Vol. B11 (1980) ACTA PHYSICA POLONICA No 7

UNIFIED FIELD, METRIC AND THE LOCAL INVARIANCE GROUP

By A. H. Klotz

Department of Applied Mathematics, University of Sydney*

(*Received October 29, 1979*)

Metric hypothesis in the nonsymmetric unified field theory of Einstein and Straus is discussed from a group theoretical and fiber bundle point of view. Theoretical background of the hypothesis is constructed. It is shown that the "hypothesis" is in fact implied by the concept of Hermitian symmetry on which the nonsymmetric theory is based.

1. *Introduction*

I have postulated recently (Ref. [1]) that the metric tensor $a_{\mu\nu}$ in the nonsymmetric unified field theory (Ref. [2]) should be determined by the equation

$$\begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix}_a = \tilde{\Gamma}^{\lambda}_{\mu\nu}, \tag{1}$$

the Christoffel brackets being constructed from $a_{\mu\nu}$ and $\tilde{\Gamma}^{\lambda}_{(\mu\nu)}$ denoting the symmetric part of Schrödinger's affine connection

$$\tilde{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} + \tfrac{2}{3}\, \delta^{\lambda}_{\mu} \Gamma_{\nu}, \tag{2}$$

where

$$\Gamma_{\nu} = \Gamma^{\sigma}_{[\nu\sigma]}$$

is the contracted, skew symmetric part of a connection $\Gamma^{\lambda}_{\mu\nu}$. The "metric hypothesis" (1) proved to have far reaching consequences (Refs. [3, 4, 5]) in the physical interpretation of the theory.

The aim of this article is to discuss group theoretical foundations of the above assumption. In particular, it will be shown that identification of the metric through equation (1) can be based on an extension of the local invariance group of the unified field from the general relativistic Lorentz group to the Poincare group. The latter operates on a bundle

* Address: Department of Applied Mathematics, University of Sydney, Sydney N.S.W., 2006, Australia.

of affine frames and factorisation of the affine group allows just enough freedom to set up all the quantities needed in the formulation of the nonsymmetric theory.

The point of view adopted here is that since the most significant aspect of the theory is its foundation on the principle of Hermitian symmetry, this must be preserved in whatever may come out of group theoretical considerations if the theory is to be physically meaningful. It turns out that this can be achieved providing a certain amount of care is exercised in defining the tetrad vectors.

Throughout this work Latin indices will denote tetrad or local vectors and Greek indices, the coordinate components where and when necessary. In a relativistic space-time manifold in which we are interested, both sets of indices go from 0 to 3. The group theoretical notation is that of Drechsler and Mayer (Ref. [6]).

Finally, I wish to express my gratitude to my colleague C. Radford for drawing my attention to the group theoretical reasoning used herein and for working out the component form of the expressions below.

2. *Group theoretical preliminaries*

Let x be a point of the space-time manifold X and $T_x(X)$, the tangent space to X at x. The (Poincare) group of affine transformations A(4, R) acts on the bundle $A(X)$ of affine frames, and can be represented as the group of all matrices of the form

$$\begin{pmatrix} g & \xi \\ 0 & 1 \end{pmatrix}, \tag{3}$$

$g \in \mathrm{GL}(4; \mathrm{R})$, $\xi \in \mathrm{R}^4$.

The tangent space itself is then the tangent affine space $TA_x(X)$ say, and an affine frame at $x \in X$, $(a; X_i)_x$, consists of

$$a \in TA_x(X)$$

together with a linear frame $(X_i)_x$. The group A(4; R) can be written (Ref. [6]) as a semi-direct product of its homogeneous and inhomogeneous parts

$$\mathrm{A}(4; \mathrm{R}) = \mathrm{GL}(4; \mathrm{R}) \cdot \mathrm{R}^4, \tag{4}$$

R^4 being assigned a Minkowski inner product. Corresponding to the product (4) is the semi-direct sum of Lie algebras

$$\mathrm{a}(4; \mathrm{R}) = \mathrm{gl}(4; \mathrm{R}) \oplus \mathrm{R}^4. \tag{5}$$

Hence, if ω denotes a gl (4; R) valued 1-form on X (a linear connection) and ψ is an R^4 valued 1-form on X (a tensor of type (1, 1)), a connection $\tilde{\omega}$ in $A(X)$ is given by

$$\tilde{\omega} = \omega \oplus \psi. \tag{6}$$

Let $\tilde{\Omega}$ be the curvature form of $\tilde{\omega}$:

$$\tilde{\Omega} = d\tilde{\omega} + \tfrac{1}{2}[\tilde{\omega}, \tilde{\omega}]. \tag{7}$$

The structure equation (7) then splits into a gl (4; R) part

$$\Omega = d\omega + \tfrac{1}{2}[\omega, \omega] \tag{8}$$

and an R^4 valued part

$$\Theta = \tfrac{1}{2}[\omega, \psi] + \tfrac{1}{2}[\psi, \omega] + d\psi = D\psi, \tag{9}$$

where D denotes the covariant derivative with respect to ω. Hence Θ is the generalised torsion form of the linear connection ψ. Equation (9) gives the standard torsion form if ψ is the canonical. 1-form.

Let us now express the above results in local coordinates (the expressions below have been worked out by Radford).

Let E^i_j be a natural basis of the Lie algebra of gl (4; R) and e_j the natural basis in R^4. Let also Y^α be the local coordinates in $TA_x(X)$; X^α_i the local coordinates in the bundle of linear frames $L(X)$; and x^α, the local coordinates in the manifold X, that is the coordinates of $x \in X$. Then $(Y^\alpha, X^\alpha_i, x^\alpha)$ are the local coordinates in the affine bundle $A(X)$ while (Y^α, X^α_i) are the coordinates in the fibre $\pi^{-1}(x)$ where

$$\pi : A(X) \to X$$

is the projection mapping of the bundle into the manifold.

In terms of these coordinates

$$\omega = E^i_j(Y^j_\alpha dX^\alpha_i + \gamma^j_{\alpha i} dx^\alpha) \tag{10}$$

and

$$\psi = e_j Y^j_\alpha (dy^\alpha + K^\alpha_\beta dx^\beta), \tag{11}$$

where K^α_β is a (1, 1) tensor, $\gamma^j_{\alpha i}$ are the local components of the Ricci rotation coefficients or, in effect a frame connection and

$$Y^j_\alpha = (X^{-1})^j_\alpha.$$

We can introduce a (nonsymmetric) affine connection $\Gamma^\lambda_{\mu\nu}$ as follows. Let X^*_α be the horizontal lift of $X_\alpha = \partial_{x^\alpha}$. Since

$$\tilde{\omega}(X^*_\alpha) = 0, \tag{12}$$

we have

$$X^*_\alpha = \partial_{x^\alpha} - \gamma^i_{\alpha j} X^\beta_i \partial_{X_j{}^\beta} - K^\beta_\alpha \partial_{y^\beta}. \tag{13}$$

Let us associate with every vector field V an A^4 valued function f:

$$f(a) = a^{-1}(V(\pi(a))), \tag{14}$$

$a \in A_x(X)$, $\pi(a) = x \in X$.

The function associated with X_α is

$$f_\alpha = a^{-1}(X_\alpha) = Y^j_\alpha e_j \tag{15}$$

and

$$X^*_\alpha(f_\beta) = (X^\gamma_j Y^j_{\beta,\alpha} + \gamma^k_{\alpha j} X^\gamma_k Y^j_\beta) f_\gamma \tag{16}$$

becomes the A^4 valued function associated with the covariant derivative $\nabla_{x^\alpha} X_\beta$. Thus

$$\nabla_{\partial X} \partial x^\beta = \Gamma^\delta_{\alpha\beta} \partial_{x^\delta}, \tag{17}$$

where

$$\Gamma^\gamma_{\alpha\beta} = X^\gamma_j Y^j_{\beta,\alpha} + \gamma^k_{\alpha j} X^\gamma_k Y^j_\beta \tag{18}$$

and represents the connection coefficients with respect to ω. $\Gamma^\gamma_{\alpha\beta}$ is related to the rotation coefficients by the $L(X)$ coordinates since it follows immediately from equation (18) that

$$\nabla_\alpha X^\beta_j = X^\beta_{j,\alpha} + \Gamma^\beta_{\alpha\sigma} X^\sigma_j - \gamma^k_{\alpha j} X^\beta_k = 0. \tag{19}$$

3. *Identification of the metric*

We are now ready to discuss the problem of identifying a metric in the base manifold X which we regard as the space-time manifold of the physical world. It turns out that there is only one way in which this can be achieved which is consistent with the principle of Hermitian symmetry. The problem is not as straightforward as in General Relativity where, by the principle of equivalence and the strong principle of geometrisation (Ref. [1]), the gravitational field is identified with the metric structure of physical manifold. In the nonsymmetric theory, the fundamental tensor $g_{\mu\nu}$ describing combined electromagnetic and gravitational fields must be related to a geometrical structure of space-time by an assumption which would replace the general relativistic axiom. Since the metric hypothesis radically affects our view of this relation by restricting the domain of solutions of the field equations, it is essential that it should possess the same Hermitian property as fundamental macrophysics.

Since the elements of the tangent bundle $T(X)$ (i.e. tanget spaces $T_x(X)$) have the same structure as R^4 endowed with a Minkowski inner product, the bundle of linear frames $L(X)$ can be always restricted to the Lorentz bundle $\mathscr{L}(X, L)$ by reducing GL (4; R) to the Lorentz group $\mathscr{L}_4$.

Transformations of the latter preserve the tensor

$$l^{\alpha\beta} = X^\alpha_i \eta^{ij} X^\beta_j, \tag{20}$$

where the Minkowski tensor

$$\eta^{ij} = \mathrm{diag}\,(1, -1, -1, -1)$$

becomes the fiber metric. The corresponding restriction of affine transformations to Poincaré transformations, reduces $A(X)$ to the Poincaré bundle of frames $P(X)$. Defining the tetrad

$$\lambda^\alpha_j = X^\alpha_j, \quad \lambda^j_\alpha = Y^j_\alpha, \tag{21}$$

Greek indices can be raised and lowered with $l^{\alpha\beta}(l_{\alpha\beta} = \lambda^i_\alpha \eta_{ij} \lambda^j_\beta)$ and Latin indices with $\eta^{ij}(\eta_{ij})$. We may observe that (Latin) skewsymmetry

$$\gamma^{ij}_\alpha + \gamma^{ji}_\alpha = 0, \quad \gamma^{ij}_\alpha = \eta^{jk} \gamma^i_{\alpha k}$$

of the rotation coefficients is a consequence of

$$\nabla_\alpha \eta^{ij} = 0. \tag{22}$$

Alternatively

$$\gamma_{\alpha ij} = -\gamma_{\alpha ji} + (-l_{\beta\gamma,\alpha} + l_{\sigma\gamma} \Gamma^\sigma_{\alpha\beta} + l_{\beta\sigma} \Gamma^\sigma_{\alpha\gamma}) \lambda^\beta_i \lambda^\gamma_j,$$

so that skew symmetry occurs if

$$\nabla_\alpha l_{\beta\gamma} = 0. \tag{23}$$

However, derivation (23) conflicts with the principle of Hermitian symmetry. Hence $l_{\alpha\beta}$ can not be chosen as the space-time metric. Since, on the other hand, equations (22) and (19) necessarily hold, the only possibility is to construct the metric tensor

$$a^{\alpha\beta} = h^\alpha_i \eta^{ij} h^\beta_j \tag{24}$$

from another tetrad h^α_i which satisfies the equations

$$\nabla_\gamma h^\alpha_i = \tilde{\Gamma}^\alpha_{[\gamma\sigma]} h^\sigma_i, \tag{25}$$

where $\tilde{\Gamma}^\alpha_{[\beta\gamma]}$ denotes the skewsymmetric part of $\tilde{\Gamma}^\alpha_{\beta\gamma}$ (with symmetric part $\tilde{\Gamma}^\alpha_{(\beta\gamma)}$). Then

$$D_\gamma h^\alpha_i \equiv h^\alpha_{i,\gamma} + \tilde{\Gamma}^\alpha_{(\gamma\sigma)} h^\sigma_i - \gamma^\kappa_{\gamma i} h^\alpha_\kappa = 0. \tag{26}$$

Also from

$$D_\gamma \eta^{ij} = \nabla_\gamma \eta^{ij} = 0$$

we get

$$D_\gamma a^{\alpha\beta} = 0 \tag{27}$$

which is equivalent to the "hypothesis" (1). We can readily verify that with

$$\gamma_{\alpha ij} = \eta_{ik} h^\sigma_{j,\alpha} h^k_\sigma + \eta_{ik} \Gamma^\sigma_{(\alpha\beta)} h^\beta_j h^k_\sigma, \tag{28}$$

$$\gamma_{\alpha ij} = -\gamma_{\alpha ji},$$

if

$$D_\gamma a_{\alpha\beta} = 0.$$

4. The Riemann and the fundamental tensors

It now follows that in order to retain Hermitian symmetry of the equation defining the metric, we must distinguish the "physical" frame h^α_j from the geometrical frame X^α_j (or λ^α_j), the rotation coefficients being given by either

$$\gamma^k_{\alpha j} = h^k_\sigma h^\sigma_{j,\alpha} + \bar{\Gamma}^k_{\alpha j} \tag{29}$$

or (because of the equation (18))

$$\gamma^k_{\alpha j} = X^k_\sigma X^\sigma_{j,\alpha} + \tilde{\Gamma}^k_{\alpha j}, \tag{30}$$

where

$$\overline{\Gamma}^k_{\alpha j} = h^k_\sigma h^\beta_j \tilde{\Gamma}^\sigma_{(\alpha\beta)}, \qquad \tilde{\Gamma}^k_{\alpha j} = Y^k_\sigma X^\beta_j \tilde{\Gamma}^\sigma_{\alpha\beta}. \tag{31}$$

Strictly speaking the rotation coefficients in the equations (29) and (30) are distinct, being 1-forms in two subbundles of the affine bundle of frames. In that case, they (or rather the h and X tetrads) will be connected by a transformation of the form

$$\omega' g = g\omega + dg. \tag{32}$$

The effect of putting

$$\omega' = \omega, \tag{33}$$

is that the transformation g should satisfy the differential exterior equation

$$dg = \omega g - g\omega. \tag{34}$$

The integrability condition of equation of equation (33) which is written in matrix form, is

$$\Omega g = g\Omega, \tag{35}$$

where Ω is the curvature form of the manifold, and can be nontrivially satisfied by $g = \Omega$ which, since Ω is skew means in general that g must belong to the symplectic group. The existence of such nontrivial (i.e. not requiring that g should be constant, in particular, unit matrix) solution ensures that equations (29) and (30) can be written down without imposing too severe restrictions on the affine bundle or the manifold.

In terms of the natural base $E^i_j = \eta^{ik} E_{kj}$, the basis of the Lorentz group is

$$L_{kj} = E_{[kj]},$$

and $\{L_{ij}, e_k\}$ generates the Poincaré group with Lie brackets

$$[L_{ij}, L_{kl}] = \eta_{jk} L_{il} - \eta_{ik} L_{jl} + \eta_{jl} L_{ki} - \eta_{li} L_{kj},$$
$$[L_{ij}, e_k] = \eta_{jk} e_i - \eta_{ki} e_j. \tag{36}$$

Let σ be the cross-section of $P(X)$ over an open neighbourhood U of a point $x \in X$, assigning to every $x \in U$ a Poincaré frame $(O_x, X^\alpha_i \delta_{x^\alpha})$, O being the origin of R^4. In terms of the coordinates $(y^\alpha, x^\alpha, X^\alpha_i)$

$$\sigma : (x^\alpha) \to ((0, 0, 0, 0), x^\alpha, X^\alpha_i). \tag{37}$$

Then we have on the base manifold X,

$$\sigma^* \omega^i_j = \gamma^i_{\alpha j} dx^\alpha,$$
$$\sigma^* \psi^j = K^j_\alpha dx^\alpha,$$
$$\sigma^* \Omega^i_j = \tfrac{1}{2} R^i_{j\mu\nu} dx^\mu \wedge dx^\nu,$$
$$\sigma^* \overline{\theta}^j = \tfrac{1}{2} \overline{T}^j_{\mu\nu} dx^\mu \wedge dx^\nu, \tag{38}$$

where

$$R^i_{j\mu\nu} = \gamma^i_{\nu j,\mu} - \gamma^i_{\mu j,\nu} + \gamma^k_{\nu j}\gamma^i_{\mu k} - \gamma^k_{\mu j}\gamma^i_{\nu k}, \tag{39}$$

and

$$\overline{T}^j_{\mu\nu} = 2[\nabla_{[\mu}K^j_{\nu]} + \tilde{\Gamma}^\sigma_{[\mu\nu]}K^j_\sigma]. \tag{40}$$

If the two frames coincide and $K^j_\nu = h^j_\nu = X^j_\nu$ (as they must not if the affine bundle is to split in the required way), $\overline{T}^j_{\mu\nu}$ becomes the usual torsion tensor of the affine connection $\tilde{\Gamma}^\sigma_{\mu\nu}$.

We can verify easily, using the equations (29) that

$$X^\alpha_i Y^j_\beta R^i_{j\mu\nu} = R^\alpha_{\beta\mu\nu}, \tag{41}$$

the Riemann–Christoffel tensor constructed from $\tilde{\Gamma}^\sigma_{\mu\nu}$. On the other hand, using the h-frame, defining the (1, 1) tensor

$$K^\alpha_\mu = h^\alpha_j K^j_\mu, \tag{42}$$

and raising and lowering Greek indices with the metric tensor $a_{\mu\nu}$, the generalised torsion becomes

$$\overline{T}^j_{\mu\nu} = h^{j\sigma}[\nabla_\mu K_{\sigma\nu} - \nabla_\nu K_{\sigma\mu} + \tilde{\Gamma}^\alpha_{[\mu\sigma]}K_{\alpha\nu} - \Gamma_{[\nu\sigma]}K_{\alpha\mu}].$$

If we identify the tensor $K_{\mu\nu}$ with the fundamental tensor $g_{\mu\nu}$ of the nonsymmetric theory for which

$$g_{\mu\nu,\lambda} - \tilde{\Gamma}^\sigma_{\mu\lambda}g_{\sigma\nu} - \tilde{\Gamma}^\sigma_{\lambda\nu}g_{\mu\sigma} = 0, \tag{43}$$

then

$$\overline{T}^j_{\mu\nu} = h^{j\sigma}g_{\alpha\beta}[\tilde{\Gamma}^\alpha_{[\nu\sigma]}\delta^\beta_\mu - \tilde{\Gamma}^\alpha_{[\mu\sigma]}\delta^\beta_\nu]. \tag{44}$$

5. *Discussion*

We have shown that the metric hypothesis of the nonsymmetric unified field theory can be given a rigorous, group theoretical foundation involving extension of local invariance of the field from a Lorentz (i.e. general relativistic) to a Poincare invariance. It may be thought that the distinction we have been forced to make between physical and geometrical frames of reference is unnatural. Nevertheless, the distinction is fully in keeping with our reinterpretation of the theory (Ref. [1]). All we need to note is that it is the physical objects, represented by the K or $g_{\mu\nu}$ field, which are expressed in terms of the physical h-frame, while geometrical objects, i.e. the affine connection and the Riemann tensor, are expressed in terms of the geometrical X-tetrad. Hence we can regard the two-frame formulation as a restatement of the weak principle of geometrisation, required in any case in order to assign correct meaning to the theory.

Let us note that vanishing of the generalised torsion implies a strong restriction on the field.

$$\mathfrak{g}^{[\mu\nu]}\tilde{\Gamma}^\sigma_{[\mu\nu]} = 0,$$

which is not satisfied by the physically meaningful solutions of the field equations. Thus, we must conclude that the torsion can not vanish.

Finally, it is interesting to observe that a formulation of the nonsymmetric theory in terms of group theoretic or fibre-bundle techniques cannot be obtained were we to identify

$$a_{\mu\nu} = g_{(\mu\nu)},$$

as Einstein thought. The reason for this is that in this formulation the metric must satisfy equations such as (23) (which we have rejected) or (27). On the other hand, equation (43) by means of which the connection is determined by $g_{\mu\nu}$ and its first derivatives, implies, as is well known, that $g_{(\mu\nu)}$ satisfies an equation depending explicitly on $g_{[\mu\nu]}$ which does not appear (and indeed, can not appear) in the equation determining the metric.

REFERENCES

[1] A. H. Klotz, *Acta Phys. Pol.* **B9**, 573 (1978).
[2] A. Einstein, E. G. Straus, *Ann. Math.* **47**, 731 (1946).
[3] A. H. Klotz, *Acta Phys. Pol.* **B9**, 595 (1978).
[4] A. H. Klotz, *Acta Phys. Pol.* **B10**, 589 (1979).
[5] A. H. Klotz, *Lett. Nuovo Cimento* **25**, 190 (1979).
[6] W. Drechsler, M. E. Mayer, *Fiber Bundle Techniques in Gauge Theories*, Lecture Notes in Physics 67, Springer 1977.