AXIALLY SYMMETRIC BRANS–DICKE SOLUTIONS

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A scheme is presented which extends Ernst's concise unified derivation of the axially symmetric solutions of Einstein's field equations to the case of the Brans–Dicke Theory. The method is applied to obtain Tomimatsu–Sato type solutions of the Brans–Dicke theory and the analogues of Schwarzschild and Kerr solutions are rediscovered.

1. *Introduction*

One of the obstacles to a better understanding of the physical implications of general relativity is the relative scarcity of exact solutions of the field equations. Hence a substantial increase in the number of known solutions would be a useful first step. A number of authors have discussed methods of generating new solutions of these equations from known solutions. Among them are Buchdahl (1954), Ehlers (1962), Perjes (1971), Harrison (1968), Geroch (1971, 1972), Kinnersley (1973), McIntosh (1974) and Goswami (1978).

Brans and Dicke (1962) have proposed a modification of Einstein's theory of gravitation through the introduction of a scalar function ϕ in the field equations to make things more consistent with Mach's principle and less reliant on the absolute properties of space. The field equations are

$$\phi(R_{ij}-\tfrac{1}{2}g_{ij}R) = 8\pi T_{ij}+\frac{\overline{\omega}}{\phi}(\phi_{,i}\phi_{,j}-\tfrac{1}{2}g_{ij}\phi_{,K}\phi^{,K})+\phi_{;i;j}-g_{ij}\phi^{;K}_{;K}, \tag{1}$$

$$\phi^{;K}_{;K} = 8\pi T/(2\overline{\omega}+3), \tag{2}$$

where $\overline{\omega}$ is a dimensionless constant and T_{ij} is the stress-energy tensor for a matter distribution.

According to the formulation of Ernst (1968), stationary, axisymmetric solutions of Einstein's field equations in empty space can be derived from a complex function ε which satisfies the following equation

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$$(\mathrm{Re}\,\varepsilon)\nabla^2\varepsilon = \nabla\varepsilon\cdot\nabla\varepsilon, \tag{3}$$

where ∇ is the three dimensional divergence operator and ε is given by

$$\varepsilon = f + i\psi, \tag{4}$$

where the line-element is given by

$$ds^2 = f(dt - \omega d\varphi)^2 + f^{-1}[e^{2\gamma}(d\varrho^2 + dz^2) + \varrho^2 d\varphi^2]. \tag{5}$$

Here f, ω and γ are functions of ϱ and z alone. ψ is a function independent of azimuth and is given by

$$\varrho^{-1} f^2 \nabla\omega = \hat{n} \times \nabla\psi, \tag{6}$$

where $\hat{n}$ is a unit vector in the azimuthal direction.

From a knowledge of ε Ernst derived the solutions of Weyl and Papapetrou and Kerr. Later Tomimatsu and Sato (1972) discovered an entirely new axisymmetric stationary solution in prolate spheroidal co-ordinates using Ernst's method. Considering the simplicity and the compactness of Ernst's formalism it seemed worthwhile to extend the method to the case of Brans–Dicke theory and interestingly enough it was found that the two key equations in Ernst's method remain the same as one goes over to the Brans–Dicke theory. This being the case one can very easily derive the Brans–Dicke analogue of Einstein's solutions. McIntosh had however given a slightly different method of generating stationary axisymmetric source-free solutions of the Brans–Dicke theory from the corresponding solutions of Einstein's theory. As an example, our method was applied to the axially symmetric stationary solution of Tomimatsu and Sato having parameter $\delta = 2$. The corresponding solutions for other values of δ can be similarly found. The method can also be used to rediscover the Schwarzschild-like and Kerr-like solutions of the Brans–Dicke theory given by Brans (1962) and McIntosh (1974).

2. *Field equations*

The vacuum Brans–Dicke equations may be written as

$$\bar{R}_{\mu\nu} = -k\Phi_{,\mu}\Phi_{,\nu}, \tag{7}$$

where Φ is a scalar field, k a coupling constant equal to $\bar{\omega}+3/2$, where $\bar{\omega}$ is the coupling coustant usually given in the Brans–Dicke theory and the conformal frame being used is the "Einstein frame" (see appendix).

The Bianchi identities give for equation (7)

$$\Box\Phi = 0, \tag{8}$$

so that the extra field equation is automatically satisfied. The scalar field Φ is also a function of ϱ and z.

For the line-element (5), we get

$$\gamma_{11} + \gamma_{22} - \gamma_1/\varrho + \frac{f_1^2}{2f^2} + \frac{f^2}{2\varrho^2}\omega_2^2 = -k\Phi_1^2, \tag{9}$$

$$\gamma_{11}+\gamma_{22}+\gamma_1/\varrho+\frac{f_2^2}{2f^2}+\frac{f^2}{2\varrho^2}\omega_1^2 = -k\Phi_2^2, \tag{10}$$

$$-\gamma_2/\varrho+\frac{f_1 f_2}{2f^2}-\frac{f^2}{2\varrho^2}\omega_1\omega_2 = -k\Phi_1\Phi_2, \tag{11}$$

$$f\nabla^2 f = \nabla f\cdot\nabla f-\frac{f^4}{\varrho^2}\nabla\omega\cdot\nabla\omega, \tag{12}$$

$$\nabla\cdot(\varrho^{-2}f^2\nabla\omega) = 0. \tag{13}$$

Here suffix 1 refers to the derivative with respect to ϱ and 2 with respect to z.

For the stationary case equation (8) reduces to

$$\nabla^2\Phi = 0, \tag{14}$$

where ∇^2 is the well known Laplacian.

From the field equations one sees that equations (12) and (13) are just the two key equations in Ernst's formalism. Hence of all the metric components f and ω would remain the same both in Einstein's and Brans–Dicke's theory. We can expect a change in γ only. Of course, there is the added scalar term Φ.

From the field equations we obtain

$$\gamma_1 = \frac{\varrho}{4f^2}(f_1^2-f_2^2)-\frac{f^2}{4\varrho}(\omega_1^2-\omega_2^2)+\frac{k\varrho}{2}(\Phi_1^2-\Phi_2^2), \tag{15}$$

$$\gamma_2 = k\varrho\Phi_1\Phi_2+\frac{\varrho f_1 f_2}{2f^2}-\frac{f^2}{2\varrho}\omega_1\omega_2. \tag{16}$$

We now make a transformation to the prolate spheroidal coordinates (x, y)

$$\varrho = a(x^2-1)^{1/2}(1-y^2)^{1/2}, \tag{17a}$$

$$z = axy. \tag{17b}$$

Hence

$$\gamma_x = \frac{1}{4f^2(x^2-y^2)}[xf_x^2(1-y^2)(x^2-1)-xf_y^2(1-y^2)^2-2yf_xf_y(1-y^2)(x^2-1)]$$

$$-\frac{f^2}{4(x^2-y^2)(x^2-1)}[x(x^2-1)\omega_x^2-x(1-y^2)\omega_y^2-2y(x^2-1)\omega_x\omega_y]$$

$$+\frac{k(1-y^2)}{z(x^2-y^2)}[x(x^2-1)\Phi_x^2-x(1-y^2)\Phi_y^2-2y(x^2-1)\Phi_x\Phi_y]. \tag{18}$$

$$\gamma_y = \frac{1}{4f^2(x^2-y^2)}[yf_x^2(x^2-1)^2 - yf_y^2(x^2-1)(1-y^2)+2xf_xf_y(x^2-1)(1-y^2)]$$
$$-\frac{f^2}{4(x^2-y^2)(1-y^2)}[y(x^2-1)\omega_x^2 - y(1-y^2)\omega_y^2+2x(1-y^2)\omega_x\omega_y]$$
$$+\frac{k(x^2-1)}{2(x^2-y^2)}[y(x^2-1)\Phi_x^2 - y(1-y^2)\Phi_y^2+2x(1-y^2)\Phi_x\Phi_y]. \quad (19)$$

Moreover the Laplacian operator assumes the following form in the prolate spheroidal co-ordinates

$$\nabla^2 = \frac{1}{a^2(x^2-y^2)}\left[\frac{\partial}{\partial x}(x^2-1)\frac{\partial}{\partial x} + \frac{\partial}{\partial y}(1-y^2)\frac{\partial}{\partial y}\right]. \quad (20)$$

From Eq. (14) it follows that Φ can be expressed as a linear superposition

$$\Phi = \sum_l \alpha_l Q_l(x) P_l(y), \quad (21)$$

where α_l are constants, $Q_l(x)$ is a Legendre function of the second kind and $P_l(y)$ is a Legendre function of the first kind.

When $l = 0$,

$$\Phi = -\beta/2 \ln\frac{x+1}{x-1}, \quad (22)$$

when $l = 1$,

$$\Phi = \eta y\left(1-\frac{x}{2}\ln\frac{x+1}{x-1}\right), \quad (23)$$

where β and η are arbitrary constants of integration.

Moreover, from equations (17) it follows that in prolate spheroidal co-ordinates the line-element reduces to

$$ds^2 = -f^{-1}\left[e^{2\gamma}a^2(x^2-y^2)\left\{\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right\} + a^2(x^2-1)(1-y^2)d\varphi^2\right]$$
$$+f(dt-a\omega d\varphi)^2. \quad (24)$$

3. *Examples*

(a) Static spherical symmetry

We may define a new variable ξ such that

$$\varepsilon = \frac{\xi-1}{\xi+1} = f+i\psi,$$

such that the differential equation (3) takes the form

$$(\xi\xi^*-1)\nabla^2\xi = 2\xi^*\nabla\xi\cdot\nabla\xi.$$

We put $\xi = x$. Hence $\psi = 0$ and $\omega = 0$. There is thus no rotation and we get

$$\gamma_x = (1+k\beta^2/2)\,\frac{x(1-y^2)}{(x^2-1)\,(x^2-y^2)}, \tag{25}$$

$$\gamma_y = (1+k\beta^2/2)\,\frac{y}{x^2-y^2}. \tag{26}$$

From these we get

$$\gamma = \frac{1+k\beta^2/2}{2}\ln\frac{x^2-1}{x^2-y^2}. \tag{27}$$

Let us assume that Φ is a function of x only i.e.

$$\Phi = -\beta/2\ln\frac{x+1}{x-1}.$$

In the asymptotic region where $x\to\infty$, γ tends to zero. Thus the metric is asymptotically flat. Moreover, when the coupling constant k vanishes we get the Schwarzschild metric back, f and ω however remain the same. The solution found becomes identical with that of Brans (1962) under certain substitutions and a conformal mapping.

(b) Kerr-like solution

With the help of the above technique we can also generate the rotating Kerr-like solution in the Brans–Dicke theory where we have to take another solution

$$\xi = x\cos\lambda+iy\sin\lambda,$$

where λ is a constant. A similar solution under a different co-ordinate system has been discussed in detail by McIntosh (1974). Hence we are not giving this solution here.

(c) Tomimatsu–Sato metric

Tomimatsu and Sato found a class of solutions of Ernst's Eq. (3) representing the gravitational fields of spinning bound masses. The co-ordinate transformation is as follows.

$$\varrho = \left(\frac{mp}{\delta}\right)(x^2-1)^{1/2}(1-y^2)^{1/2},$$

$$z = \left(\frac{mp}{\delta}\right)xy, \tag{28}$$

where δ is a dimensionless parameter. They obtained

$$f = A/B, \qquad \omega = \frac{2mqC(1-y^2)}{A}, \tag{29}$$

where

$$A = [p^2(x^2-1)^2+q^2(1-y^2)^2]^2-4p^2q^2(x^2-1)(1-y^2)(x^2-y^2)^2, \tag{30}$$

$$B = (p^2x^4+q^2y^4-1+2px^3-2px)^2+4q^2y^2(px^3-pxy^2+1-y^2), \tag{31}$$

$$C = p^2(x^2-1)[(x^2-1)(1-y^2)-4x^2(x^2-y^2)]$$
$$-p^3x(x^2-1)[2(x^4-1)+(x^3+3)(1-y^2)]+q^2(1+px)(1-y^2)^3, \tag{32}$$

$$e^{2\gamma} = \frac{A}{p^4(x^2-y^2)^4}. \tag{33}$$

Here p and q are the parameters defined by the spin factor J and gravitational mass m as $q = J/m^2$ and $p = (1-q^2)^{1/2}$. The line-element is given by equation (5).

Let us find the Brans–Dicke analogue of the above solutions with the help of our new technique. We take the value of $l = 1$. Hence $\Phi = \eta y\left(1-\frac{x}{2}\ln\frac{x+1}{x-1}\right)$. As in case (a) both ω and f remain the same. The only change will be in the equation (33). From equations (18) and (19) we get after a somewhat lengthy but straight forward calculation

$$\gamma = \tfrac{1}{2}\ln A-2\ln p^4(x^2-y^2)-\frac{k\eta}{8mp}(x^2-1)(1-y^2)\left|\ln\frac{x+1}{x-1}\right|^2$$
$$+\frac{k\eta}{2mp}y^2+\frac{k\eta}{2mp}\ln\frac{x^2-1}{x^2-y^2}+\frac{k\eta x}{2mp}(1-y^2)\ln\frac{x+1}{x-1}. \tag{34}$$

Similarly we can find the Brans–Dicke analogues of other Tomimatsu–Sato solutions with $\delta > 2$.

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APPENDIX

The Brans–Dicke theory was originally expressed in the conformal frame ("The Brans–Dicke frame") in which the masses of the small particles remain constant while the locally measured gravitational constant G is $\phi^{-1}\times(4+2\omega)(3+2\omega)^{-1}$, where ϕ is the scalar field.

Dicke pointed out that the theory can also be expressed in another conformal frame in which the field equations resemble Einstein's. In this frame (the "Einstein frame") the value of G is defined to be a constant G_0 everywhere but the masses of particles vary as $\phi^{-1/2}$. The conformal transformation defining this frame is

$$\bar{g}_{ab} = \phi G_0 g_{ab}, \quad \bar{T}_{ab} = \phi^{-1}G_0^{-1}T_{ab} \quad \text{and} \quad \Phi = \ln\phi.$$

Under these transformations the equation (1) reduces to

$$\bar{R}_{\mu\nu} = -(\bar{\omega}+3/2)\Phi_{,\mu}\Phi_{,\nu},$$

where $\bar{R}_{\mu\nu}$ = Ricci tensor formed out of $\bar{g}_{\mu\nu}$.

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