RELATIVISTIC THREE-BODY EQUATION FOR ONE DIRAC AND TWO KLEIN-GORDON PARTICLES

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A new relativistic three-body wave equation is derived for a system of one Dirac and two Klein–Gordon particles interacting through a potential. The original motivation for considering such an equation came from a preon model of leptons and quarks, where it was conjectured that $\nu_e = \delta\bar{\chi}$, $e = \delta\chi\chi$ and $u = \bar{\delta}\bar{\chi}$, $d = \bar{\delta}\chi\chi$, the preons δ and χ being colour triplets of spin $\frac{1}{2}$ and 0, respectively, and of equal charge $-\frac{1}{3}$.

As is well known, the three-body problem played an equally important role in quantum physics as in classical physics. Especially, the development and check-up of the approximate methods of solving the wave equations were greatly stimulated by this problem (cf. e.g. helium atom or tritium nucleus).

While in the non-relativistic limit the three-body wave equations have a universal character of the Schrödinger equation for three particles with an appropriate potential, their relativistic form must become specific for a given kind of particles, much depending on their spins and interactions. For instance, the relativistic one-time [1] wave equation for three Dirac particles has the form

$$(E-V-D_1-D_2-D_3)\psi = 0, \tag{1}$$

where

$$D_i = \vec{\alpha}_i \cdot \vec{p}_i + \beta_i m_i, \tag{2}$$

if the potential V transforms as the time-component of a four-vector. Obviously, putting $D_3 \equiv 0$ and taking for V the two-body potential one gets from Eq. (1) the Breit [2] or Salpeter [3] equation for two Dirac particles (depending on the form of the effective interaction V).

In the present note we derive a relativistic one-time wave equation for one Dirac and two Klein–Gordon particles. Although the obtained equation applies generally to

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any such system of particles, our original motivation for its developing was a preon model of leptons and quarks, where it was conjectured that $\nu_e = \delta\bar{\chi}$, $e = \delta(\chi\chi)_P$ (colour singlets) and $u = \bar{\delta}\bar{\chi}$, $d = \bar{\delta}(\chi\chi)_P$ (colour triplets), the symbols δ and χ denoting colour triplets of preons with spin $\frac{1}{2}$ and 0, respectively, and with equal charge $-\frac{1}{3}$ [4]. Thus, the electron was supposed to be a three-body system of one Dirac and two Klein–Gordon subelementary particles bound by a colour antitriplet potential, e.g.

$$V = -\tfrac{2}{3}\alpha_S\left(\frac{1}{|\vec{r}_1-\vec{r}_2|}+\frac{1}{|\vec{r}_1-\vec{r}_3|}+\frac{1}{|\vec{r}_2-\vec{r}_3|}\right). \tag{3}$$

Such a system is likely to be relativistic in its internal motion because of the small electron mass resulting from large binding forces (in spite of very large preon masses). There seems to be an analogy with the massless neutrino which is certainly relativistic (in its internal motion) if it is considered as a system of one Dirac and one Klein-Gordon subelementary particles bound by the colour singlet potential

$$V = -\tfrac{4}{3}\alpha_S\frac{1}{|\vec{r}_1-\vec{r}_2|},$$

where $(4/3)\alpha_S = 2$ and $m_1 = m_2$ [5]. The relativistic three-body wave equation obtained here, though relatively simple, is in fact (unavoidably) a very difficult system of four fourth-order differential equations, so that we cannot yet offer for it any well justified solution of a physical interest.

To start with let us consider a system of one Dirac and two Klein–Gordon free particles. It is evident that in this case the relativistic one- time wave equation should factorize as follows:

$$(E_1-D_1-\sqrt{K_2^2}-\sqrt{K_3^2})(E-D_1+\sqrt{K_2^2}+\sqrt{K_3^2})$$

$$\times(E-D_1-\sqrt{K_2^2}+\sqrt{K_3^2})(E-D_1+\sqrt{K_2^2}-\sqrt{K_3^2})\,\psi_0 = 0, \tag{5}$$

where

$$K_i^2 = \vec{p}_i^{\,2}+m_i^2. \tag{6}$$

After a simple manipulation, this equation can be rewritten in the form

$$\{[(E-D_1)^2-K_2^2-K_3^2]^2-4K_2^2K_3^2\}\psi_0 = 0 \tag{7}$$

or

$$[(E-D_1)^2-4K^2]\psi_0 = 0 \quad \text{if} \quad K_2^2 = K_3^2 \equiv K^2. \tag{8}$$

Putting in Eq. (7) $K_3^2 \equiv 0$ one obtains the relativistic one-time wave equation for one Dirac and one Klein–Gordon free particles

$$[(E-D_1)^2-K_2^2]\psi_0 = 0. \tag{9}$$

Similarly, putting in Eq. (7) or (8) $D_1 \equiv 0$ one gets such an equation for two Klein–Gordon free particles

$$[(E^2-K_2^2-K_3^2)^2-4K_2^2K_3^2]\psi_0 = 0. \tag{10}$$

or

$$(E^2-4K^2)\psi_0 = 0 \quad \text{if} \quad K_2^2 = K_3^2 \equiv K^2. \tag{11}$$

Obviously, the condition $K_2^2 = K_3^2$ can be exactly realized only if $D_1 \equiv 0$ and $m_2 = m_3$ (and the centre-of-mass frame is used).

Now, in the case of interactions described by a potential V transforming as the time--component of a four-vector, we make in Eqs. (7) and (9), and (10) or (11) the substitution

$$E \to E-V. \tag{12}$$

Then we obtain, respectively, the following relativistic one-time wave equations for interacting particles:

$$\{[(E-V-D_1)^2-K_2^2-K_3^2]^2-4K_2^2K_3^2\}\psi = 0 \tag{13}$$

and [5]

$$[(E-V-D_1)^2-K_2^2]\psi = 0, \tag{14}$$

and

$$\{[(E-V)^2-K_2^2-K_3^2]^2-4K_2^2K_3^2\}\psi = 0 \tag{15}$$

or [6]

$$[(E-V)^2-4K^2]\psi = 0 \quad \text{if} \quad K_2^2 = K_3^2 \equiv K^2. \tag{16}$$

In the case of $m_1 = m_2 = m_3$ we can introduce into Eq. (13) the centre-of-mass and internal co-ordinates by the relations [7]

$$\begin{aligned} \vec{r}_1 &= \vec{R}+\tfrac{2}{3}\vec{\varrho}, \\ \vec{r}_2 &= \vec{R}-\tfrac{1}{3}\vec{\varrho}+\tfrac{1}{2}\vec{r}, \\ \vec{r}_3 &= \vec{R}-\tfrac{1}{3}\vec{\varrho}-\tfrac{1}{2}\vec{r} \end{aligned} \tag{17}$$

and

$$\begin{aligned} \vec{p}_1 &= \tfrac{1}{3}\vec{P}+\vec{\pi}, \\ \vec{p}_2 &= \tfrac{1}{3}\vec{P}-\tfrac{1}{2}\vec{\pi}+\vec{p}, \\ \vec{p}_3 &= \tfrac{1}{3}\vec{P}-\tfrac{1}{2}\vec{\pi}-\vec{p}, \end{aligned} \tag{18}$$

where $\vec{P}$, $\vec{\pi}$ and $\vec{p}$ are canonical momenta conjugate to the position vectors $\vec{R}$, $\vec{\varrho}$ and $\vec{r}$, respectively. Then Eq. (13) gets the following form in the centre-of-mass frame:

$$\left(\left\{[E-V-(\vec{\alpha}_1\cdot\vec{\pi}+\beta_1 m)]^2-2\left(\frac{\vec{\pi}^2}{4}+\vec{p}^2+m^2\right)\right\}^2 \right.$$
$$\left. -4\left(\frac{\vec{\pi}^2}{4}+\vec{p}^2+m^2\right)^2+4(\vec{\pi}\cdot\vec{p})^2\right)\psi = 0. \tag{19}$$

We can see that it is a very difficult equation indeed, in contrast to Eq. (14) which in the case of $m_1 = m_2$ reduces in the centre-of-mass frame to a Dirac-like equation for the internal motion

$$\left\{E-V-2(\vec{\alpha}_1 \cdot \vec{p}+\beta_1 m)+\frac{\vec{\alpha}_1 \cdot [\vec{p}, V]}{E-V}\right\} \psi = 0 \tag{20}$$

or equivalently

$$[E-V-2(\vec{\alpha}_1 \cdot \vec{p}+\beta_1 m)] \sqrt{E-V}\, \psi = 0. \tag{21}$$

Here

$$\vec{r}_1 = \vec{R}+\tfrac{1}{2}\vec{r}, \quad \vec{r}_2 = \vec{R}-\tfrac{1}{2}\vec{r} \tag{22}$$

and

$$\vec{p}_1 = \tfrac{1}{2}\vec{P}+\vec{p}, \quad \vec{p}_2 = \tfrac{1}{2}\vec{P}-\vec{p}. \tag{23}$$

Eq. (21) is, of course, analytically solvable for a Coulomb-like potential $V = -\alpha/r$. For the critical $\alpha = 2$ it gives a well-defined zero-energy level $E = 0$ [5] corresponding in our model to the massless neutrino.

If it is possible to consider approximately the pair of Klein–Gordon particles 2 and 3 as a single integer-spin particle of mass $2m$–b and momentum $\vec{p}_2+\vec{p}_3 = (2/3)\vec{P}-\vec{\pi}$, then applying Eq. (14) one obtains in the centre-of-mass frame

$$\{[E-V-(\vec{\alpha}_1 \cdot \vec{\pi}+\beta_1 m)]^2-[\vec{\pi}^2+(2m-b)^2]\}\psi \simeq 0, \tag{24}$$

where $V = V(\vec{\varrho})$. Hence

$$\left\{E-V-2(\vec{\alpha}_1 \cdot \vec{\pi}+\beta_1 m)+\frac{\vec{\alpha}_1 \cdot [\vec{\pi}, V]+m^2-(2m-b)^2}{E-V}\right\} \psi \simeq 0 \tag{25}$$

or equivalently

$$\left[E-V-2(\vec{\alpha}_1 \cdot \vec{\pi}+\beta_1 m)+\frac{m^2-(2m-b)^2}{E-V}\right] \sqrt{E-V}\, \psi \simeq 0. \tag{26}$$

The approximate equation (26) takes the form of Eq. (21) if the binding energy $b = m$. Then it gives $E \simeq 0$ for $V = -\alpha/\varrho$ with the critical $\alpha = 2$, suggesting the possible mechanism of keeping the small electron mass in our model (in spite of very large preon masses). In fact, one gets

$$b \sim -\langle V_{23}\rangle = \left\langle \frac{1}{r} \right\rangle \sim m \tag{27}$$

because $\alpha = 1$ in the colour-antitriplet potential $V_{23} = -\alpha/r$ for the $\chi\chi$ pair, while $\alpha = 2$ in the colour-singlet potential $V = -\alpha/\varrho$ for the $\delta(\chi\chi)$ quasi-pair. The explicit solution of Eq. (16) for the $(\chi\chi)_P$ pair with the potential $V_{23} = -1/r$ gives, however, much smaller binding energy (in the $n_r = 0$, $l = 1$ state)

$$b = 2m-E_{23} = 2m\left(1-\frac{1+\sqrt{8}}{\sqrt{10+2\sqrt{8}}}\right) \simeq 0.06\, m. \tag{28}$$

So the $(\chi\chi)_P$ pair must be bound stronger than through the Coulomb-like potential $V_{23} = -1/r$ only. For an illustration, the potential $V_{23} = -\alpha/r$ with $\alpha = (1+\sqrt{33}) \times \sqrt{3}/4 \simeq 2.9$ gives $b = m$ exactly (in the $n_r = 0$, $l = 1$ state). Let us note that the χ preons, being spin-0 objects, should possess also quartic couplings besides the gluon Yukawa coupling [4]. Thus the last term V_{23} in formula (3) certainly requires modifications.

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