu u^\nu u^\sigma W_{\sigma\mu\nu} = 0. \tag{15}$$

We now see that in the absence of the electromagnetic field $\Gamma_\mu = 0$ when $\tilde{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu}$ (which does not by itself mean reversal to General Relativity) the curves (13), (14) and (15) coincide if $a_{\mu\nu}$ is defined by (2) since then

$$u^\mu u^\nu u^\sigma W_{\sigma\mu\nu} = u^\mu u^\nu u^\sigma a_{\mu\nu;\sigma} = 0.$$

We may also note that, as in General Relativity, we have

$$\frac{d}{ds}(a_{\mu\nu}u^\mu u^\nu) = 0,$$

while the equation

$$g_{\underset{+}{\mu}\,\underset{-}{\nu}\,;\lambda}(\tilde{\Gamma}) = 0,$$

implies that

$$\frac{d}{ds}(\underline{g}_{\mu\nu}u^\mu u^\nu) = 0,$$

too.

This, however, does not mean that

$$a_{\mu\nu} = \underline{g}_{\mu\nu}$$

because the vector u^μ cannot be chosen arbitrarily.
[For example in Klotz 1978a, we indeed have

$$g_{44} = a_{44},$$

in the spherically symmetric static case, but the remaining components of $a_{\mu\nu}$ do not take their general relativistic values except, of course, in a local approximation. This deviation from a general relativistic metric is the origin of the cosmological interpretation of the resulting space-time, (Klotz 1978c, 1979 and also in *Lett. Il Nuovo Cim.*).

We now turn to the main object of this article which is to investigate spinor algebra and analysis associated with the non-symmetric theory.

3. *Spinors in the nonsymmetric theory*

The preliminary results are independent of the nonsymmetric background and we follow the account of Bade and Jehle. Thus, since an hermitian spinor

$$h_{\dot{A}B} = h_{B\dot{A}},$$

and a four-vector h_μ are both determined by four real functions of the coordinates, there exist mixed (vector-spinor) quantities σ such that

$$h_{\dot{A}B} = \sigma^\mu{}_{\dot{A}B}h_\mu, \quad h^\mu = \sigma^{\mu\dot{A}B}h_{\dot{A}B} \tag{16}$$

which are hermitian with respect to their spinor indices. Then, arbitrary identification of the metric forms

$$a^{\mu\nu}h_\mu h_\nu \quad \text{and} \quad \gamma^{\dot{A}\dot{B}}\gamma^{CD}h_{\dot{A}C}h_{\dot{B}D},$$

where γ_{AB} is the fundamental (skewsymmetric) spinor, yields the defining equations of spinor algebra:

$$\sigma^{\mu\dot{A}B}\sigma^{\nu}{}_{\dot{A}B} = a^{\mu\nu}, \tag{17}$$

$$\sigma^{\mu}{}_{\dot{A}B}\sigma_{\mu}{}^{\dot{C}D} = \delta^{\dot{C}}{}_{\dot{A}}\delta^{D}{}_{B}, \tag{18}$$

and

$$\sigma^{\mu}{}_{\dot{A}B}\sigma^{\nu\dot{A}C}+\sigma^{\nu}{}_{\dot{A}B}\sigma^{\mu\dot{A}C} = a^{\mu\nu}\delta^{C}{}_{B}. \tag{19}$$

It is interesting to note that it is the metric tensor $a_{\mu\nu}$ and not, for example, $g_{\underline{\mu\nu}}$ which necessarily appears in the definitions. Spinor algebra is linked to the underlying Riemannian space and not to its nonsymmetric generalisation. This, of course, is as expected since spinors constitute representations of the Lorentz group.

Let us now define covariant derivatives of a spinor ψ^A by

$$D_\lambda\psi^A = \partial_\lambda\psi^A+\psi^B\begin{Bmatrix} A \\ \lambda B\end{Bmatrix},$$

$$\nabla_\lambda\psi^A = \partial_\lambda\psi^A+\psi^B\Gamma^A_{\lambda B},$$

$$\tilde{\nabla}_\lambda\psi^A = \partial_\lambda\psi^A+\psi^B\tilde{\Gamma}^A_{\lambda B}. \tag{20}$$

If the quantity

$$\psi_A\psi^A = -\psi^A\psi_A$$

is regarded as scalar, we also have

$$D_\lambda\psi_A = \partial_\lambda\psi_A-\psi_B\begin{Bmatrix} B \\ \lambda A\end{Bmatrix}, \text{ etc.}$$

Regarding now $\sigma^{\lambda}{}_{\dot{A}B}$ for the purpose of differentiation as an outer product $h^\lambda\psi_{\dot{A}}\psi_B$, it follows from (17) that

$$D_\mu\sigma^{\lambda}{}_{\dot{A}B} = D_\mu\sigma^{\lambda\dot{A}B} = 0. \tag{21}$$

From now on we shall write ∇_λ instead of ∇^+_λ when operating on tensor indices. (Since spinor and tensor indices are basically distinct there is no merit in distinguishing $\Gamma^A_{\lambda B}$ from something like $\Gamma^A_{B\lambda}$ which we would have to do if we wanted to preserve the ∇^+_λ and ∇^-_λ operators.) Also the tensor $W_{\lambda\mu\nu}$ cannot in general vanish in the nonsymmetric theory and hence $\nabla_\lambda\sigma^{\mu}{}_{\dot{A}B}$ must be a nonvanishing linear combination of the σ-spinors:

$$\nabla_\lambda\sigma^{\mu}{}_{\dot{A}B} = \partial_\lambda\sigma^{\mu}{}_{\dot{A}B}+\Gamma^{\mu}_{\nu\lambda}\sigma^{\nu}{}_{\dot{A}B}-\Gamma^{\dot{C}}_{\lambda\dot{A}}\sigma^{\mu}{}_{\dot{C}B}-\Gamma^{B}_{\lambda B}\sigma^{\mu}{}_{\dot{A}C} = H^{\mu}_{\lambda\nu}\sigma^{\nu}{}_{\dot{A}B}, \tag{22}$$

where $H^{\mu}_{\lambda\nu}$ is a tensor.

It now follows from the equations (17) and (12) that

$$H^{\mu}_{\lambda e}\sigma^{e}{}_{\dot{A}B}\sigma^{\nu\dot{A}B}+H^{\nu}_{\lambda e}\sigma^{\mu}{}_{\dot{A}B}\sigma^{e\dot{A}B} = -W_{\lambda}{}^{\mu\nu}$$

or

$$H_{\lambda}{}^{\underline{\mu\nu}} = -\tfrac{1}{2}\, W_{\lambda}{}^{\mu\nu}. \tag{23}$$

If we write similarly

$$\tilde{\nabla}_{\lambda}\sigma^{\mu}{}_{\dot{A}B} = \tilde{H}^{\mu}_{\lambda\nu}\sigma^{\nu}{}_{\dot{A}B}, \tag{24}$$

where $\tilde{H}^{\mu}_{\lambda\nu}$ is also (another) tensor, then

$$\tilde{H}_{\lambda}{}^{\underline{\mu\nu}} = -\tfrac{1}{2}\, \tilde{W}_{\lambda}{}^{\mu\nu}. \tag{25}$$

Equations (23) and (25) are of great importance in the sequel. They are independent of any assumptions about γ_{AB} except that we must have

$$D_{\lambda}(\gamma_{\dot{A}\dot{B}}\gamma_{CD}) = \nabla_{\lambda}(\gamma_{\dot{A}\dot{B}}\gamma_{CD}) = \tilde{\nabla}_{\lambda}(\gamma_{\dot{A}\dot{B}}\gamma_{CD}) = 0. \tag{26}$$

In the van der Waerden-Infeld analysis, these equations would lead to non vanishing solutions for

$$\begin{Bmatrix} A \\ \lambda A \end{Bmatrix}, \quad \Gamma^{A}_{\lambda A} \quad \text{and} \quad \tilde{\Gamma}^{A}_{\lambda A},$$

but as each of these would then introduce arbitrary vectors (presumably distinct) into the theory (see Bade and Jehle 1953 or Klotz 1962) we would have confusion with no obvious physical interpretation. Let us therefore assume that γ_{AB} is a constant spinor and that

$$D_{\lambda}\gamma_{AB} = 0. \tag{27}$$

Then

$$\begin{Bmatrix} A \\ \lambda A \end{Bmatrix} = 0,$$

and, from equation (21)

$$\begin{Bmatrix} C \\ \lambda B \end{Bmatrix} = \tfrac{1}{2}\, \sigma_{\mu}{}^{\dot{A}C}\left(\partial_{\lambda}\sigma^{\mu}{}_{\dot{A}B}+\sigma^{\nu}{}_{\dot{A}B}\begin{Bmatrix} \mu \\ \nu\lambda \end{Bmatrix}\right). \tag{28}$$

Using this equation together with equation (6) (and writing $N^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu}-\{{}^{\lambda}_{\mu\nu}\}$) it follows from the equations (22) and (24) respectively that

$$\Gamma^{C}_{\lambda B} = \begin{Bmatrix} C \\ \lambda B \end{Bmatrix} +\tfrac{1}{2}\, \sigma_{\mu}{}^{\dot{A}C}\sigma^{\nu}{}_{\dot{A}B}(N^{\mu}_{\nu\lambda}-H^{\mu}_{\lambda\nu})-\tfrac{1}{2}\, \Gamma^{\dot{A}}_{\lambda\dot{A}}\delta^{C}{}_{B}, \tag{20}$$

and

$$\tilde{\Gamma}^{C}_{\lambda B} = \begin{Bmatrix} C \\ \lambda B \end{Bmatrix} +\tfrac{1}{2}\, \sigma_{\mu}{}^{\dot{A}C}\sigma^{\nu}{}_{\dot{A}B}(\tilde{N}^{\mu}_{\nu\lambda}-\tilde{H}^{\mu}_{\lambda\nu})-\tfrac{1}{2}\, \tilde{\Gamma}^{\dot{A}}_{\lambda\dot{A}}\delta^{C}{}_{B}. \tag{30}$$

Let us now consider, the relations between the "twiddled" and untwiddled tensors. From their definitions (9) and (10) and the relation (4) we immediately deduce that

$$\tilde{W}_{\lambda\mu\nu} = W_{\lambda\mu\nu} - \tfrac{4}{3}\,\Gamma_\lambda a_{\mu\nu}, \tag{31}$$

and, therefore, from equations (23) and (25), that

$$\tilde{H}_{\lambda\underline{\mu\nu}} = H_{\lambda\underline{\mu\nu}} + \Gamma_\lambda a_{\mu\nu}, \tag{32}$$

and, of course, we also have

$$\tilde{N}^{\lambda}_{\mu\nu} = N^{\lambda}_{\mu\nu} + \tfrac{2}{3}\,\delta^{\lambda}_{\mu}\Gamma_\nu,$$

so that, if we define

$$N_{\mu\lambda\nu} = a_{\lambda\sigma}N^{\sigma}_{\mu\nu}, \qquad \tilde{N}_{\mu\lambda\nu} = a_{\lambda\sigma}\tilde{N}^{\sigma}_{\mu\nu},$$

then

$$\tilde{N}_{\mu\lambda\nu} = N_{\mu\lambda\nu} + \tfrac{2}{3}\,a_{\lambda\mu}\Gamma_\nu. \tag{33}$$

It follows from (33) that

$$\tilde{N}_{\overset{\vee}{\mu}\lambda\overset{\vee}{\nu}} = N_{\overset{\vee}{\mu}\lambda\overset{\vee}{\nu}}. \tag{34}$$

The relation (32) implies that we can write

$$\tilde{H}_{\lambda\mu\nu} = H_{\lambda\mu\nu} + \tfrac{2}{3}\,\Gamma_\lambda a_{\mu\nu} + A_{\lambda\mu\nu}, \tag{35}$$

where the tensor $A_{\lambda\mu\nu}$ is skew symmetric in its last pair of indices:

$$A_{\lambda\mu\nu} = -A_{\lambda\nu\mu}.$$

Let us return now to equations (29) and (30). We have

$$\sigma_\mu{}^{\dot{A}C}\sigma^{\nu}{}_{\dot{A}B}(N_\nu\mu_\lambda - H_\lambda\mu_\nu) = \sigma^{\mu\dot{A}C}\sigma^{\nu}{}_{\dot{A}B}(N_{\nu\mu\lambda} - H_{\lambda\mu\nu}). \tag{37}$$

We now show that only the part of this expression skew symmetric in μ and ν contributes to it. In fact the first of the equations (10) can be rewritten as

$$N_{\mu\nu\lambda} + N_{\nu\mu\lambda} = -W_{\lambda\mu\nu}. \tag{10}$$

Permuting the indices λ, μ, ν cyclically, adding two of the resulting equations and subtracting the third (and using round and square brackets for symmetric and skew symmetric parts respectively)

$$N_{(\mu|\nu|\lambda)} - N_{[\mu|\lambda|\nu]} - N_{[\lambda|\mu|\nu]} = -\tfrac{1}{2}\,(W_{\lambda\mu\nu} + W_{\mu\nu\lambda} - W_{\nu\lambda\mu}),$$

or

$$N_{\mu\nu\lambda} - H_{\lambda\nu\mu} = -\tfrac{1}{2}\,(W_{\lambda\mu\nu} + W_{\mu\nu\lambda} - W_{\nu\mu\lambda}) + S_{\mu\lambda\nu} + S_{\lambda\mu\nu} - S_{\lambda\nu\mu} - H_{\lambda\nu\mu}, \tag{38}$$

where we have put

$$S_{\mu\lambda\nu} = N_{[\mu|\lambda|\nu]} = a_{\lambda\sigma}N^{\sigma}_{\overset{\vee}{\mu\nu}} = a_{\lambda\sigma}\Gamma^{\sigma}_{\overset{\vee}{\mu\nu}} = -S_{\nu\lambda\mu}. \tag{39}$$

Symmetrising equation (38) over μ and ν, we get, in view of (39)

$$N_{\underline{\mu}\nu\lambda}-H_{\lambda\nu\underline{\mu}} = -\tfrac{1}{2}\, W_{\lambda\mu\nu}-H_{\lambda\nu\underline{\mu}} = 0, \tag{40}$$

by equation (23). Similarly of course,

$$\tilde{N}_{\underline{\mu}\nu\lambda}-\tilde{H}_{\lambda\nu\underline{\mu}} = 0. \tag{41}$$

We may note also that defining the tensors $T_{\lambda\mu\nu}$ and $\tilde{T}_{\lambda\mu\nu}$, by

$$T_{\lambda\mu\nu} = H_{\lambda\mu\nu}-S_{\nu\mu\lambda}, \qquad \tilde{T}_{\lambda\mu\nu} = \tilde{H}_{\lambda\mu\nu}-\tilde{S}_{\nu\mu\lambda}, \tag{42}$$

we can write equation (38) and the identical "twiddled" equation in the form

$$N_{\mu\nu\lambda}-H_{\lambda\nu\mu} = \tfrac{1}{2}\,[T_{\dot{\lambda}\dot{\mu}\dot{\nu}}-T_{\dot{\lambda}\dot{\nu}\dot{\mu}}+2(T_{\mu\lambda\nu}-T_{\nu\lambda\mu})], \tag{43}$$

and

$$\tilde{N}_{\mu\nu\lambda}-\tilde{H}_{\lambda\nu\mu} = \tfrac{1}{2}\,[\tilde{T}_{\dot{\lambda}\dot{\mu}\dot{\nu}}-\tilde{T}_{\dot{\lambda}\dot{\nu}\dot{\mu}}+2(\tilde{T}_{\mu\lambda\nu}-\tilde{T}_{\lambda\nu\mu})], \tag{44}$$

(using (23) and (25)) from which equations (40) and (41) follow at once. We shall need these forms later.

Finally, let us define (with Bade and Jehle) a curvature spinor

$$P^{A}{}_{B\lambda x} = -2\partial_{[x}\Gamma_{\lambda]}{}^{A}{}_{B}+2\Gamma^{C}{}_{[x|B|}\Gamma_{\lambda]}{}^{A}{}_{C}. \tag{45}$$

Then, using the equation (22) and the commutator

$$(\nabla_x\nabla_\lambda-\nabla_\lambda\nabla_x)\sigma^{\mu}{}_{\dot{A}B},$$

we readily find that

$$P^{A}{}_{B\lambda x} = \tfrac{1}{2}\, R_{\mu\nu\lambda x}\sigma^{\mu A\dot{C}}\sigma^{\nu}{}_{\dot{C}B}+(\tfrac{1}{2}\, S^{\mu}{}_{\nu\lambda x}+\Gamma^{\varrho}_{\lambda x}H^{\mu}_{\varrho\check{\nu}})\sigma_{\mu}{}^{A\dot{C}}\sigma^{\nu}{}_{\dot{C}B}-\tfrac{1}{2}\, P^{\dot{C}}{}_{\dot{C}\lambda x}\delta^{A}{}_{B}, \tag{46}$$

where the tensor $S^{\mu}{}_{\nu\lambda x}$ is given by

$$S^{\mu}{}_{\nu\lambda x} = \nabla_x H^{\mu}_{\lambda\nu}-\nabla_\lambda H^{\mu}_{x\nu}-H^{\varrho}_{\lambda\nu}H^{\mu}_{x\varrho}+H^{\varrho}_{x\varrho}H^{\mu}_{\lambda\varrho}. \tag{47}$$

The above are purely formal results. We must now consider their application to physics and in particular to the problem of constructing an analogue of the Dirac equations in the curved space of the nonsymmetric theory.

4. Dirac equations in the non-symmetric theory

In some ways the problem of writing down a covariant form of the Dirac equations is easier in the generalised field theory background than in General Relativity. In the latter "minimal coupling" method is somewhat high-handed not only because of the almost total back of empirical data on the interaction of the gravitational field and microphysics but also because of the local flatness of the Riemannian space. For example, one could easily envisage addition of a term like

$$R_{\mu\nu\lambda x}\sigma^{\mu\dot{A}C}\sigma^{\nu}{}_{\dot{C}B}\sigma^{\lambda B\dot{D}}\sigma^{x}{}_{\dot{D}C}\psi^{E}$$

which automatically vanishes in the flat space-time, to the invariant equations. Also the transition to the case when there is interaction with an exterior electromagnetic field

$$\partial_\mu \to \partial_\mu + i\psi_\mu$$

involves additional assumptions which are foreign to the macrophysical theory. One can, of course, drop the assumption that γ_{AB} is a constant (Bade and Jehle); write

$$\gamma_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \gamma^{1/2} \varrho^{i\theta}, \ \gamma, \theta \text{ real;}$$

and observe that equations (21) are invariant under the substitution

$$\begin{Bmatrix} A \\ \lambda B \end{Bmatrix} \to \begin{Bmatrix} A \\ \lambda B \end{Bmatrix} + i\psi_\lambda \delta^A{}_B,$$

where ψ_λ is an arbitrary (real) vector. Then ψ_λ can be interpreted as proportional to the electromagnetic vector potential, the gradient $\theta_\lambda = \partial_\lambda \theta$ represents a gauge transformation, and if the trace $\{{}^{A}_{\lambda A}\}$ is not pure imaginary

$$\begin{Bmatrix} A \\ \lambda A \end{Bmatrix} + \begin{Bmatrix} \dot{A} \\ \lambda \dot{A} \end{Bmatrix} = \partial_\lambda ln\gamma.$$

But since ψ_λ is arbitrary this procedure hardly corresponds to a geometrisation of electromagnetism.

In the nonsymmetric theory it is impossible to establish a priori, that is before the (macrophysical) field equations are solved, what might be a flat space approximation, or indeed what is meant by flat space.

Hence, in the nonsymmetric theory, we have little choice but to take the Dirac equations in the form

$$\sqrt{2}\, \sigma^{\mu\dot{A}B}\nabla_\mu \psi_B + m\psi^{\dot{A}} = 0,$$

$$\sqrt{2}\, \sigma^{\mu\dot{A}B}\nabla_\mu \psi_{\dot{A}} + m\psi^{B} = 0, \tag{48}$$

or, perhaps,

$$\sqrt{2}\, \sigma^{\mu\dot{A}B}\tilde{\nabla}_\mu \psi_B + m\psi^{\dot{A}} = 0. \tag{49}$$

(together with its conjugate equation). We shall see that rejection of the second alternative leads not only to an unequivocal fixing of the Dirac equations with correct electromagnetic and spin interaction terms but also back to our metric hypothesis.

Thus, a theory in which the latter is adopted becomes particularly suitable for providing a link between macrophysics of electromagnetism and gravitation and quantum electrodynamics.

From equations (33) and (35), we have

$$N^{\mu}_{\nu\lambda}-H^{\mu}_{\lambda\nu} = \tilde{N}^{\mu}_{\nu\lambda}-\tilde{H}^{\mu}_{\lambda\nu}-A^{\mu}_{\lambda\nu}, \tag{50}$$

so that, substituting into the first of the equations (48) from equations (50) and (29) we obtain

$$\sigma^{\lambda\dot{A}B}(D_{\lambda}\psi_B-\tfrac{1}{2}\,\psi_C\sigma^{\mu\dot{D}C}\sigma^{\nu}{}_{\dot{D}B}(\underset{\vee}{\tilde{N}^{\Gamma}_{\nu\mu\lambda}}-\underset{\vee}{\tilde{H}_{\lambda\mu\nu}})+\tfrac{1}{2}\,\sigma^{\mu\dot{D}C}\sigma^{\nu}{}_{\dot{D}B}A_{\lambda\mu\nu}\psi_C+\tfrac{1}{2}\,\psi_B\Gamma^{\dot{D}}_{\lambda\dot{D}})+m\psi^{\dot{A}} = 0. \tag{51}$$

Let us now return to the equations (26). A natural way to avoid uncertainty whether "twiddled" or "untwiddled" operators are to be used is to assume that

$$\nabla_{\lambda}\gamma_{AB} = u^{\Gamma}{}_{\lambda}\gamma_{AB}, \tag{52}$$

where u is a numerical, possibly complex, multiplier. Since

$$\tilde{\Gamma}_{\lambda} \equiv 0,$$

we shall then have

$$D_{\lambda}\gamma_{AB} = \tilde{\nabla}_{\lambda}\gamma_{AB} = 0$$

automatically, and so (γ_{AB} being a constant spinor)

$$\begin{Bmatrix} A \\ \lambda A \end{Bmatrix} = \tilde{\Gamma}^{A}_{\lambda A} = 0.$$

Moreover, the second of the equations (26)

$$\nabla_{\lambda}(\gamma_{\dot{A}\dot{B}}\gamma_{CD}) = (\nabla_{\lambda}\gamma_{\dot{A}\dot{B}})\gamma_{CD}+\gamma_{\dot{A}\dot{B}}\nabla_{\lambda}\gamma_{CD} = 0,$$

will be satisfied if

$$\nabla_{\lambda}\gamma_{\dot{A}\dot{B}} = -u\Gamma_{\lambda}\gamma_{\dot{A}\dot{B}},$$

and this will hold if

$$u = iw,\ w \text{ real}, \tag{53}$$

is pure imaginary, that is if ∇_{λ} is an antihermitian operator. Since then

$$\nabla_{\lambda}\gamma_{AB} = -\Gamma^{C}_{\lambda A}\gamma_{CB}-\Gamma^{C}_{\lambda B}\gamma_{AC} = iw\Gamma_{\lambda}\gamma_{AB}$$

we find that

$$\Gamma^{A}_{\lambda A} = -iw\Gamma_{\lambda}, \tag{54}$$

and the Dirac equation (51) becomes

$$\sqrt{2}\,\sigma^{\lambda\dot{A}B}(D_{\lambda}\psi_B+\tfrac{1}{2}\,\sigma^{\mu\dot{D}C}\sigma^{\nu}{}_{\dot{D}B}A_{\lambda\mu\nu}\psi_C+\tfrac{1}{2}\,iw\Gamma_{\lambda}\psi_B)+m\psi^{\dot{A}} = 0, \tag{55}$$

if

$$\tilde{N}_{\underset{\vee}{\nu}\mu\lambda} - \tilde{H}_{\lambda\mu\underset{\vee}{\nu}} = 0. \tag{56}$$

In the nonsymmetric theory this term clearly refers to an electromagnetic coupling which in the equation (55) is already expressed by Γ_λ. Hence its vanishing implies the assumption that apart from the appearance of the electromagnetic vector potential there should be no further coupling with the electromagnetic field. This is then in full accordance with the empirically well verified structure of quantum electrodynamics. The $A_{\lambda\mu\nu}$ term must then express spin-properties of matter (or perhaps rather of the particle—electron—described by equation (56)). As a correction to the Dirac equation it will presumably be very small and there may be no harm in supposing that

$$A_{\lambda\mu\nu} = 0, \tag{57}$$

although this is not forced by the theory. It suggests on the other hand, that equation (56) could likewise be strengthened by replacing it with

$$\tilde{N}_{\nu\mu\lambda} - \tilde{H}_{\lambda\mu\nu} = 0. \tag{58}$$

If we assume this in equation (44), multiply it by $\varepsilon^{\alpha\lambda\mu\nu}$ (the Levi–Civita permutator) and define the vector density

$$\mathcal{T}^\alpha = \varepsilon^{\alpha\lambda\mu\nu}\tilde{T}_{\lambda\mu\nu},$$

we get

$$6\mathcal{T}^\alpha + 2(-\mathcal{T}^\alpha - \mathcal{T}^\alpha) = 2\mathcal{T}^\alpha = 0,$$

so that

$$\tilde{T}_{\mu\lambda\nu} - \tilde{T}_{\nu\lambda\mu} = 0,$$

or

$$\tilde{H}_{\lambda\mu\nu} - \tilde{H}_{\nu\mu\lambda} = 2\tilde{S}_{\nu\mu\lambda}$$

and, from the definition (6)

$$\tfrac{1}{2}(\tilde{H}_{\lambda\mu\nu} + \tilde{H}_{\nu\mu\lambda}) = a_{\mu\sigma}\left(\tilde{\Gamma}^\sigma_{\underline{\nu\lambda}} - \left\{ {\sigma \atop \nu\lambda} \right\}\right).$$

Hence, the assumption (57) together with the unforced but plausible condition

$$H^\mu_{\lambda\nu} = H^\mu_{\lambda\underset{\vee}{\nu}} \tag{59}$$

not only establishes concretely our spinor analysis (without (58) we have no means of identifying uniquely the tensor $H^\mu_{\lambda\nu}$ which by the equation (22) is crucial to the theory), but also leads unequivocally to the metric hypotheses (2) with which we have started.

If the background Riemannian space reduces to a flat, Minkowski space-time then, under the assumption (57), equations (48) iterate to the standard Schrödinger–Klein–Gordon equation of second order. The scalar w is then identified as

$$w = \tfrac{3}{4}\varrho, \tag{60}$$

where ϱ is the charge which gives rise to the external electromagnetic field (eg. Gregory and Klotz 1977).

5. *Conclusions*

We have seen that the metric hypothesis (2) which already led to a reinterpretation of the nonsymmetric unified field theory in a macrophysical sense, is consistent with the assumption that geodesics, paths of extremal length and paths of no acceleration should coincide with each other. Nevertheless, the hypothesis comes even more into its own in an attempt to set up a concurrent spinor analysis. It is difficult to say at present whether the results lead to an empirically verifiable prediction. The problem is that we have no independent way for determining the $A_{\lambda\mu\nu}$ term in equation (51). We have tentatively called it a spin interaction but the Schrödinger–Klein–Gordon equation resulting from equations (48) without this term already contains an electromagnetic spin quantity. It is for this reason that we have put $A_{\lambda\mu\nu}$ equal to zero. However, the assumption that it vanishes is not forced by the formalism and, hence, becomes an independent hypothesis which, of course, it would be better to avoid.

On the other hand, our method (equation (52)) of introducing the electromagnetic four-vector potential, which is essentially a macrophysical concept being a description of the external field, seems preferable to that of van der Waerden's and Infeld's. In the latter, the potential vector is purely arbitrary whereas we have endowed it, again through the equation (52) with a definite geometrical meaning. Its components appear as a kind of eigenvalues when the geometrical covariant operator ∇_μ acts on the components of the fundamental spinor. The correct place where the potential vector makes its appearance is the macrophysical unified field theory where its curl is proportional to the skew symmetric part of the Ricci tensor identified with the electromagnetic field (Gregory and Klotz 1977).

Similarly to the case if the $A_{\lambda\mu\nu}$ term, the assumption that the $H^{\nu}_{\lambda\mu}$ tensor should be skew symmetric in its covariant indices which led us to the metric hypothesis (2) is not strictly forced by the sp nor formalism. It is merely extremely plausible.

What we have shown is that the same hypothesis which offers a possibility of an observational verification of the unified field theory through cosmological considerations (Klotz 1978c), enables us to construct a spinor analysis particularly well suited to the derivation of the correct equations of quantum electrodynamics. It should be noted that these results are independent of any variational procedures which are always somewhat arbitrary. Our methods provide us also with an unequivocal support and explanation for the concept of minimal coupling. Equations (48) are the only ones which can be sensibly postulated within the context of the nonsymmetric theory.

The link which we have found between the macrophysics of relativistic gravitation and electromagnetism on the one hand and the quantum electrodynamics of Dirac on the other, may be regarded as a powerful argument in favour of the unified field theory of Einstein and Straus. In this sense it is also an explanation of the correct meaning of the metric hypothesis which hitherto was purely arbitrary.

REFERENCES

Bade, W. L., Jehle, H., *Rev. Mod. Phys.* **25**, 714 (1953).
Einstein, A., *Ann. Math.* **46**, 578 (1945).
Einstein, A., Straus, E. G., *Ann. Math.* **47**, 731 (1946).
Gregory, L. J., Klotz, A. H. *Acta Phys. Pol.* **B8**, 595, 601 (1977).
Klotz, A. H., *Acta Phys. Pol.* **B9**, 573 (1978) (a); **B9**, 589 (1978) (b); **B9**, 595 (1978) (c); **B10**, 295 (1979) (a); **B10**, 307 (1979) (b); **B10**, 469 (1979) (c).
Klotz, A. H., *Lett. Nuovo Cimento*, in print.
Klotz, A. H., *Nuovo Cimento* **23**, 1054 (1962).
Klotz, A. H., Russell, G. K., *Acta Phys. Pol.* **B4**, 579 (1973).