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THE RELATIVISTIC TWO-FERMION EQUATIONS (I)

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The relativistic radial equations for two spin 1/2 particles interacting via an instantaneous potential are derived. These equations are solved for the case of positronium. The solution obtained is similar to that of Schrödinger's equation for a hydrogen like atom in the ground state.

1. Introduction

In quantum field theory the work on the two-body problem is based on two approaches:
i) three-dimensional one-time formulation based on the work of Fock and Podolsky [1], in which the Coulomb law is derived from the basic equations of QED;
ii) four-dimensional approach which is based on the many-time formulation of Dirac, Fock and Podolsky [2], in which a time variable is given to each particle.

Breit [3] proposed an equation for two fermions to describe the interaction between the electrons in helium. Later on, a covariant approach — the so called Bethe–Salpeter (BS) equation [4] — was formulated to describe the relativistic two-body systems. In QED, where precise comparisons with experiments are made, and in other applications, the BS equation has been found akward to work with. The unphysical variable of relative time (relative energy) gives rise to redundant unphysical solutions [5, 6].

Hence, in order to give a physical interpretation for the two-body amplitude several one-time formulations have been proposed. Logunov and Tavkhelidze [7] developed a one-time approach which has been applied by Faustov, Todorov and others [8] in different aspects of the two-body problem. Królikowski and Rzewuski [9, 10] developed

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a one-time formulation for the relativistic two-body problem. Partovi [11] derived a two--time covariant, functional differential equation, which reduces to a single-time Schrödinger-type equation in the CM frame. Suura [12] has also proposed an equal-time two-body equation.

In fact, a realistic analogue of the Dirac equation to describe the two-body problem so far does not exist. The purpose of the present work is to study the formal structure, self-consistency and solutions of a new set of equations for the two-fermion system. These equations are generalization to the field equations derived previously by one of us [13] to describe particles of definite mass m and definite spin s. Hereby, we follow the same notations. In the next section the relativistic two-body equations are presented. In Section 2 the case of the e^+e^- bound system is discussed.

2. *The relativistic two-body equations*

Consider the free-particle energy-momentum equations

$$(P_\mu^{(1)}+P_\mu^{(2)})\psi = (p_\mu^{(1)}+p_\mu^{(2)})\psi, \tag{2.1}$$

where

$$P_\mu^i = S_{\mu\nu}^i p_\nu^{(i)}+m_i\gamma_\mu^{(i)}. \tag{2.2}$$

In order to introduce the electromagnetic interaction, we consider the one-particle equations

$$(S_{\mu\nu}^{(1)}p_\nu^{(1)}+m\gamma_\mu^{(1)})\psi_1 = P_\mu^{(1)}\psi_1 \tag{2.3}$$

and introduce the minimal interaction $P_\mu^{(1)} \to P_\mu^{(1)} - e_1A_\mu^{(2)}$, where $A_\mu^{(2)}$ is the 4-vector potential of the second particle. Then

$$\begin{aligned} P_\mu^{(1)}\psi_1 &= [S_{\mu\nu}^{(1)}p_\nu^{(1)}+m_1\gamma_\mu^{(1)}+e_1(A_\mu^{(2)}-S_{\mu\nu}^{(1)}A_\nu^{(2)})]\psi_1 \\ &= [S_{\mu\nu}^{(1)}p_\nu^{(1)}+m_1\gamma_\mu^{(1)}-e_1\gamma_\mu^{(1)}\gamma_\nu^{(1)}A_\nu^{(2)}]\psi_1 \end{aligned} \tag{2.4}$$

and $A_\mu^{(2)}$ satisfies the equation

$$\Box_2 A_\mu^{(2)} = -4\pi e_2\psi_2^\dagger\gamma_0^{(2)}\gamma_\mu^{(2)}\psi_2,$$

with the retarded solution

$$A_\mu^{(2)} = e_2\int\frac{dV_2}{r}\,[\psi_2^\dagger\gamma_0^{(2)}\gamma_\mu^{(2)}\psi_2]_{t_2-r}, \tag{2.5}$$

where $r = |\boldsymbol{r}_1-\boldsymbol{r}_2|$.

Integrating equation (2.3) with respect to V_2, and inserting $\int\psi_1^\dagger\psi_1 dV_1 = 1$, whenever no ψ_2 is present, and writing the two-body wave function $\psi(\boldsymbol{r}_1, t_1; \boldsymbol{r}_2, t_2)$ in place of $\psi_1\otimes\psi_2$, we get

$$\begin{aligned} \int dV_1dV_2\psi^\dagger P_\mu^{(1)}\psi &= \int dV_1dV_2\psi^\dagger(S_{\mu\nu}^{(1)}p_\nu^{(1)}+m_1\gamma_\mu^{(1)})\psi \\ &-e_1e_2\int\frac{dV_1dV_2}{r}\,\psi^\dagger(\boldsymbol{r}_1, t_1; \boldsymbol{r}_2, t_2)\gamma_0^{(2)}\gamma_\mu^{(1)}\gamma_\nu^{(1)}\gamma_\nu^{(2)}\psi(\boldsymbol{r}_1, t_1; \boldsymbol{r}_2, t_2). \end{aligned} \tag{2.6}$$

A similar equation for the second particle may also be obtained. To write down the symmetrical two-particle equation, we average the interaction over the two particles, since it is a mutual interaction, hence

$$(P_\mu^{(1)}+P_\mu^{(2)})\psi = (S_{\mu\nu}^{(1)}p_\nu^{(1)}+S_{\mu\nu}^{(2)}p_\nu^{(2)}+m_1\gamma_\mu^{(1)}+m_2\gamma_\mu^{(2)})\psi$$

$$-\frac{e_1e_2}{r}[\gamma_0^{(1)}\gamma_\mu^{(2)}\gamma_\nu^{(1)}\gamma_\nu^{(2)}\psi_{t_1-r}+\gamma_0^{(2)}\gamma_\mu^{(1)}\gamma_\nu^{(1)}\gamma_\nu^{(2)}\psi_{t_2-r}]. \tag{2.7}$$

If we consider the instantaneous interaction, we approximate $\psi_{t_1-r} = \psi_{t_2-r} = \psi$, and get

$$(P_\mu^{(1)}+P_\mu^{(2)})\psi = [S_{\mu\nu}^{(1)}p_\nu^{(1)}+S_{\mu\nu}^{(2)}p_\mu^{(2)}+m_1\gamma_\mu^{(1)}+m_2\gamma_\mu^{(2)}+V_\mu]\psi, \tag{2.8}$$

where

$$V_\mu = -\frac{e_1e_2}{2r}(\gamma_0^{(1)}\gamma_\mu^{(2)}+\gamma_0^{(2)}\gamma_\mu^{(1)})\gamma_\nu^{(1)}\gamma_\nu^{(2)}. \tag{2.9}$$

Here,

$$V_0 = \frac{e_1e_2}{r}(1-\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2) \tag{2.10a}$$

and

$$\boldsymbol{V} = -\frac{e_1e_2}{r}(\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2)(1-\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2). \tag{2.10b}$$

In this approximation we obtain a local interaction. The factor $\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2$ reminds us of the Breit interaction

$$V_0^{\mathrm{B}} = \frac{e_1e_2}{r}\left[1-\tfrac{1}{2}\left(\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2+\frac{(\boldsymbol{\alpha}_1\cdot\boldsymbol{r})(\boldsymbol{\alpha}_2\cdot\boldsymbol{r})}{r^2}\right)\right]$$

$$= V_0+\frac{1}{2r^2}(\boldsymbol{\alpha}_1\wedge\boldsymbol{r})\cdot(\boldsymbol{\alpha}_2\wedge\boldsymbol{r}). \tag{2.11}$$

To evaluate $V_0^{\mathrm{B}}-V_0$, we take the expectation value between free particle states, where in the rest-system $\boldsymbol{P}_1 = -\boldsymbol{P}_2 = \boldsymbol{p}$ is the relative momentum, $\langle\boldsymbol{\alpha}_1\rangle = \boldsymbol{P}/m_1$, $\langle\boldsymbol{\alpha}_2\rangle = -\boldsymbol{P}/m_2$, such that

$$\langle V_0^{\mathrm{B}}-V_0\rangle \approx \frac{-e_1e_2}{2m_1m_2}\langle j^2/r^2\rangle, \tag{2.12}$$

where $\boldsymbol{j} = \boldsymbol{r}\wedge\boldsymbol{p}$ is the relative angular momentum. Thus $V_0^{\mathrm{B}}-V_0$ gives spin-dependent splitting of the energy levels.

Since our first concern is to examine the self-consistency of the energy-momentum equations, we shall consider now the instantaneous minimal interaction and with equal mass case $m_1 = m_2 = m$.

The Schrödinger equation $(P_0^{(1)}+P_0^{(2)}+V_0)\psi = P_0\psi$, where $P_\mu = P_\mu^{(1)}+P_\mu^{(2)}$, was used already by Breit [3], and also by Fulton and Karplus [14] and Fulton and Martin [15], derived from quantum field theory. What is new here, is the momentum equations $(\boldsymbol{P}_1+\boldsymbol{P}_2+\boldsymbol{V})\psi = \boldsymbol{P}\psi$. One wants to see, whether they are consistent with the energy equation, and whether they yield additional supplementary conditions. For equal masses the energy-momentum equations read

$$P_\mu\psi = [S_{\mu\nu}P_\nu+(S_{\mu\nu}^{(1)}-S_{\mu\nu}^{(2)})p_\nu+m(\gamma_\mu^{(1)}+\gamma_\mu^{(2)})+V_\mu]\psi, \tag{2.13}$$

where

$$P_\mu = P_\mu^{(1)}+P_\mu^{(2)}, \quad p_\mu = \tfrac{1}{2}(P_\mu^{(1)}-P_\mu^{(2)}), \quad X_\mu = \tfrac{1}{2}(x_\mu^{(1)}+x_\mu^{(2)}),$$

$$x_\mu = x_\mu^{(1)}-x_\mu^{(2)}, \quad S_{\mu\nu} = \tfrac{1}{2}(S_{\mu\nu}^{(1)}+S_{\mu\nu}^{(2)}).$$

Furthermore, if we consider the rest-system $P_k = 0$, $P_4 = iM$, where M is the mass of the two particles. We seek the static solution, such that $p_0\psi = 0$, which is independent of the relative time. Thus the energy-momentum equations read, for the instantaneous Coulomb interaction

$$M\psi = \left[(\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2)\cdot\boldsymbol{p}+m(\gamma_0^{(1)}+\gamma_0^{(2)})+\frac{e_1e_2}{r}(1-\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2)\right]\psi \tag{2.14a}$$

and

$$0 = \left[\frac{M}{2}(\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2)+i(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\wedge\boldsymbol{p}+m(\boldsymbol{\gamma}_1+\boldsymbol{\gamma}_2)\right.$$
$$\left.-\frac{e_1e_2}{2r}(\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2)(1-\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2)\right]\psi. \tag{2.14b}$$

Considering the direct product representation

$$S_{AB}^{(1)} = S_{AB}\otimes I, \quad S_{AB}^{(2)} = I\otimes S_{AB}, \tag{2.15}$$

where S_{AB} is the Dirac algebra, with

$$S_{kl}^{(i)} = -i\varepsilon_{kln}\sigma_n^{(i)}, \quad S_{4k}^{(i)} = i\alpha_k^{(i)} = i\gamma_5\sigma_k^{(i)},$$
$$S_{\mu5}^{(i)} = \gamma_\mu^{(i)}, \quad S_{\mu6}^{(i)} = \beta_\mu^{(i)} = i\gamma_5\gamma_\mu^{(i)},$$
$$S_{56} = \beta_5^{(i)} = i\gamma_5^{(i)}, \quad \gamma_5^{(i)} = \gamma_1^{(i)}\gamma_2^{(i)}\gamma_3^{(i)}\gamma_4^{(i)}. \tag{2.16}$$

The ψ has the matrix elements $\psi_{\alpha\beta}$, such that

$$(S_{AB}^{(1)}\psi)_{\alpha\beta} = (S_{AB}^D)_{\alpha\alpha'}\psi_{\alpha'\beta} \tag{2.17a}$$

and

$$(S_{AB}^{(2)}\psi)_{\alpha\beta} = (S_{AB}^D)_{\beta\beta'}\psi_{\alpha\beta'}. \tag{2.17b}$$

For convenience, we write

$$\psi = aC+\tfrac{1}{2}S_{AB}C\psi_{AB}, \tag{2.18}$$

where C is the complex conjugation matrix, such that [16]

$$C^{\mathrm{T}} = -C, \quad S_{ab}^{\mathrm{T}} = -C^{-1}S_{ab}C \quad \text{and} \quad S_{a6}^{\mathrm{T}} = C^{-1}S_{a6}C. \tag{2.19}$$

The 16 independent components of ψ are thus expressed in terms of the scalar a and the 6-dimensional 15-component antisymmetric tensor $\psi_{AB} = -\psi_{BA}$. We denote its components in three dimensions as follows:

$$\psi_{kl} = \varepsilon_{kln}H_n, \quad \psi_{4k} = iE_k, \quad \psi_{k5} = \chi_k, \quad \psi_{45} = i\chi_0$$

$$\psi_{k6} = \phi_k, \quad \psi_{46} = i\phi_0, \quad \psi_{56} = \eta, \tag{2.20}$$

which are four vectors $\boldsymbol{H}$, $\boldsymbol{E}$, $\boldsymbol{\chi}$, $\boldsymbol{\phi}$ and three scalars χ_0, ϕ_0, η such that

$$\psi = (a - i\boldsymbol{\sigma}\cdot\boldsymbol{H} - \boldsymbol{\alpha}\cdot\boldsymbol{E} + \boldsymbol{\gamma}\cdot\boldsymbol{\chi} - \gamma_0\chi_0 + \boldsymbol{\beta}\cdot\boldsymbol{\phi} - \beta_0\phi_0 + \beta_5\eta)C. \tag{2.21}$$

Equations (2.17a) and (2.17b) read, in favour of (2.19)

$$S_{AB}^{(1)}\psi = S_{AB}Ca + \tfrac{1}{2}S_{AB}S_{CD}C\psi_{CD} \tag{2.22}$$

and

$$S_{AB}^{(2)}\psi = CS_{AB}^{T}a + \tfrac{1}{2}S_{CD}CS_{AB}^{\mathrm{T}}\psi_{CD}$$

such that

$$S_{ab}^{(2)}\psi = -S_{ab}Ca - \tfrac{1}{2}S_{CD}S_{ab}C\psi_{CD} \tag{2.22a}$$

and

$$S_{a6}^{(2)}\psi = S_{a6}Ca + \tfrac{1}{2}S_{CD}S_{a6}C\psi_{CD}. \tag{2.22b}$$

Here the indices a and b run over 1, ..., 5.

The energy equation (2.14a) reads, after multiplying by C^{-1} from the right,

$$\begin{aligned}
&M\{a - i\boldsymbol{\sigma}\cdot\boldsymbol{H} - \boldsymbol{\alpha}\cdot\boldsymbol{E} + \boldsymbol{\gamma}\cdot\boldsymbol{\chi} - \gamma_0\chi_0 + \boldsymbol{\beta}\cdot\boldsymbol{\phi} - \beta_0\phi_0 + \beta_5\eta\} \\
&= -2i\boldsymbol{\alpha}\cdot\nabla a + 2i\beta_5 \operatorname{div}\boldsymbol{H} + 2i\operatorname{div}\boldsymbol{E} - 2i\boldsymbol{\beta}\cdot\operatorname{curl}\boldsymbol{\chi} \\
&\quad -2i\boldsymbol{\gamma}\cdot\operatorname{curl}\boldsymbol{\phi} + 2\boldsymbol{\sigma}\cdot\nabla\eta + 2m(-\boldsymbol{\gamma}\cdot\boldsymbol{E} + \boldsymbol{\alpha}\cdot\boldsymbol{\chi} + \beta_5\phi_0 - \beta_0\eta) \\
&\quad + \frac{e_1e_2}{r}\{4a + 2\boldsymbol{\gamma}\cdot\boldsymbol{\chi} + 2\gamma_0\chi_0 + 2\boldsymbol{\beta}\cdot\boldsymbol{\phi} + 2\beta_0\phi_0 + 4\beta_5\eta\}.
\end{aligned} \tag{2.23}$$

From the linear independence of the S_{AB}, we obtain the equivalent equations

$$\chi_0 = 0, \tag{2.24a}$$

$$2m\eta = \left(M + \frac{2e_1e_2}{r}\right)\phi_0, \tag{2.24b}$$

$$2i\nabla\eta = M\boldsymbol{H}, \tag{2.24c}$$

$$2i\operatorname{div}\boldsymbol{H} = \left[M - \frac{4e_1e_2}{r}\right]\eta - 2m\phi_0, \tag{2.24d}$$

$$2i\nabla a = 2m\boldsymbol{\chi}+M\boldsymbol{E}, \tag{2.25a}$$

$$2i \operatorname{div} \boldsymbol{E} = \left(M - \frac{4e_1e_2}{r}\right) a, \tag{2.25b}$$

$$-2i \operatorname{curl} \boldsymbol{\chi} = \left(M - \frac{2e_1e_2}{r}\right) \boldsymbol{\phi}, \tag{2.25c}$$

$$2i \operatorname{curl} \boldsymbol{\phi} = 2m\boldsymbol{E} + \left(M - \frac{2e_1e_2}{r}\right) \boldsymbol{\chi} \tag{2.25d}$$

and the momentum equations (2.14b) read

$$\begin{aligned} &M[-\boldsymbol{\alpha} \wedge \boldsymbol{H} + i\boldsymbol{\sigma} \wedge \boldsymbol{E} - \gamma_0\boldsymbol{\chi} + \boldsymbol{\gamma}\chi_0 - \beta_0\boldsymbol{\phi} + \boldsymbol{\beta}\phi_0] + 2\boldsymbol{\sigma} \wedge \nabla a + 2i \operatorname{curl} \boldsymbol{H} \\ &-2i\beta_5 \operatorname{curl} \boldsymbol{E} - 2i\beta_0 \operatorname{curl} \boldsymbol{\chi} - 2i\boldsymbol{\beta} \wedge \nabla\chi_0 + 2i\gamma_0 \operatorname{curl} \boldsymbol{\phi} + 2i\boldsymbol{\gamma} \wedge \nabla\phi_0 + 2i\boldsymbol{\alpha} \wedge \nabla\eta \\ &+2m[-\boldsymbol{\gamma} \wedge \boldsymbol{H} - \gamma_0\boldsymbol{E} + i\boldsymbol{\sigma} \wedge \boldsymbol{\chi} + \boldsymbol{\alpha}\chi_0 + \beta_5\boldsymbol{\phi} - \boldsymbol{\beta}\eta] \\ &- \frac{2e_1e_2}{r}[\gamma_0\boldsymbol{\chi} + \boldsymbol{\gamma}\chi_0 + \beta_0\boldsymbol{\phi} + \boldsymbol{\beta}\phi_0] = 0. \end{aligned} \tag{2.26}$$

This gives

$$\chi_0 = 0, \tag{2.27a}$$

$$2m\eta = \left(M + \frac{2e_1e_2}{r}\right) \phi_0, \tag{2.27b}$$

$$2i\nabla\eta = M\boldsymbol{H}, \tag{2.27c}$$

$$i\nabla\phi_0 = m\boldsymbol{H}, \tag{2.27d}$$

$$\operatorname{curl} \boldsymbol{H} = 0, \tag{2.27e}$$

$$2i\nabla a = 2m\boldsymbol{\chi}+M\boldsymbol{E}, \tag{2.28a}$$

$$i \operatorname{curl} \boldsymbol{E} = m\boldsymbol{\phi}, \tag{2.28b}$$

$$-2i \operatorname{curl} \boldsymbol{\chi} = \left(M - \frac{2e_1e_2}{r}\right) \boldsymbol{\phi}, \tag{2.28c}$$

$$2i \operatorname{curl} \boldsymbol{\phi} = 2m\boldsymbol{E} + \left(M - \frac{2e_1e_2}{r}\right) \boldsymbol{\chi}. \tag{2.28d}$$

We can see that equations (2.27a, b, c) and (2.28a, c, d) are identical with (2.24a, b, c) and (2.25a, c, d) showing the consistency of the energy and the momentum equations. Also (2.27e) follows identically from either of the equations (2.24c) or (2.27c).

If we consider the energy equations alone, we find that equations (2.24) suffice to determine H, ϕ_0 and η, where η satisfies the differential equation

$$\nabla^2\eta + \left\{\frac{M}{4}\left(M - \frac{4e_1e_2}{r}\right) - \frac{m^2 M}{\left(M + \frac{2e_1e_2}{r}\right)}\right\}\eta = 0.$$

On the other hand equations (2.25) do not suffice to determine the three vectors χ, ϕ and E. Thus the momentum equations give necessary supplementary conditions. Furthermore, the momentum equations (2.27) alone lead to $H = \phi_0 = \eta = 0$, which are satisfied automatically by (2.24). In fact, equations (2.27c and d) give $2M\eta = m\phi_0$, which together with (2.27b) give $\eta = \phi_0 = 0$. Hence also

$$\boldsymbol{H} = 0. \tag{2.29}$$

From equations (2.25) and (2.28)

$$\boldsymbol{\phi} = 0. \tag{2.30}$$

In fact, by taking the curl of equation (2.25a), we obtain

$$2\text{m curl}\, \boldsymbol{\chi} + M \,\text{curl}\, \boldsymbol{E} = 0.$$

Substituting equations (2.28b and c) into the above equation, we obtain equation (2.30). Thus we are left with equation (2.25a) and

$$\text{curl}\, \boldsymbol{E} = 0, \qquad \text{curl}\, \boldsymbol{\chi} = 0, \tag{2.31}$$

$$2m\boldsymbol{E} + \left(M - \frac{2e_1e_2}{r}\right)\boldsymbol{\chi} = 0. \tag{2.32}$$

Taking curl (2.32) and using (2.31), we get

$$\boldsymbol{r} \wedge \boldsymbol{\chi} = \boldsymbol{r} \wedge \boldsymbol{E} = 0. \tag{2.33}$$

Hence

$$\boldsymbol{\chi} = \chi\frac{\boldsymbol{r}}{r} \qquad \mathbf{E} = E\frac{\boldsymbol{r}}{r}$$

and

$$E = -\frac{1}{2m}\left(M - \frac{2e_1e_2}{r}\right)\chi. \tag{2.34}$$

It follows from (2.25a) that $a = a(r)$ is a function of r alone,

$$i\frac{da}{dr} = \left[m - \frac{M}{4m}\left(M - \frac{2e_1e_2}{r}\right)\right]\chi \tag{2.35}$$

showing that χ and also E are functions of r only. Further, equation (2.25b) reads

$$\frac{2i}{r^2}\frac{d}{dr}(r^2E) = \left(M - \frac{4e_1e_2}{r}\right)a.$$

Hence

$$\left(M - \frac{2e_1e_2}{r}\right)\frac{d\chi}{dr} + \frac{2(Mr - e_1e_2)}{r^2}\chi = m\left(M - \frac{4e_1e_2}{r}\right)ia. \tag{2.36}$$

Also, from equations (2.35) and (2.36) we can obtain, by eliminating a, the differential equation for χ.

3. *The mass of the positronium*

We consider now the positronium $e_1 = -e_2 = e$ (or also the muonium). We solve equations (2.35) and (2.36), which now read

$$i\frac{da}{dr} = \left[m - \frac{M}{4m}\left(M + \frac{2e^2}{r}\right)\right]\chi, \tag{3.1}$$

$$m\left(M + \frac{4e^2}{r}\right)ia = \left(M + \frac{2e^2}{r}\right)\frac{d\chi}{dr} + \frac{2(Mr + e^2)}{r^2}\chi. \tag{3.2}$$

We seek solutions which vanish at infinity. Consider the asymptotic functions χ_∞ and a_∞ as $r \to \infty$. Equations (3.1) and (3.2) become

$$i\frac{da_\infty}{dr} = \left(m - \frac{M^2}{4m}\right)\chi_\infty$$

and

$$\frac{d\chi_\infty}{dr} = ima_\infty,$$

hence,

$$\frac{d^2\chi_\infty}{dr^2} - K^2\chi_\infty = 0, \tag{3.3}$$

where

$$K = \tfrac{1}{2}(4m^2 - M^2)^{1/2}. \tag{3.4}$$

When $M < 2m$ we obtain the asymptotic solution $\chi_\infty \sim e^{-Kr}$. Thus, a solution of this form may be sought,

$$\chi = e^{-Kr}\sum_{n=0}^{N} B_n(Kr)^{n+\nu} = e^{-Kr}B(r)$$

and

$$a = ie^{-Kr} \sum_{n=0}^{N} A_n(Kr)^{n+\nu} = ie^{-Kr}A(r),$$

where the sums terminate at the same $n = N$.

Denoting

$$\varrho = Kr, \quad e^2 = \alpha, \quad m^* = m/K, \quad M^* = M/K,$$

where α is the fine structure constant, then $A(\varrho)$ and $B(\varrho)$ satisfy the following equations

$$\left(M^* + \frac{2\alpha}{\varrho}\right)\frac{dB}{d\varrho} + \left[\frac{2\alpha}{\varrho^2} + \frac{2(M^*-\alpha)}{\varrho} - M^*\right]B = -m^*\left(M^* + \frac{4\alpha}{\varrho}\right)A \tag{3.5}$$

and

$$A - \frac{dA}{d\varrho} = \left[\frac{1}{m^*} - \frac{M^*\alpha}{2m^*\varrho}\right]B. \tag{3.6}$$

We obtain the following recurrence relations

$$2\alpha(\nu+n+3)B_{n+2} + [-2\alpha + M^*(\nu+n+3)]B_{n+1} - M^*B_n = -m^*M^*A_n - 4\alpha m^*A_{n+1} \tag{3.7}$$

and

$$A_n - (\nu+n+1)A_{n+1} = \frac{1}{m^*}B_n - \frac{M^*\alpha}{2m^*}B_{n+1}. \tag{3.8}$$

From the above equations we get

$$(2\nu+1)B_0 = 0, \quad \nu A_0 = -\frac{M^*\alpha}{2m^*}B_0 \tag{3.9}$$

hence, $\nu = 0$, $B_0 = 0$, $A_0 \neq 0$ and $B_1 = -m^*A_0$ is a solution and the other solution

$$\nu = -1, \quad A_0 = \frac{M^*\alpha}{2m^*}B_0$$

is singular, and thus is excluded. On the other hand, if we take $n = N$, such that

$$A_{N+1} = A_{N+2} = B_{N+1} = B_{N+2} = 0$$

then both equations (3.7) and (3.8) give the same relation

$$B_N = m^*A_N \tag{3.10}$$

showing the consistency that both series for A and B terminate at the same $n = N$. Furthermore, in equations (3.7) and (3.8) by taking $n = N-1$, we get

$$[M^*(N+2) - 2\alpha]B_N - M^*B_{N-1} = -m^*M^*A_{N-1} - 4\alpha m^*A_N$$

and

$$m^*(A_{N-1}-NA_N) = B_{N-1}-\frac{M^*\alpha}{2}B_N. \tag{3.11}$$

From these equations and equation (3.10), we get

$$[M^*(N+2)+2\alpha]B_N = M^*(B_{N-1}-m^*A_{N-1})$$

and

$$B_{N-1}-m^*A_{N-1} = \left(\frac{M^*\alpha}{2}-N\right)B_N.$$

The self-consistency of these equations gives the eigenvalue for M in the form

$$2MK(N+1) = \alpha(M^2-2m^2). \tag{3.12}$$

This holds naturally for $N \geqslant 2$, because condition (3.10) does not hold for $N = 1$. Squaring we get equation

$$[(N+1)^2+\alpha^2]M^4-4m^2[(N+1)^2+\alpha^2]M^2+4\alpha^2m^2 = 0$$

which gives the mass spectrum

$$\left(\frac{M_N}{m}\right)^2 = 2+2\left[1-\frac{\alpha^2}{(N+1)^2+\alpha^2}\right]^{1/2}. \tag{3.13}$$

The other root of the equation is excluded. In fact

$$M_N^2-2m^2 = \pm 2m^2\left[1-\frac{\alpha^2}{(N+1)^2+\alpha^2}\right]^{1/2}.$$

Since $MK > 0$, then equation (3.12) requires that $M_n^2-2m^2 > 0$. Hence only the root with the positive sign is admissible. Expanding in powers of α^2, we get to the first degree

$$M_N \approx 2m-\frac{\alpha^2 m}{4(N+1)^2}. \tag{3.14}$$

Here

$$M_N-2m \approx -\mathrm{Ry}/2(N+1)^2$$

which is the same as for the hydrogen atom where $\mathrm{Ry} = \dfrac{\alpha^2 m}{2}$ is the Rydberg constant with the reduced mass $\dfrac{m}{2}$, except that $N+1$ replaces N.

The ground state $N = 1$ is given by

$$-m^*A_0 = \left(\frac{M^*\alpha}{2}-1\right)B_1, \quad B_1 = -m^*A_0.$$

Hence,

$$\frac{M^*\alpha}{2} - 1 = 1, \quad M\alpha = 4K, \quad M^2\alpha^2 = 16m^2 - 4M^2$$

or

$$M^2(\alpha^2+4) = 16m^2,$$

from which we obtain

$$M_1 = \frac{4m}{\sqrt{4+\alpha^2}} \approx 2m - \frac{\alpha^2 m}{4}$$

which is the same as Schrödinger's hydrogen atom for the ground state.

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REFERENCES

[1] V. Fock, B. Podolsky, *Sov. Phys.* **1**, 798 (1932); *Sov. Phys.* **1**, 801 (1932); *Sov. Phys.* **2**, 275 (1932); V. Fock, *Sov. Phys.* **6**, 425 (1934).

[2] P. A. M. Dirac, V. Fock, B. Podolsky, *Sov. Phys.* **2**, 468 (1932).

[3] G. Breit, *Phys. Rev.* **34**, 553 (1929); *Phys. Rev.* **35**, 569 (1930); *Phys. Rev.* **36**, 383 (1930); *Phys. Rev.* **39**, 616 (1932).

[4] E. E. Salpeter, H. A. Bethe, *Phys. Rev.* **84**, 1232 (1951).

[5] G. C. Wick, *Phys. Rev.* **96**, 1124 (1954); R. E. Cutkosky, *Phys. Rev.* **96**, 1135 (1954).

[6] M. Günther, *Phys. Rev.* **D9**, 2411 (1974).

[7] A. A. Logunov, A. N. Tavkhelidze, *Nuovo Cimento* **29**, 380 (1963); A. A. Logunov, A. N. Tavkhelidze, I. T. Todorov, O. A. Khrushtalev, *Nuovo Cimento* **30**, 134 (1963); R. Blankenbecler, R. Sugar, *Phys. Rev.* **142**, 1051 (1966).

[8] I. T. Todorov, in *Properties of the Fundamental Interactions*, edited by A. Zichichi, Editrice Compositori, Bologna 1973, Vol. 9, part C, p. 953; *Phys. Rev.* **D3**, 2351 (1971); R. N. Faustov, *Ann. Phys.* **78**, 176 (1973); V. A. Rizov, I. T. Todorov, *Sov. J. Part. Nucl.* **6**, 269 (1975).

[9] W. Królikowski, J. Rzewuski, *Nuovo Cimento* **2**, 203 (1955); *Nuovo Cimento* **3**, 260 (1956); *Nuovo Cimento* **4**, 975 (1956); *Nuovo Cimento* **25B**, 739 (1975).

[10] W. Królikowski, J. Rzewuski, *Acta Phys. Pol.* **B7**, 487 (1976).

[11] M. H. Partovi, *Phys. Rev.* **D12**, 3887 (1975).

[12] H. Suura, *Phys. Rev. Lett.* **38**, 636 (1977).

[13] M. M. Bakri, *Nuovo Cimento* **51A**, 864 (1967); *J. Math. Phys.* **10**, 298 (1969).

[14] T. Fulton, R. Karplus, *Phys. Rev.* **93**, 1109 (1954).

[15] T. Fulton, P. C. Martin, *Phys. Rev.* **95**, 811 (1954).

[16] Confer L. C. Biedenharn, J. Nuyts, H. Ruegg, *On the Generalization of Isoparity*, CERN Report No. 65-3, Geneva 1965, p. 14 and p. 15.