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SINGLE-TIME FORM OF THE FOKKER-TYPE RELATIVISTIC DYNAMICS. I

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It is shown that the many-time Fokker-type action integral corresponds to a single-time action with a Lagrangian depending on higher derivatives (in exact theory — up to infinite order). The expressions for such Lagrangians are found and the corresponding equations of motion, forming a set of ordinary differential equations of infinite order are given. The particular cases corresponding to manifestly invariant Fokker-type action, arbitrary tensor interaction and electrodynamics are considered.

1. Introduction

The first relativistic theories of direct interactions which do not employ the concept of field as an independent object explicitly were proposed by Schwarzshild, Tetrode and Fokker at the beginning of the twentieth century as an alternative to Maxwell's field electrodynamics. They were supplemented and extended on the radiation processes by Wheeler and Feynman [1]. Theories of this type were proposed later for other interactions described usually by means of field methods. All of them are based on the Fokker-type action integrals of the following form [2–4]

$$S = -\sum_a m_a \int_{-\infty}^{\infty} d\tau_a \sqrt{u_a^2} - \sum_{a<b}\sum \iint_{-\infty}^{\infty} d\tau_a d\tau_b \Lambda_{ab}. \tag{1}$$

Here $a, b = 1, \ldots, N$ (N denotes the number of the particles in the system), m_a being the rest-masses of the particles, τ_a being invariant parameters of their world-lines $x_a(\tau_a) \equiv (t_a, \mathbf{x}_a)$, $u_a \equiv dx_a/d\tau_a$, $\Lambda_{ab} = \Lambda_{ab}(x_a - x_b, u_a, u_b)$ being some Poincaré-invariant functions including the coupling constants; velocity of light $c = 1$. The infinite limits for the integrations will no longer be written, but will be understood.

The manifestly relativistic covariant equations of motion obtained from the variational principle for the Fokker-type action (1) have a very complicated mathematical structure, and the presence of the individual time parameter for each particle inherent in

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four-dimensional formalism complicates the physical interpretation of their solutions As a consequence the employing of the Fokker-type dynamics for the investigation of the particle system with direct interaction presents difficulties.

On the other hand great attention was paid recently to the problem of the construction of the Hamiltonian [5] and Newtonian [6, 7] relativistic mechanics for the interacting particle system in frames of the single-time formalism which is not manifestly Poincaré--invariant. Therefore, the question of the connection of the single-time formalism with the many-time one based on the action (1) arises. The investigation of the problem of the possibility to transform the Fokker-type action integral to single-time form

$$S = \int dt\, L, \tag{2}$$

seems to be the most natural way to analyse the above question.

The action integrals of the type (2) with a Lorentz-invariant "time" and Lagrangian L, depending on four-dimensional particle coordinates and velocities obeying some constraints, were considered in [8, 9]. However, it is more customary to operate with a variational principle in which these constraints are already taken into account and the action is a functional on the three-dimensional particle variables, taken (in each Lorentz reference frame) in the same coordinate time moment t. Problems concerning the construction of the relativistic theory for direct interactions, based on the action of this form were discussed in [10, 11]. The case considered there, in which the Lagrange function depends only on the three-dimensional coordinates and velocities, is applicable, as has been shown by recent investigations, no further than in the second approximation in c^{-2}. Assuming the dependence of the Lagrangian on higher derivatives (in exact theory to infinite order) expression (2) may be taken as a basis for Lagrangian formulation of the relativistic particle system dynamics which, in turn, makes it possible to move on to the Hamiltonian and Newtonian formulation. The aim of this paper is to formulate the rule of obtaining the single-time relativistic Lagrangians L and equations of motion corresponding to arbitrary functions Λ_{ab} in Eq. (1). These equations are, in the general case, ordinary differential equations of infinite order with respect to the single coordinate time t. The method of transition from expression (1) to (2), proposed here, is the generalization of the approximate approaches presented in [12–15]. The consideration of some special cases (e.g. electromagnetic interaction) will allow us to obtain, in the frame of the general approach, the results of other authors found for concrete interactions in different ways (including the employing of field theories). The symmetry properties of such a theory and corresponding conservation laws will be studied in the next work.

2. *Derivation of the single-time Lagrangian*

One of the possible ways to transform the expression (1) to form (2) was outlined in [12, 13, 16]. Choosing as parameters τ_a proper times of the particles and carrying out the transitions to integration over the time coordinates t_a by means of the relation

$$d\tau_a = dt_a/\gamma_a(t_a), \quad \gamma_a(t_a) \equiv [1-v_a^2(t_a)]^{-1/2},$$

where $\boldsymbol{v}_a(t_a) \equiv d\boldsymbol{x}_a(t_a)/dt_a$ are three-dimensional particles velocities, we transform the action (1) to the form

$$S = -\sum_a m_a \int dt_a \gamma_a^{-1}(t_a) - \sum_{a<b}\sum \iint dt_a dt_b \gamma_a^{-1}(t_a)$$

$$\times \gamma_b^{-1}(t_b) \Lambda_{ab}(x_a(t_a) - x_b(t_b), u_a(t_a), u_b(t_b)). \tag{3}$$

Since the limits of integration are the same we may here identify the integration variables in each term and therefore obtain

$$S = -\sum_a m_a \int dt \gamma_a^{-1}(t) - \sum_{a<b}\sum \iint dt_1 dt_2 \chi_{ab}(t_1, t_2), \tag{4}$$

where

$$\chi_{ab}(t_1, t_2) \equiv \gamma_a^{-1}(t_1)\gamma_b^{-1}(t_2)\Lambda_{ab}(x_a(t_1) - x_b(t_2), u_a(t_1), u_b(t_2)). \tag{5}$$

Changing the integration variables in the double sum of Eq. (4) [16]

$$t_1 = t - (1-\lambda)\theta, \qquad t_2 = t + \lambda\theta, \tag{6}$$

(λ is an arbitrary real number) we get, after moving on to three-dimensional denotations, the expression (2) with the Lagrangian

$$L \equiv L_f - U = -\sum_a m_a \gamma_a^{-1}(t) - \sum_{a<b}\sum \int d\theta \chi_{ab}(\theta, \boldsymbol{x}_a(t_1) - \boldsymbol{x}_b(t_2), \boldsymbol{v}_a(t_1), \boldsymbol{v}_b(t_2)), \tag{7}$$

where t_1 and t_2 are shortened denotations for expressions (6) rather than integration variables. The Lagrangian (7) contains three-dimensional particle coordinates and velocities with shifted time arguments. The other change of variables which was more complicated than (6) was proposed in [14, 15].

The obtained form of the Lagrangian was considered in [16] as final and in [12, 13–15] the shift of the time arguments was eliminated by some approximate approach. To eliminate that shift in a general case we write the function U from Eq. (7) in the form

$$U = \sum_{a<b}\sum \int d\theta \exp[(\lambda-1)\theta D_a] \exp(\lambda\theta D_b) \chi_{ab}(\theta, \boldsymbol{r}_{ab}, \boldsymbol{v}_a, \boldsymbol{v}_b), \tag{8}$$

where $\boldsymbol{r}_{ab} \equiv \boldsymbol{x}_a - \boldsymbol{x}_b$,

$$D_a \equiv \sum_{s=0}^{\infty} x_{aj}^{(s+1)} \frac{\partial}{\partial x_{aj}^{(s)}}, \qquad x_{aj}^{(s)} \equiv \frac{d^s x_{aj}}{dt^s} \equiv D^s x_{aj}, \qquad j = 1, 2, 3,$$

and a lack of an explicitly written time argument here and henceforth indicates that it is equal to t. Taking into account the commutativity of the operators D_a and D_b we have

$$U = \sum_{a<b}\sum \int d\theta \exp[\theta(\lambda D - D_a)] \chi_{ab}(\theta, \boldsymbol{r}_{ab}, \boldsymbol{v}_a, \boldsymbol{v}_b), \tag{9}$$

or, expanding the exponents in the power series

$$U = \sum_{a<b}\sum \sum_{s=0}^{\infty} (s!)^{-1}(\lambda D - D_a)^s \int d\theta \theta^s \chi_{ab}(\theta, \boldsymbol{r}_{ab}, \boldsymbol{v}_a, \boldsymbol{v}_b). \tag{10}$$

In Eq. (9) the equality $(D_a + D_b)\chi_{ab} = D\chi_{ab}$ was used, where D is the operator of the total time derivative:

$$D = \partial/\partial t + \sum_a D_a.$$

It is Eq. (10) that gives the needed Lagrangian depending on particle coordinates and their derivatives to infinite order taken in a single time moment t. It is sufficient for existence of the integral in (10) that for an arbitrary nonnegative integer s the equality

$$\lim_{\theta \to \pm\infty} \theta^s \chi_{ab} = 0 \tag{11}$$

should be satisfied.

The arbitrary number λ entering Eq. (10) as a factor before the operator D does not affect the observable characteristics of the system because the terms including the total time derivative in the Lagrange function are not essential.

We note that, so far, among the symmetry properties of the function Λ_{ab} only its translational invariance was used.

3. *Equations of motion*

The equations of motion corresponding to Lagrangians of the form (10) including derivatives to an infinite order are

$$\mathscr{L}_{ai} L \equiv \sum_{s=0}^{\infty} (-D)^s \frac{\partial L}{\partial x_{ai}^{(s)}} = 0, \qquad i = 1, 2, 3. \tag{12}$$

To calculate the derivative in Eq. (12) it is convenient to go back to the expression (7) and employ the method given in [16]. We have directly from (7)

$$\frac{\partial U}{\partial x_{ai}^{(s)}} = \int d\theta \left\{ \sum_{b(>a)} \left[\frac{\partial \chi_{ab}(t_1, t_2)}{\partial x_{aj}(t_1)} \frac{\partial x_{aj}(t_1)}{\partial x_{ai}^{(s)}} + \frac{\partial \chi_{ab}(t_1, t_2)}{\partial v_{aj}(t_1)} \frac{\partial v_{aj}(t_1)}{\partial x_{ai}^{(s)}} \right] \right.$$

$$\left. + \sum_{b(<a)} \left[\frac{\partial \chi_{ba}(t_1, t_2)}{\partial x_{aj}(t_2)} \frac{\partial x_{aj}(t_2)}{\partial x_{ai}^{(s)}} + \frac{\partial \chi_{ba}(t_1, t_2)}{\partial v_{aj}(t_2)} \frac{\partial v_{aj}(t_2)}{\partial x_{ai}^{(s)}} \right] \right\}.$$

Taking into account the Taylor expansions of $x_{aj}(t_1)$ and $x_{aj}(t_2)$ in the neighbourhood of t and eliminating the shift of the time argument we find:

$$\frac{\partial U}{\partial x_{ai}^{(s)}} = \int d\theta \exp(\theta \lambda D) \left\{ \sum_{b(>a)} \exp(-\theta D_a) \left[(\lambda - 1)^s \theta^s \frac{1}{s!} \frac{\partial \chi_{ab}}{\partial x_{ai}} \right. \right.$$

$$+(\lambda-1)^{s-1}\theta^{s-1}\frac{1}{(s-1)!}\frac{\partial\chi_{ab}}{\partial v_{ai}}(1-\delta_{0,s})\Bigg]+\sum_{b(<a)}\exp(-\theta D_b)$$

$$\times\left[\lambda^s\theta^s\frac{1}{s!}\frac{\partial\chi_{ba}}{\partial x_{ai}}+\lambda^{s-1}\theta^{s-1}\frac{1}{(s-1)!}\frac{\partial\chi_{ba}}{\partial v_{ai}}(1-\delta_{0,s})\right]\Bigg\}. \tag{13}$$

Substituting the total Lagrangian (7) in Eq. (12) and using Eq. (13) we shall have, after the formal summing of the series, the following equations of motion:

$$Dm_a\gamma_a v_{ai} = -\int d\theta\{\sum_{b(>a)}\exp(\theta D_b)\mathscr{L}_{ai}\chi_{ab}+\sum_{b(<a)}\exp(-\theta D_b)\mathscr{L}_{ai}\chi_{ba}\}$$

$$\equiv -\sum_{s=0}^{\infty}(s!)^{-1}\int d\theta\theta^s\{\sum_{b(>a)}D_b^s\mathscr{L}_{ai}\chi_{ab}+\sum_{b(<a)}(-D_b)^s\mathscr{L}_{ai}\chi_{ba}\}. \tag{14}$$

For symmetric interactions ($\chi_{ab} = \chi_{ba}$) they may be written in the simpler form

$$Dm_a\gamma_a v_{ai} = -\sum_{b(\neq a)}\int d\theta\exp(\eta_{ab}\theta D_b)\mathscr{L}_{ai}\chi_{ab}$$

$$\equiv -\sum_{b(\neq a)}\sum_{s=0}^{\infty}(s!)^{-1}\int d\theta(\eta_{ab}\theta D_b)^s\mathscr{L}_{ai}\chi_{ab}, \tag{15}$$

where $\eta_{ab} \equiv \operatorname{sgn}(b-a)$.

As it is seen from the expressions (14)–(15) they do not include the number λ. They form a set of ordinary differential equations of infinite order replacing the integral-differential (or differential-difference) equations following from the action (1). The connection between these two forms of the equations of motion was discussed qualitatively in [17]

The selection of the physically meaningful solutions of such equations constitutes a separate problem. As it was pointed out in [16, 18, 19], as a criterion for such a selection the stability of solutions may be used, i.e. the analytical character of a solution's dependence on small parameters entering into (14)–(15): coupling constants or c^{-1}, respectively. For such solutions the second order equations of motion can be derived by means of successively excluding on the right-hand side of Eqs (14)–(15) all derivatives of coordinates of order higher than the first, using equations of lower order in the chosen small parameter and supposing convergence for this procedure. Detailed discussion of this problem together with examples are intended for a separate work.

4. *The case of manifest covariance of the action* (1)

In most applications the functions Λ_{ab} entering the Fokker-type action (1) depend on their arguments through four-dimensional scalar products only (see Refs. [2–4]).

$$\tilde{\varrho}_{ab} \equiv (x_a-x_b)^2, \quad \tilde{\omega}_{ab} \equiv u_a\cdot u_b, \quad \tilde{\sigma}_{ab} \equiv \eta_{ab}(x_a-x_b)\cdot u_a, \tag{16}$$

i.e. they are manifestly Poincaré-invariant. Moving on to three-dimensional denotations the corresponding functions χ_{ab} entering Eq. (8) may be written, according to (5), in the form

$$\chi_{ab} = (1-\boldsymbol{v}_a\cdot\boldsymbol{v}_b)F_{ab}(\varrho_{ab}, \omega_{ab}, \sigma_{ab}, \sigma_{ba}), \tag{17}$$

where $F_{ab} \equiv \omega_{ab}^{-1}\Lambda_{ab}$ is a function of the following arguments:

$$\varrho_{ab} \equiv \theta^2 - r_{ab}^2, \qquad \omega_{ab} \equiv \gamma_a\gamma_b(1-\boldsymbol{v}_a\cdot\boldsymbol{v}_b), \qquad \sigma_{ab} \equiv -\gamma_a[\theta+\eta_{ab}\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a]. \tag{18}$$

Substitution of this expression for χ_{ab} in Eq. (14) gives the following equations of motion:

$$\begin{aligned} Dm_a\gamma_a\boldsymbol{v}_a = \int d\theta\Bigg\{\sum_{b(>a)} \exp(\theta D_b)\Bigg\{(1-\boldsymbol{v}_a\cdot\boldsymbol{v}_b)\left(2\boldsymbol{r}_{ab}\frac{\partial F_{ab}}{\partial\varrho_{ab}}\right. \\ \left.+\boldsymbol{v}_a\gamma_a\frac{\partial F_{ab}}{\partial\sigma_{ab}}+\boldsymbol{v}_b\gamma_b\frac{\partial F_{ab}}{\partial\sigma_{ba}}\right) - D\left[(\boldsymbol{v}_b-\boldsymbol{v}_a\gamma_a^2(1-\boldsymbol{v}_a\cdot\boldsymbol{v}_b))\left(F_{ab}+\omega_{ab}\frac{\partial F_{ab}}{\partial\omega_{ab}}\right)\right. \\ \left.+\boldsymbol{r}_{ab}\gamma_a(1-\boldsymbol{v}_a\cdot\boldsymbol{v}_b)\frac{\partial F_{ab}}{\partial\sigma_{ab}}\right]\Bigg\} + \sum_{b(<a)}\exp(-\theta D_b) \\ \times\Bigg\{(1-\boldsymbol{v}_a\cdot\boldsymbol{v}_b)\left(2\boldsymbol{r}_{ab}\frac{\partial F_{ba}}{\partial\varrho_{ab}} - \boldsymbol{v}_a\gamma_a\frac{\partial F_{ba}}{\partial\sigma_{ab}} - \boldsymbol{v}_b\gamma_b\frac{\partial F_{ba}}{\partial\sigma_{ba}}\right) \\ -D\left[(\boldsymbol{v}_b-\boldsymbol{v}_a\gamma_a^2(1-\boldsymbol{v}_a\cdot\boldsymbol{v}_b))\left(F_{ba}+\omega_{ab}\frac{\partial F_{ba}}{\partial\omega_{ab}}\right) - \boldsymbol{r}_{ab}\gamma_a(1-\boldsymbol{v}_a\cdot\boldsymbol{v}_b)\frac{\partial F_{ba}}{\partial\sigma_{ab}}\right]\Bigg\}\Bigg\}. \end{aligned} \tag{19}$$

For symmetric interactions with $F_{ab} = F_{ba}$ two sums may be united into one with the condition $b \neq a$ by introducing in the exponent a sign factor η_{ab} (cf. Eq. (25) below).

We will illustrate the results obtained above by an important example corresponding to interactions which in the framework of the classical field theory may be described through a tensor field of integer rank n. Then the function Λ_{ab} has the form (see Refs [4, 14])

$$\Lambda_{ab} = g_a g_b \tilde{\omega}_{ab}^n G(\tilde{\varrho}_{ab}), \tag{20}$$

g_a being coupling constants, $G(\varrho)$ being some (symmetrical, retarded or advanced) Green's function of the corresponding field equation. Then $F_{ab} = g_a g_b \omega_{ab}^{n-1} G(\varrho_{ab})$ and after substituting expression (17) into Eq. (10) and taking into account that that integral has symmetric limits we get the following expression for the interaction Lagrangian (the constant c is inserted for later convenience)

$$U = \sum\sum_{a<b} g_a g_b \sum_{s=0}^{\infty} \frac{(\lambda D - D_a)^{2s}}{c^{2s}(2s)!}\left(1-\frac{\boldsymbol{v}_a\cdot\boldsymbol{v}_b}{c^2}\right)\omega_{ab}^{n-1}W_s(r_{ab}). \tag{21}$$

Functions $W_s(r)$ introduced here are defined by the integrals

$$W_s(r) \equiv \int d\theta\,\theta^{2s} G(\theta^2 - r^2). \tag{22}$$

They exist if function G secures the fulfilment of the condition (11). Taking it into account we obtain from expression (22) the recurrence relations

$$\frac{1}{r}\frac{dW_s(r)}{dr} = (2s-1)W_{s-1}(r), \qquad s = 1, 2, \ldots . \tag{23}$$

It follows from consideration of the limit $c \to \infty$ in Eq. (21) that $W_0(r_{ab})$ is (up to the factor $g_a g_b$) the nonrelativistic interaction potential of the particles a and b.

Since functions (20) are symmetric in indices a and b the observable characteristics obtained from the Lagrangian (21) will also be symmetric at arbitrary λ. Nevertheless, it is more convenient to symmetrize the Lagrangian itself. This can be achieved in different ways, for instance, by putting in (21) $\lambda = 1/2$ or, otherwise, choosing $\lambda = 0$ and taking into account that the operator $D_a^{2s} = [(D-D_b)D_a]^s$ acting on the two-particle expressions differs from $(-D_aD_b)^s$ in terms including the total time derivative D [19]. In the last case we have

$$U = \sum_{a<b}\sum g_a g_b \sum_{s=0}^{\infty} \frac{(-D_aD_b)^s}{c^{2s}(2s)!}\left(1-\frac{\boldsymbol{v}_a\cdot\boldsymbol{v}_b}{c^2}\right)\omega_{ab}^{n-1}W_s(r_{ab}). \tag{24}$$

Equations of motion corresponding to the Lagrangian above are obtained by means of substituting the foregoing functions F_{ab} into Eq. (19). Because of the symmetry of F_{ab} with respect to the indices a and b they have the form

$$\begin{aligned} Dm_a\gamma_a\boldsymbol{v}_a = g_a \sum_{b(\neq a)} g_b \int d\theta \exp(\eta_{ab}\theta c^{-1}D_b)\,\{2\boldsymbol{r}_{ab}(1-\boldsymbol{v}_a\cdot\boldsymbol{v}_b c^{-2})\omega_{ab}^{n-1}G'(\varrho_{ab}) \\ -c^{-2}D[(n\boldsymbol{v}_b+(1-n)\boldsymbol{v}_a\gamma_a^2(1-\boldsymbol{v}_a\cdot\boldsymbol{v}_b c^{-2}))\omega_{ab}^{n-1}G(\varrho_{ab})]\}, \end{aligned} \tag{25}$$

where the prime on function G denotes the derivative with respect to its argument.

When the exponent in (25) is expanded in power series, the terms with odd powers of θ vanish and therefore the sign factor η_{ab} may be omitted. Performing the expansion and using definition (22) for the function W_s we obtain

$$\begin{aligned} Dm_a\gamma_a\boldsymbol{v}_a = -g_a \sum_{b(\neq a)} g_b \sum_{s=0}^{\infty} \frac{D_b^{2s}}{c^{2s}(2s)!}\left\{\boldsymbol{r}_{ab}\left(1-\frac{\boldsymbol{v}_a\cdot\boldsymbol{v}_b}{c^2}\right)\omega_{ab}^{n-1}\frac{1}{r_{ab}}\frac{dW_s(r_{ab})}{dr_{ab}}\right. \\ \left. +\frac{1}{c^2}D\left[n\boldsymbol{v}_b+(1-n)\boldsymbol{v}_a\gamma_a^2\left(1-\frac{\boldsymbol{v}_a\cdot\boldsymbol{v}_b}{c^2}\right)\right]\omega_{ab}^{n-1}W_s(r_{ab})\right\}. \end{aligned} \tag{26}$$

Another form of the equations of motion obtained from the set of three equations (26) through multiplying by the matrix $\gamma_a^{-1}\times(\delta_{ij}-v_{ai}v_{aj}/c^2)$ will be useful below:

$$m_a\dot{\boldsymbol{v}}_a = g_a \sum_{b(\neq a)} g_b \sum_{s=0}^{\infty} \frac{D_b^{2s}}{c^{2s}(2s)!}\gamma_a^{-1}\omega_{ab}^{n-1}\left\{\left[n\frac{\boldsymbol{v}_{ab}}{c^2}\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_{ab}\right.\right.$$

$$+\left(\frac{v_a}{c^2}\,r_{ab}\cdot v_b-r_{ab}\right)\left(1-\frac{v_a\cdot v_b}{c^2}\right)\Bigg]\frac{1}{r_{ab}}\frac{dW_s(r_{ab})}{dr_{ab}}+\frac{1}{c^2}\,[(n-1)\dot{v}_a\gamma_a^2$$

$$\times(1-v_a\cdot v_b c^{-2})-n\dot{v}_b+n(n-1)v_{ab}c^{-2}(\gamma_a^2 v_a\cdot\dot{v}_a+\gamma_b^2 v_b\cdot\dot{v}_b$$

$$-(1-v_a\cdot v_b c^{-2})^{-1}(v_b\cdot\dot{v}_a+v_a\cdot\dot{v}_b))+v_a c^{-2}(n v_a\cdot\dot{v}_b+(1-n)v_b\cdot\dot{v}_b$$

$$\times\gamma_a^2(1-v_a\cdot v_b c^{-2}))]W_s(r_{ab})\}, \qquad v_{ab}\equiv v_a-v_b, \qquad \dot{v}_a\equiv Dv_a. \tag{27}$$

5. *Application to electrodynamics*

In the case of the electromagnetic interaction transferred through a massless vector field, in expression (20) $n = 1$, G is Green's function of the wave equation and $g_a = e_a$ are the particle charges.

In Wheeler–Feynman's electrodynamics [1] the symmetric Green's function $G^{\mathrm{sym}}(\varrho) = \delta(\varrho)$ is chosen. Then from Eq. (22)

$$W_s(r) = \int d\theta\theta^{2s}\delta(\theta^2-r^2) = r^{2s-1} \tag{28}$$

and the substitution of these expressions (and $n = 1$) into Eq. (24) gives Kerner's Lagrangian

$$U_e = \sum_{a<b}\sum e_a e_b \sum_{s=0}^{\infty}[(2s)!]^{-1}(-D_a D_b)^s(1-v_a\cdot v_b)r_{ab}^{2s-1} \tag{29}$$

obtained in [19] for the system of two particles by means of a series expansion of the half-sum of the retarded and advanced Lienard–Wiechert potentials.

The equations of motion (25) in this case have the following form

$$Dm_a\gamma_a v_a = e_a\sum_{b(\neq a)} e_b\int d\theta\exp(\theta D_b)\{2r_{ab}(1-v_a\cdot v_b)\delta'(\varrho_{ab})-Dv_b\delta(\varrho_{ab})\}. \tag{30}$$

After some transformations they get into the standard form with the Lorentz force

$$Dm_a\gamma_a v_a = e_a\sum_{b(\neq a)}\{E_b^{\mathrm{sym}}(x_a)+v_a\times H_b^{\mathrm{sym}}(x_a)\}, \tag{31}$$

where $E_a^{\mathrm{sym}}(x)$, $H_a^{\mathrm{sym}}(x)$ are vectors of electric and magnetic fields due to the particle a at the point x.:

$$E_a^{\mathrm{sym}}(x) = 2e_a\int d\theta\exp(\theta D_a)(R_a+\theta v_a)\delta'(\theta^2-R_a^2),$$

$$H_a^{\mathrm{sym}}(x) = 2e_a\int d\theta\exp(\theta D_a)v_a\times R_a\delta'(\theta^2-R_a^2), \qquad R_a\equiv x-x_a. \tag{32}$$

These fields may be obtained in the standard way from scallar and vector potentials

$$\varphi_a^{\mathrm{sym}}(x) = e_a\int d\theta\exp(\theta D_a)\delta(\theta^2-R_a^2),$$

$$A_a^{\mathrm{sym}}(x) = e_a\int d\theta\exp(\theta D_a)v_a\delta(\theta^2-R_a^2), \tag{33}$$

which are, as one may be easily convinced through the direct calculations of integrals, the half-sum of the retarded and advanced Lienard-Wiechert's potentials. Equations (31) agree both with general conclusions of the work [1] and with the results of [17] where the concept of Fokker-type action was not used explicitly.

Let us consider now the case of electromagnetic interaction described in expression (20) by the retarded Green's function (see [2, 8]) $G^{\text{ret}} = 2\Theta(x_b^0 - x_a^0)\delta(\tilde{\varrho}_{ab})$ ($\Theta(x)$ is Heaviside's function). Although this function depends on particle four-coordinates not only through the invariant $\tilde{\varrho}_{ab}$ it is, as it is known, Poincaré-invariant. In our three-dimensional denotations it is equal to $2\Theta(\theta)\delta(\varrho_{ab})$ and the corresponding interaction Lagrangian has the form

$$U = \sum_{a<b}\sum e_a e_b \sum_{s=0}^{\infty} (s!)^{-1}(\lambda D - D_a)^s (1 - \boldsymbol{v}_a \cdot \boldsymbol{v}_b) r_{ab}^{s-1}. \tag{34}$$

It should be noted that unlike Eq. (29) it contains both even and odd powers of the velocities $\boldsymbol{v}_a$. It can be symmetrized by taking $\lambda = 1/2$ only. The equations of motion for this case differ from Eqs (30) by the factor 2 by the presence in the exponent of η_{ab} and by the fact that integration is performed from 0 to ∞. They can also be reduced to the form (31) with the Lorentz force, however, the common formula for the fields due to an arbitrary particle in a given space point can no longer be written. These fields are determined by the potentials

$$\varphi_b(\boldsymbol{x}_a) = 2e_b \int_0^{\infty} d\theta \exp(\eta_{ab}\theta D_b)\delta(\theta^2 - r_{ab}^2), \tag{35}$$

$$\boldsymbol{A}_b(\boldsymbol{x}_a) = 2e_b \int_0^{\infty} d\theta \exp(\eta_{ab}\theta D_b)\boldsymbol{v}_b\delta(\theta^2 - r_{ab}^2),$$

which in the case of $b > a (\eta_{ab} = 1)$ are purely advanced

$$\varphi_b^{\text{adv}}(\boldsymbol{x}) = 2e_b \int_0^{\infty} d\theta \exp(\theta D_b)\delta(\theta^2 - R_b^2) \equiv \sum_{s=0}^{\infty} (s!)^{-1} D_b^s R_b^{s-1},$$

$$\boldsymbol{A}_b^{\text{adv}}(\boldsymbol{x}) = 2e_b \int_0^{\infty} d\theta \exp(\theta D_b)\boldsymbol{v}_b\delta(\theta^2 - R_b^2) \equiv \sum_{s=0}^{\infty} (s!)^{-1} D_b^s \boldsymbol{v}_b R_b^{s-1}, \tag{36}$$

and in the case of $b < a (\eta_{ab} = -1)$ are purely retarded:

$$\varphi_b^{\text{ret}}(\boldsymbol{x}) = 2e_b \int_0^{\infty} d\theta \exp(-\theta D_b)\delta(\theta^2 - R_b^2) \equiv \sum_{s=0}^{\infty} (s!)^{-1}(-D_b)^s R_b^{s-1},$$

$$\boldsymbol{A}_b^{\text{ret}}(\boldsymbol{x}) = 2e_b \int_0^{\infty} d\theta \exp(-\theta D_b)\boldsymbol{v}_b\delta(\theta^2 - R_b^2) \equiv \sum_{s=0}^{\infty} (s!)^{-1}(-D_b)^s \boldsymbol{v}_b R_b^{s-1}. \tag{37}$$

Thus the indicated choice of Green's function corresponds to the model in which each pair of particles a and b (suppose for definiteness that $a < b$) interacts in the following way: the advanced field of particle a acts on particle b and the retarded field of particle b acts on particle a. Such models for a two-particle system were considered in [8, 20]

Naturally, if in expression (20) one takes the advanced Green's function $G^{\mathrm{adv}} = 2\Theta(x_a^0 - x_b^0)\delta(\varrho_{ab})$ instead of the retarded one, then particles a and b ($a < b$) will change their roles.

6. Expansion in c^{-1}

The structure of the expressions found in the present work, which have the form of a power series, enables us to obtain the expansions in powers of c^{-1}, a few terms of which may be used for the description of weakly relativistic systems.

Let us consider here the case of the tensor interaction (20), for which all series contain only even powers of c^{-1}. For the interaction Lagrangian U we have from (24) up to the terms of order c^{-4}

$$
\begin{aligned}
U = \sum\sum_{a<b} g_a g_b \Big\{ & W_0(r_{ab}) + \frac{1}{2c^2}\Big\{[-\boldsymbol{v}_a\cdot\boldsymbol{v}_b + (n-1)v_{ab}^2] W_0(r_{ab}) \\
& + (\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a)(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b)\frac{1}{r_{ab}}\frac{dW_0(r_{ab})}{dr_{ab}}\Big\} + \frac{1}{8c^4}\Big\{(4n-1)\dot{\boldsymbol{v}}_a\cdot\dot{\boldsymbol{v}}_b W_1(r_{ab}) \\
& + [(2n^2-4n+3)v_a^2 v_b^2 + 2(2n^2-4n+1)(\boldsymbol{v}_a\cdot\boldsymbol{v}_b)^2 + (n^2-1)(v_a^4+v_b^4) \\
& + 2(n-1)(1-2n)(\boldsymbol{v}_a\cdot\boldsymbol{v}_b)(v_a^2+v_b^2) + 2(2n-1)(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a)(\boldsymbol{v}_a\cdot\boldsymbol{v}_b) \\
& - 4(n-1)(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a)(\dot{\boldsymbol{v}}_b\cdot\dot{\boldsymbol{v}}_b) + 4(n-1)(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b)(\boldsymbol{v}_a\cdot\dot{\boldsymbol{v}}_a) - 2(2n-1)(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b)(\boldsymbol{v}_b\cdot\dot{\boldsymbol{v}}_a) \\
& - v_a^2(\boldsymbol{r}_{ab}\cdot\dot{\boldsymbol{v}}_b) + v_b^2(\boldsymbol{r}_{ab}\cdot\dot{\boldsymbol{v}}_a) - (\boldsymbol{r}_{ab}\cdot\dot{\boldsymbol{v}}_a)(\boldsymbol{r}_{ab}\cdot\dot{\boldsymbol{v}}_b)] W_0(r_{ab}) \\
& + [2(n-1)v_{ab}^2(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a)(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b) + (v_a^2 + \boldsymbol{r}_{ab}\cdot\dot{\boldsymbol{v}}_a)(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b)^2 \\
& + (v_b^2 - \boldsymbol{r}_{ab}\cdot\dot{\boldsymbol{v}}_b)(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a)^2]\frac{1}{r_{ab}}\frac{dW_0(r_{ab})}{dr_{ab}} \\
& + (\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a)^2(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b)^2\left(\frac{1}{r_{ab}}\frac{d}{dr_{ab}}\right)^2 W_0(r_{ab})\Big\} + O(c^{-6})\Big\}. \qquad (38)
\end{aligned}
$$

Here the dot denotes the derivative with respect to t.

The terms of order c^{-2} written above were found by many authors in different ways. (The method of the works [12, 14] is similar to that used here). These terms give Bopp's and Bagge's Lagrangians, respectively, for scalar and vector interactions if $n = 0,1$. The latter, after the substitution of Eq. (28), gives Darwin's Lagrangian for electromagnetic interaction and the linear combination

$$U_{\mathrm{gr.}} \equiv 2U|_{n=0} - U|_{n=2} \qquad (39)$$

gives the linear terms of the Einstein-Infeld-Hoffmann Lagrangian for gravitational interaction (more detailed discussion of similar correspondence problems for terms of the order c^{-2} is given in [14, 21]).

The terms of order c^{-4} for arbitrary n are written here, as far as we know, for the first time. They give the second order corrections for the interaction Lagrangians mentioned above. For electrodynamics, putting $n = 1$ and taking $W_s(r)$ from (28), we obtain the expression coinciding (if all charges $e_a = e$) with the Lagrangian found in [22] from classical electrodynamics. Finally, the linear combination (39) is in agreement with the linear approximation of Lagrangians of the order c^{-4} for the gravitational interactions proposed in [23, 24].

The equations of motion (27) up to terms of the order c^{-4} may be reduced by recurrence relations (23) to the following form:

$$m_a\dot{\boldsymbol{v}}_a = g_a \sum_{b(\neq a)} g_b \left\{ -\frac{\boldsymbol{r}_{ab}}{r_{ab}} \frac{dW_0(r_{ab})}{dr_{ab}} + \frac{1}{2c^2}\left\{ [(1-2n)\dot{\boldsymbol{v}}_b + 2(n-1)\dot{\boldsymbol{v}}_a] W_0(r_{ab}) \right.\right.$$

$$+ [2\boldsymbol{v}_{ab}(n\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a + (1-n)\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b) + \boldsymbol{r}_{ab}((2-n)v_a^2 - nv_b^2 + 2n\boldsymbol{v}_a\cdot\boldsymbol{v}_b + \boldsymbol{r}_{ab}\cdot\dot{\boldsymbol{v}}_b]\frac{1}{r_{an}}\frac{dW_0(r_{ab})}{dr_{ab}}$$

$$\left. - \boldsymbol{r}_{ab}(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b)^2 \left(\frac{1}{r_{ab}}\frac{d}{dr_{ab}}\right)^2 W_0(r_{ab})\right\} + \frac{1-4n}{8c^4}\dddot{\boldsymbol{v}}_b W_1(r_{ab}) + \frac{1}{8c^4}\{4\ddot{\boldsymbol{v}}_b[-n\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a$$

$$+ (3n-1)\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b] + 4(n-1)\dot{\boldsymbol{v}}_a(nv_{ab}^2 - \boldsymbol{r}_{ab}\cdot\dot{\boldsymbol{v}}_b) + 2(2n-1)\dot{\boldsymbol{v}}_b[(2-n)v_a^2 - (n+2)v_b^2$$

$$+ 2n\boldsymbol{v}_a\cdot\boldsymbol{v}_b + 3\boldsymbol{r}_{ab}\cdot\dot{\boldsymbol{v}}_b] + 4\boldsymbol{v}_{ab}[2n(n-1)\boldsymbol{v}_{ab}\cdot\dot{\boldsymbol{v}}_a + n(3-2n)\boldsymbol{v}_a\cdot\dot{\boldsymbol{v}}_b$$

$$+ (n-1)(2n+1)\boldsymbol{v}_b\cdot\dot{\boldsymbol{v}}_b - (n-1)\boldsymbol{r}_{ab}\cdot\ddot{\boldsymbol{v}}_b] + \boldsymbol{r}_{ab}[(1-4n)\dot{\boldsymbol{v}}_b^2 + 4n\boldsymbol{v}_{ab}\cdot\ddot{\boldsymbol{v}}_b$$

$$+ \boldsymbol{r}_{ab}\cdot\dddot{\boldsymbol{v}}_b]\} W_0(r_{ab}) + \frac{1}{8c^4}\{4(n-1)\dot{\boldsymbol{v}}_a(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b)^2 + 2\dot{\boldsymbol{v}}_b(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b)[4n\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a$$

$$+ 3(1-2n)\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b] + 4n\boldsymbol{v}_{ab}(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a)[(n-2)v_a^2 + nv_b^2 + 2(1-n)\boldsymbol{v}_a\cdot\boldsymbol{v}_b - \boldsymbol{r}_{ab}\cdot\dot{\boldsymbol{v}}_b]$$

$$+ 4(n-1)\boldsymbol{v}_{ab}(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b)[(2-n)v_a^2 - (n+2)v_b^2 + 2n\boldsymbol{v}_a\cdot\boldsymbol{v}_b + 3\boldsymbol{r}_{ab}\cdot\dot{\boldsymbol{v}}_b]$$

$$+ \boldsymbol{r}_{ab}[2n(2-n)v_a^2v_b^2 - 4n(n-1)(\boldsymbol{v}_a\cdot\boldsymbol{v}_b)^2 + 4n(n-2)(\boldsymbol{v}_a\cdot\boldsymbol{v}_b)v_a^2 + 4n^2(\boldsymbol{v}_a\cdot\boldsymbol{v}_b)v_b^2$$

$$+ n(2-n)v_a^4 - n(n+2)v_b^4 + 4(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b)(-2n\boldsymbol{v}_a\cdot\dot{\boldsymbol{v}}_b + (2n+1)\boldsymbol{v}_b\cdot\dot{\boldsymbol{v}}_b - \boldsymbol{r}_{ab}\cdot\ddot{\boldsymbol{v}}_b)$$

$$+ (\boldsymbol{r}_{ab}\cdot\dot{\boldsymbol{v}}_b)(2(n-2)v_a^2 + 2(n+2)v_b^2 - 4n\boldsymbol{v}_a\cdot\boldsymbol{v}_b - 3\boldsymbol{r}_{ab}\cdot\dot{\boldsymbol{v}}_b)]\}\frac{1}{r_{ab}}\frac{dW_0(r_{ab})}{dr_{ab}}$$

$$+ \frac{1}{8c^4}\{4\boldsymbol{v}_{ab}[n\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a + (1-n)\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b] + 2\boldsymbol{r}_{ab}[(3-n)v_a^2 - (n+2)v_b^2$$

$$+ 2n\boldsymbol{v}_a\cdot\boldsymbol{v}_b + 3\boldsymbol{r}_{ab}\cdot\dot{\boldsymbol{v}}_b]\}(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b)^2\left(\frac{1}{r_{ab}}\frac{d}{dr_{ab}}\right)^2 W_0(r_{ab})$$

$$\left. - \frac{\boldsymbol{r}_{ab}}{8c^4}(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b)^4\left(\frac{1}{r_{ab}}\frac{d}{dr_{ab}}\right)^3 W_0(r_{ab}) + O(c^{-6})\right\}. \tag{40}$$

Naturally, equations (40) can be obtained directly from the Lagrangian (38). They form the set of ordinary differential equations of the fourth order. Selecting from the complete set of solutions only those which are analytical in c^{-1} one can obtain the second order Newton-type equations of motion by excluding on the right-hand side of (40) all derivatives of order higher than one by means of the lower order equations of motion. We shall indicate only some distinctive features of the expressions obtained in that way without writing them explicitly, namely: 1) They satisfy the Poincaré-invariance conditions of motion (Currie-Hill conditions) [6, 7] up to terms of the order c^{-4}. 2) Excluding higher derivatives on the right-hand side of Eq. (40) one obtains double and triple sums, besides the single ones, which agree with the general conclusion as to the necessity of many-particle interactions in the relativistic action-at-a distance theory on the level of Newton-type equations of motion. 3) For values $n = 0,1$ these expressions are in agreement with equations of motion obtained in various approximations for scalar [25] and electromagnetic [26] interactions in the framework of four-dimensional manifestly covariant formalism of predictive relativistic mechanics.

It should be noted finally that the equations under consideration cannot be obtained by excluding higher derivatives directly in the Lagrangian (38) and employing to it after that the Euler–Lagrange operator (12), as was proposed, for instance, in [22]. This is connected with the fact that the Lagrange function is determined not only on the real particle trajectories, but also on virtual ones for which the equations of motion must not necessarily be satisfied.

7. Conclusions

The obtained single-time representation of the Fokker-type action in which the Lagrangian of the system depends on the higher derivatives up to infinite order opens new possibilities for the investigation of the direct interacting particle systems. The introduction of the unique time parameter allows us to bring the theory based on the Fokker-type action closer to the form of the classical nonrelativistic mechanics and to establish its relationship with other approaches to the theory of direct interactions.

It should be noted that although the limits of integration in Eq. (2) are formally infinite, the corresponding equations of motion will be valid if these limits are finite. This constitutes one of the main differences of our approach from the many-time one based on action (1), in which to obtain the correct equations of motion it is necessary to consider the double integral limits to be infinite, as a consequence of that, however, its mathematical correctness becomes problematic [8, 27]. Thus, from the point of view of its formal structure the relativistic action (2) obtained by us differs from its nonrelativistic analogue only by the dependence of the Lagrangian on higher derivatives reflecting the finite interaction velocity. We have found the general form of such a Lagrangian (10), corresponding to the arbitrary interaction (1) and the Lagrangian, corresponding to manifestly Poincaré-invariant many-time action (see Eq. (17)). These results may be regarded as a fairly general solution of the problem of the Lagrangian relativistic single-time description of the directly interacting particle system formulated in [10, 11] as well as a proof of the relationship of the latter with the Fokker formalism.

It seems to us that there exists one more reason why employing the Fokker-type integrals for finding relativistic particle Lagrangians is important, namely, as we have seen in Sections 4 and 5, it is possible in this way to establish the relationship of the theory based on the concept of direct action-at-a-distance with the field description. In fact, as was indicated in [4, 14], integrals (1) for tensor interaction (20) have quite definite field theoretical analogs.

Knowledge of the Lagrangian makes it possible to obtain the complete description of a system (at least classical) by means of more or less standard methods. In the present work the single-time equations of motion (14), (15), (19) are given which may be transformed to Newton's form using some selection criteria [18]. In that way the relationship between the Fokker-type action formalism and Currie-Hill's approach is outlined as well as the four-dimensional manifestly covariant formalism of the predictive relativistic mechanics.

The expressions found in our work may be used as a basis for various approximate approaches. Besides the expansion in c^{-2} for tensor interactions considered in the previous section, one may investigate the expansions in the coupling constant (the first approximation is considered in [13]) or in particle masses ratio.

The question of the symmetry properties of the obtained single-time Lagrangian description is very important. Although the Poincaré-invariance of the proposed formalism following from the manifestly invariant Fokker-type action, is undoubtful, it is desirable to perform the independent investigation of the invariance of the equation of motion (19) with respect to the Poincaré group representation in the particle system configuration space (more precisely in a certain continuation of it), which, specifically, makes it possible to find the explicit formulae for the corresponding single-time integrals of motion. This problem will be solved in the following work.

Further, one may hope to obtain a self-consistent quantum description of the directly interacting particle system by means of hamiltonization of the theory and its subsequent quantization.

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