SYMMETRIES AND CONSERVATION LAWS IN THE SINGLE--TIME LAGRANGIAN FORM OF THE FOKKER-TYPE RELATIVISTIC DYNAMICS. II

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Symmetry properties of the single-time relativistic Lagrangian of an N-particle-system corresponding to the many-time action of the Fokker-type, which are a function of derivatives of particle coordinates with respect to time up to infinite order, are investigated. The conditions for quasi-invariance for such a Lagrangian, with respect to a representation of an arbitrary group in infinite continuation of configuration space of the system, are discussed. Using these conditions a general expression for the Lagrangian, securing Poincaré covariance of corresponding equations of motion, is found, and the conservation laws related to this covariance are formulated. In the case of tensor interaction, the expansion of conserved quantities in c^{-1} up to terms of the order c^{-4} is performed.

1. *Introduction*

The authors' previous work [1] shows that an arbitrary Fokker-type action integral may be put in correspondence with an action in the form of a single integral over the coordinate time t of a relativistic Lagrangian function L. The closed expression was found for the latter, which is an infinite series including derivatives of the particle coordinates up to the infinite order. The agreement of this approach with known results, obtained in different ways, was seen from some special cases considered.

However, the transition to the single-time formalism, allowing one to regard the ordinary differential equations as equations of motion (in general case of infinite order) was performed at the cost of loss of the manifest Poincaré invariance, inherent to many--time Fokker-type formalism. Therefore, in spite of the expected relativistic covariance of description of a particle system constructed in such a way, its symmetry properties should be formulated in terms of the group-theoretical analysis and should be studied irrespective of the invariance of the initial many-time expressions. The necessity of such

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analysis is conditioned also by the fact that the conservation laws obtained in the framework of the Fokker-type action approach [2, 3] are expressed in the form which is essentially distinct from one typical for the single-time theory. To derive the motion integrals in the latter, it is more convenient to start from the invariance of the single-time action and corresponding equations of motion rather than from the conservation laws in many-time formalism.

The purpose of the present paper is to prove the relativistic invariance of Lagrangian dynamics based on the single-time representation of the manifestly relativistic invariant Fokker-type action integral, as well as to find the corresponding integrals of motion. This invariance, as will be shown in Sections 3 and 4, consists in the covariance of equations of motion with respect to the representation of the Poincaré group in space where the Lagrangian L is defined. Before regarding these special questions constituting the main subject of the paper, we shall give in Section 2 the most significant results of the symmetry theory of particle system dynamics for the general case of arbitrary single-time Lagrangians depending on derivatives of arbitrarily high order. (A more detailed discussion of these questions together with all proofs will be published in *Teor. Mat. Fiz.*). By means of these results in Section 3 the quasi-invariance conditions of such Lagrangians for any r-parametric group $\mathscr{G}_r$ are written, and also the formulae for the corresponding motion integrals. The following sections contain the discussion of the Poincaré invariance, formulae for energy, linear momentum, angular momentum and the integral of the center-of-mass of the particle system as well as the investigation of the special case of the tensor interaction.

Inasmuch as this paper is a direct continuation of [1] we shall use many definitions and notations introduced there without explanation. References to the formulae of [1] will be marked by I before the formula number.

2. *Symmetries and conservation laws in the single-time Lagrangian formalism with higher derivatives*

As was shown in [1], the single-time relativistic interaction Lagrangian, corresponding to a given action integral of the Fokker-type (formula (I. 10)), depends on the particle coordinates and their derivatives up to infinite order, taken at the same time moment t. Analytical mechanics for Lagrangians including higher derivatives to some finite order n has been constructed by Ostrogradsky as far back as 1848 [4] and thereafter it was rediscovered and generalized by many authors. Many of its results may be generalized for the case $n \to \infty$ (see Ref. [5]). For investigation of the symmetry properties of such a theory under some r-parametric group $\mathscr{G}_r$ one needs to have a representation of that group in the space $\boldsymbol{E}_\infty^{3N}$ of the particle coordinates $x_a^i(t)$ and their derivatives $x_a^{i(s)}(t)$ up to infinite order[1]

$$\boldsymbol{E}_\infty^{3N} = \{x_a^{i(s)}(t) \,|\, a = 1, \dots, N;\ i = 1, 2, 3;\ s = 0, 1, \dots;\ t \in \boldsymbol{R}\}.$$

[1] Here and henceforth the three-vector subscripts and superscripts $i, j, k = 1, 2, 3$ are completely equivalent. Summation over repeated indices is meant.

The general tangent transformation of the space E_∞^{3N} into itself (called Lie–Bäcklund transformation [6, 7]) in the infinitesimal neighbourhood of the identity transformation may be written as follows

$$x_a^{i(s)\prime} = x_a^{i(s)} + \sum_{\alpha=1}^{r} \delta\lambda^\alpha D^s \zeta_{\alpha a}^i(z), \qquad z \in E_\infty^{3N}, \tag{1}$$

where $\delta\lambda^\alpha$ are infinitesimal group parameters, D is an operator of the total time derivative. A more general case of Lie–Bäcklund transformations, when the independent variable t is also transformed, can be reduced to Eq. (1) (see for instance [8]) and, therefore, it will not be considered. In order transformations (1) should determine the representation of the group $\mathcal{G}_r$ in the space E_∞^{3N}, the vector fields $\zeta_{\alpha a}^i$ should satisfy the set of equations securing for generators

$$X_\alpha = \sum_a \sum_{s=0}^{\infty} (D^s \zeta_{\alpha a}^i) \frac{\partial}{\partial x_a^{i(s)}}; \qquad \alpha = 1, \ldots, r, \tag{2}$$

the fulfilment of the commutation relations

$$X_\alpha X_\beta - X_\beta X_\alpha = \sum_{\gamma=1}^{r} c_{\alpha\beta}^\gamma X_\gamma, \qquad \alpha, \beta = 1, \ldots, r, \tag{3}$$

where $c_{\alpha\beta}^\gamma$ is the structure constant tensor of the group $\mathcal{G}_r$.

For the invariance of the Euler–Lagrange equations $\mathcal{L}_{ai}L = 0$, corresponding to the single-time Lagrangian $L : E_\infty^{3N} \to \mathbf{R}$, the existence of such r functions $\Omega_\alpha : E_\infty^{3N} \to \mathbf{R}$, $\alpha = 1, \ldots, r$, for which relations

$$X_\alpha L = D\Omega_\alpha \tag{4}$$

representing the conditions of quasi-invariance of the Lagrangian L under $\mathcal{G}_r$ are satisfied, will be sufficient. These relations, regarded as linear nonhomogeneous equations for L, will be consistent if functions Ω_α satisfy the relations:

$$X_\alpha \Omega_\beta - X_\beta \Omega_\alpha = \sum_{\gamma=1}^{r} c_{\alpha\beta}^\gamma \Omega_\gamma. \tag{5}$$

Let Ω_α^1 and Ω_α^2 be two different solutions of this set of equations. It may be shown that for any Lagrangian L_1 satisfying Eqs. (4) with $\Omega_\alpha = \Omega_\alpha^1$ there exists Lagrangian $L_2 = L_1 + DV$, $V: E_\infty^{3N} \to \mathbf{R}$, satisfying Eqs (4) with $\Omega_\alpha = \Omega_\alpha^2$ dynamically equivalent to it. Therefore, any partial solution of set (5) may be substituted without any loss of generality into Eq. (4) to obtain the equations for L expressing the invariance conditions under the group $\mathcal{G}_r$.

Finally, from the identity [4, 7]

$$X_\alpha L - D\Omega_\alpha \equiv \sum_a \zeta_{\alpha a}^i \mathcal{L}_{ai} L + DG_\alpha, \tag{6}$$

where

$$G_\alpha \equiv \sum_a \sum_{s=0}^{\infty} \xi_{ai,s} D^s \zeta^i_{\alpha a} - \Omega_\alpha, \tag{7}$$

and

$$\xi_{ai,s} \equiv \sum_{n=0}^{\infty} (-D)^n \frac{\partial L}{\partial x_a^{i(n+s+1)}}, \tag{8}$$

we see that the set (4) of r invariance conditions corresponds to r conservation laws of quantities G_α which are expressed by formula (7) through the single-time Lagrangian of the system. This constitutes the subject of the Noether theorem for our case.

3. General symmetry properties of the single-time Lagrangian corresponding to the Fokker-type action

Let us employ the above facts to the single-time Lagrangian corresponding to Fokker--type action (I. 9)

$$\begin{aligned} L &\equiv L_{\mathrm{f}} - U \\ &= -\sum_a m_a \gamma_a^{-1} - \sum\sum_{a<b} \int d\theta \exp\{\theta(\lambda D - D_a)\} \chi_{ab}(\boldsymbol{r}_{ab}, \boldsymbol{v}_a, \boldsymbol{v}_b, \theta). \end{aligned} \tag{9}$$

When substituting expression (9) into Eqs (4) we assume that functions Ω_α are decomposed in two terms $\Omega_\alpha = \Omega_{\mathrm{f}\alpha} - \psi_\alpha$ where $\Omega_{\mathrm{f}\alpha}$ is the free-particle term and ψ_α involves the coupling--constants. Then Eqs (4) are replaced by a pair of equations as follows

$$X_\alpha L_{\mathrm{f}} = D\Omega_{\mathrm{f}\alpha}, \qquad X_\alpha U = D\psi_\alpha. \tag{10}$$

Both systems of functions $\Omega_{\mathrm{f}\alpha}$ and ψ_α should satisfy separately a set of equations of form (5). Making use of an explicit form of operators (2) as well as of a free-particle Lagrangian L_{f} one may write Eqs (10) as

$$\sum_a m_a \gamma_a v_{ai} D\zeta^i_{\alpha a} = D\Omega_{\mathrm{f}\alpha}, \tag{11}$$

$$\sum_a \sum_{s=0}^{\infty} \frac{\partial U}{\partial x_a^{i(s)}} D^s \zeta^i_{\alpha a} = D\psi_\alpha. \tag{12}$$

Assuming (as it is in fact in most interesting cases) that vector fields $\zeta^i_{\alpha a}$ depend on variables of particle a only (and, eventually, on time t) and using formulae (I. 13) for derivative of U one can transform Eq. (12) into the form

$$\begin{aligned} \sum\sum_{a<b} \int d\theta \exp\{\theta(\lambda D - D_a)\} \Big\{ & \frac{\partial \chi_{ab}}{\partial x_a^i} \exp\left(-\theta \frac{\partial}{\partial t}\right) \zeta^i_{\alpha a} + \frac{\partial \chi_{ab}}{\partial x_b^i} \zeta^i_{\alpha b} \\ & + \frac{\partial \chi_{ab}}{\partial v_a^i} \exp\left(-\theta \frac{\partial}{\partial t}\right) D\zeta^i_{\alpha a} + \frac{\partial \chi_{ab}}{\partial v_b^i} D\zeta^i_{\alpha b} \Big\} = D\psi_\alpha. \end{aligned} \tag{13}$$

The set of equations (13) expresses the conditions of quasi-invariance of Lagrangian (9) under the group $\mathscr{G}_r$.

Let us suppose that for some group $\mathscr{G}_r$ equations (11) and (13) are satisfied. Then we derive the formulae for the corresponding integrals of motion (7). To start with, we find the auxiliary functions $\xi_{ai,s}$ (8) (the so-called "Ostrogradsky momenta"). By means of Eq. (I. 13) we have, after some transformations

$$\xi_{ai,s} = m_a \gamma_a v_{ai} \delta_{0,s} - \int d\theta \exp(\theta\lambda D) \left\{ \sum_{b(>a)} \exp(-\theta D_a) \right.$$

$$\times \left[\frac{(\lambda-1)^s \theta^s}{s!} \frac{\partial \chi_{ab}}{\partial v_a^i} + \sum_{n=0}^{\infty} \frac{(\lambda-1)^{n+s+1} \theta^{s+1}}{(n+s+1)!} (-\theta D)^n \mathscr{L}_{ai} \chi_{ab} \right]$$

$$\left. + \sum_{b(<a)} \exp(-\theta D_b) \left[\frac{\lambda^s \theta^s}{s!} \frac{\partial \chi_{ba}}{\partial v_a^i} + \sum_{n=0}^{\infty} \frac{\lambda^{n+s+1} \theta^{s+1}}{(n+s+1)!} (-\theta D)^n \mathscr{L}_{ai} \chi_{ba} \right] \right\}. \tag{14}$$

Substituting this formula in (7) we obtain a rather bulky expression for G_α. It becomes considerably simpler if one acts formally on the involved in it terms with χ_{ab} by the unit operator represented in the form $D^{-1}D$, and if one uses Eq. (13). Then

$$G_\alpha = \sum_a m_a \gamma_a v_{ai} \zeta^i_{\alpha a} - \Omega_{f\alpha}$$

$$+ D^{-1} \sum_{a<b} \sum \int d\theta \{ \zeta^i_{\alpha a} \exp(\theta D_b) \mathscr{L}_{ai} \chi_{ab} + \zeta^i_{\alpha b} \exp(-\theta D_a) \mathscr{L}_{bi} \chi_{ab} \}. \tag{15}$$

The operator D^{-1} was introduced here for compactness of the notes only: the expression on which it acts is a total time derivative as one can easily verify by means of Eq. (13). A similar formal method with operator D was employed in [5] when calculating the motion integrals in classical electrodynamics.

The expression (15), as it should be, does not contain an arbitrary number λ, entering the Lagrangian (9) (it was pointed out in [1] that two Lagrangians (9) which differ by a value of λ only are dynamically equivalent). It is interesting that integrals of motion written in our form (15), do not include explicitly the functions ψ_α. Making use of the equations of motion (I. 15) one easily verifies directly that on their solutions $DG_\alpha = 0$.

4. *Poincaré invariance conditions*

All considerations in Sections 2 and 3 referred to any r-parametric Lie group $\mathscr{G}_r$. For the ten-parameter Poincaré group $\mathscr{P}(1, 3)$, vector fields $\zeta^i_{\alpha a}$ have the following form [9]

$$\zeta^{Ti}_a = v^i_a, \quad \zeta^{Si}_{ja} = \delta^i_j, \quad \zeta^{Ri}_{ja} = \varepsilon^i_{jk} x^k_a, \quad \zeta^{Li}_{ja} = -t\delta^i_j + x_{aj} v^i_a \tag{16}$$

for time and space translations, and space and Lorentz rotations, respectively. It is not difficult to verify that generators X_α (2), as defined by Eqs (16), satisfy the commutation relations (3), where $c^\gamma_{\alpha\beta}$ is the structure constant tensor of the group $\mathscr{P}(1,3)$.

Substituting these vector fields in Eqs. (10) we get the Poincaré-invariance conditions for the Lagrangian (9). It is easily seen that for the free-particle Lagrangian L_f equations (11), when substituting in them relations (16), convert to identities by the following choice of functions $\Omega_{f\alpha}$:

$$\Omega_f^T = L_f, \quad \Omega_{fi}^S = 0, \quad \Omega_{fi}^R = 0, \quad \Omega_{fi}^L = \sum_a m_a x_{ai} \gamma_a^{-1}, \tag{17}$$

which satisfy, as it is easily seen, the conditions of the form (5). Substitution of expressions (16) in Eqs. (13) leads to Poincaré-invariance conditions which have the form of the equations for functions χ_{ab}. For definitness of these equations one needs to make the form of the functions ψ_α concrete. In the case of time and space translations and space rotations we put, respectively

$$\psi^T = U, \quad \psi_i^S = 0, \quad \psi_i^R = 0. \tag{18}$$

Then corresponding to this choice Eqs. (13) are satisfied if functions χ_{ab} do not include explicitly time t, and depend on particle coordinates through r_{ab} only and are three-scalars.

The investigation of the Lorentz-invariance conditions is more complicated. First of all we note that functions

$$\psi_i^L = \sum_{a<b}\sum \int d\theta \exp\{\theta(\lambda D - D_a\} [\lambda x_{ai} + (1-\lambda) x_{bi}] \chi_{ab} \tag{19}$$

together with (18) satisfy the set of equations of the form (5) for the Poincaré group. Using this function in the case of Lorentz transformations[2] one can write Eqs. (13) in the form

$$\sum_{a<b}\sum \int d\theta \exp(-\theta D_a) M_{ab}^i = 0, \tag{20}$$

where

$$M_{ab}^i \equiv \frac{\partial \chi_{ab}}{\partial v_{ak}}(-\delta_k^i + v_a^i v_{ak}) + \frac{\partial \chi_{ab}}{\partial v_{bk}}(-\delta_k^i + v_b^i v_{bk}) - (v_a^i + v_b^i)\chi_{ab} + \theta \frac{\partial \chi_{ab}}{\partial r_{ab}^i} + r_{ab}^i \frac{\partial \chi_{ab}}{\partial \theta}. \tag{21}$$

Since the functions χ_{ab} and, consequently, M_{ab}^i depend, for different pairs a, b ($a < b$) on different variables, Eq. (20) demands that each term of the sum on its l. h.s. be equal to zero. A sufficient condition for this will be the fulfilment of the following equations

$$M_{ab}^i = 0. \tag{22}$$

Let us find the general solution of the Eq. (22). Because of the translation and rotation invariance conditions indicated above, the functions χ_{ab} may depend on their variables only through the following seven expressions

$$\boldsymbol{r}_{ab}^2, \boldsymbol{v}_a^2, \boldsymbol{v}_b^2, \boldsymbol{v}_a \cdot \boldsymbol{v}_b, \boldsymbol{r}_{ab} \cdot \boldsymbol{v}_a, \boldsymbol{r}_{ab} \cdot \boldsymbol{v}_b, \theta. \tag{23}$$

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[2] According to the affirmation, stated after Eqs. (5), the special choice of the functions ψ_α in the form (18) and (19) does not restrict the generality of the results.

Treating quantities (23) as independent variables we obtain from Eqs (22) a set of three equations, the general solution of which found by the Jacobi method [10] has the form

$$\chi_{ab} = (1-\boldsymbol{v}_a\cdot\boldsymbol{v}_b)F_{ab}(\varrho_{ab}, \omega_{ab}, \sigma_{ab}, \sigma_{ba}), \tag{24}$$

where F_{ab} are arbitrary functions of the arguments indicated, the expressions for which through variables (23) were given by Eq. (I. 18).

The solution found coincides with the expression (I. 17) for χ_{ab} obtained in [1] from the functions $\Lambda_{ab} = \omega_{ab}F_{ab}$ describing the two-particle interaction in Fokker-type action Therefore, we have proved that the single-time interaction Lagrangian corresponding to manifestly invariant action of the Fokker-type determines equations of motion which are covariant under representation of the Poincaré group in $\boldsymbol{E}_\infty^{3N}$ (16).

We note that expression (24) is not a general solution of Eqs (20). The latter are satisfied also by the expressions of the type

$$\Theta(\theta)\delta(\varrho_{ab})\tilde{\chi}_{ab} \tag{25}$$

($\Theta(\theta)$ is Heaviside's function), where $\tilde{\chi}_{ab}$ is a regular function of variable θ and $\delta(\varrho_{ab})\tilde{\chi}_{ab}$ satisfies Eqs. (22), i.e. it has the form (24). Really then the only extra term in M^i_{ab} will be the expression, following from $\partial\chi_{ab}/\partial\theta$ containing $\delta(\theta)\delta(\theta^2-r^2_{ab})$; when integrated over θ it will give a term proportional to $\delta(r^2_{ab})$ and equal to zero if there are no collisions ($r^2_{ab}\neq 0$). Single-time Lagrangians considered in [1] and corresponding to Fokker's formulation of classical electrodynamics with a choice of a purely retarded (advanced) Green's function are examples of the expressions (25). So, in this case a correspondence (in a sense of the invariance of description) of the single-time formalism and a four-dimensional one is stated also (four-dimensional functions of the type $\Theta(x^0_a-x^0_b)\delta[(x^\nu_a-x^\nu_b)(x_{a\nu}-x_{b\nu})]$ are known to be Lorentz-invariant).

5. Motion integrals corresponding to Poincaré-invariance

Formulae for ten motion integrals corresponding to Poincaré quasi-invariance of the Lagrangian (9) are obtained according to the general theory through the substitution of the vector fields (16) and functions $\Omega_{t\alpha}$ (17) in Eq. (15). Using the properties of the functions χ_{ab} securing Poincaré invariance of the theory as stated above after some transformations we obtain:

$$E \equiv G^{\mathrm{T}} = \sum_a m_a\gamma_a + \sum_{a<b}\sum \int d\theta\left\{\exp(\theta D_b)\left(\chi_{ab}-\boldsymbol{v}_a\cdot\frac{\partial\chi_{ab}}{\partial\boldsymbol{v}_a}\right)\right.$$
$$\left.+\exp(-\theta D_a)\left(\chi_{ab}-\boldsymbol{v}_b\cdot\frac{\partial\chi_{ab}}{\partial\boldsymbol{v}_b}\right)+D^{-1}[D_a\exp(-\theta D_a)+D_b\exp(\theta D_b)]\chi_{ab}\right\}, \tag{26}$$

$$\boldsymbol{P} \equiv \boldsymbol{G}^{\mathrm{S}} = \sum_a m_a\gamma_a\boldsymbol{v}_a - \sum_{a<b}\sum \int d\theta\left\{\exp(\theta D_b)\frac{\partial\chi_{ab}}{\partial\boldsymbol{v}_a}\right.$$
$$\left.+\exp(-\theta D_a)\frac{\partial\chi_{ab}}{\partial\boldsymbol{v}_a}+D^{-1}[\exp(-\theta D_a)-\exp(\theta D_b)]\frac{\partial\chi_{ab}}{\partial\boldsymbol{r}_{ab}}\right\}, \tag{27}$$

$$J \equiv G^{\mathrm{R}} = \sum_a m_a \gamma_a \boldsymbol{x}_a \times \boldsymbol{v}_a - \sum_{a<b}\sum \int d\theta \Big\{ \exp(\theta D_b) \boldsymbol{x}_a \times \frac{\partial \chi_{ab}}{\partial \boldsymbol{v}_a}$$

$$+\exp(-\theta D_a)\boldsymbol{x}_b \times \frac{\partial \chi_{ab}}{\partial \boldsymbol{v}_b} + D^{-1}[\exp(-\theta D_a) - \exp(\theta D_b)] \left(\boldsymbol{x}_a \times \frac{\partial \chi_{ab}}{\partial \boldsymbol{x}_a} + \boldsymbol{v}_a \times \frac{\partial \chi_{ab}}{\partial \boldsymbol{v}_a} \right) \Big\}, \tag{28}$$

$$K \equiv G^{\mathrm{L}} = -t\boldsymbol{P} + \sum_a m_a \gamma_a \boldsymbol{x}_a + \sum_{a<b}\sum \int d\theta \Big\{ \exp(\theta D_b) \boldsymbol{x}_a \left(\chi_{ab} - \boldsymbol{v}_a \cdot \frac{\partial \chi_{ab}}{\partial \boldsymbol{v}_a} \right)$$

$$+\exp(-\theta D_a) x_b \left(\chi_{ab} - \boldsymbol{v}_b \cdot \frac{\partial \chi_{ab}}{\partial \boldsymbol{v}_b} \right) + D^{-1}[\exp(\theta D_b)$$

$$-\exp(-\theta D_a)] \left[\boldsymbol{x}_a \frac{\partial \chi_{ab}}{\partial \theta} - \frac{\partial \chi_{ab}}{\partial \boldsymbol{v}_a} - \boldsymbol{v}_a \left(\chi_{ab} - \boldsymbol{v}_a \cdot \frac{\partial \chi_{ab}}{\partial \boldsymbol{v}_a} \right) \right]$$

$$+D^{-1}[D^{-1}(\exp(\theta D_b) - \exp(-\theta D_a)) - \theta \exp(-\theta D_a)] \frac{\partial \chi_{ab}}{\partial \boldsymbol{r}_{ab}} \Big\}. \tag{29}$$

When picking out in $\boldsymbol{K}$ the term $-t\boldsymbol{P}$ the identity

$$D^{-1} t f = t D^{-1} f - D^{-2} f, \tag{30}$$

(where f is an arbitrary function) was used.

As usual the quantities, the conservation of which is related to invariance under time and space translations, space and Lorentz rotations, are identified, with energy E, linear momentum $\boldsymbol{P}$, angular momentum $\boldsymbol{J}$ and integral of center-of-mass of the system $\boldsymbol{K}$, respectively.

The impression may arise that quantities (26)–(29) are strongly asymmetrical with respect to particle variables. However, one may show that in the case of symmetrical functions χ_{ab} satisfying the conditions of Section 4 the expressions (26)–(29) are symmetrical also. We note also that, as it is seen from these formulae, in all the operator expressions containing D^{-1} the latter is cancelled out when exponents are expanded in series. For example, for the expression entering $\boldsymbol{P}$ and $\boldsymbol{J}$, when acting on the two-particle terms we have the following:

$$D^{-1}[\exp(-\theta D_a) - \exp(\theta D_b)] = D^{-1} \sum_{n=0}^{\infty} \frac{\theta^n}{n!} [(-D_a)^n - D_b^n]$$

$$= -\sum_{n=0}^{\infty} \frac{\theta^n}{n!} \sum_{s=1}^{n} D_b^{s-1} (-D_a)^{n-s}.$$

Corresponding expressions in E and $\boldsymbol{K}$ are transformed in a similar manner (although the latter is somewhat more complicated).

Substituting in formulae (26)–(29) the solutions of the Poincaré invariance conditions for functions χ_{ab} (formula (24)) one can obtain the integrals of motion in terms of functions F_{ab} from which, alongside, their symmetry under particle transposition will be seen. The corresponding formulae, which are rather bulky, shall not be written here and we confine ourselves to consideration of the case of tensor interaction of the rank n, discussed in [1], to which in (24) the function F_{ab} of the form [1]

$$F_{ab} = g_a g_b \omega_{ab}^{n-1} G(\theta^2 - r_{ab}^2) \tag{31}$$

corresponds. In this case by expanding exponents in (26)–(29) in series, taking into account the parity of function G with respect to θ, using the definition (1.22) of the functions W_s and restoring the constant c we get:

$$\begin{aligned}E = E_f + \sum_{a<b}\sum g_a g_b \sum_{s=0}^{\infty} c^{-2s}[(2s)!]^{-1}\{D_a^{2s}\gamma_b^2[1-nv_b^2c^{-2}+(n-1)\boldsymbol{v}_a\cdot\boldsymbol{v}_b c^{-2}]\omega_{ab}^{n-1}W_s(r_{ab})\\ + D_b^{2s}\gamma_a^2[1-nv_a^2c^{-2}+(n-1)\boldsymbol{v}_a\cdot\boldsymbol{v}_b c^{-2}]\omega_{ab}^{n-1}W_s(r_{ab})\\ - D^{-1}(D_a^{2s+1}+D_b^{2s+1})(1-\boldsymbol{v}_a\cdot\boldsymbol{v}_b c^{-2})\omega_{ab}^{n-1}W_s(r_{ab})\},\end{aligned} \tag{32}$$

$$\begin{aligned}\boldsymbol{P} = \boldsymbol{P}_f + \sum_{a<b}\sum g_a g_b \sum_{s=0}^{\infty} c^{-2s}[(2s)!]^{-1}\Big\{c^{-2}D_a^{2s}[n\boldsymbol{v}_a+(1-n)\boldsymbol{v}_b\gamma_b^2(1\\ -\boldsymbol{v}_a\cdot\boldsymbol{v}_b c^{-2})]\omega_{ab}^{n-1}W_s(r_{ab}) + c^{-2}D_b^{2s}[n\boldsymbol{v}_b+(1-n)\boldsymbol{v}_a\gamma_a^2(1-\boldsymbol{v}_a\cdot\boldsymbol{v}_b c^{-2})]\omega_{ab}^{n-1}W_s(r_{ab})\\ + D^{-1}(D_b^{2s}-D_a^{2s})\boldsymbol{r}_{ab}(1-\boldsymbol{v}_a\cdot\boldsymbol{v}_b c^{-2})\omega_{ab}^{n-1}\frac{1}{r_{ab}}\frac{dW_s(r_{ab})}{dr_{ab}}\Big\},\end{aligned} \tag{33}$$

$$\begin{aligned}\boldsymbol{J} = \boldsymbol{J}_f + \sum_{a<b}\sum g_a g_b \sum_{s=0}^{\infty} c^{-2s}[(2s)!]^{-1}\Big\{c^{-2}D_a^{2s}[n\boldsymbol{x}_b\times\boldsymbol{v}_a+(1-n)\gamma_b^2(1\\ -\boldsymbol{v}_a\cdot\boldsymbol{v}_b c^{-2})\boldsymbol{x}_b\times\boldsymbol{v}_a]\omega_{ab}^{n-1}W_s(r_{ab}) + c^{-2}D_b^{2s}[n\boldsymbol{x}_a\times\boldsymbol{v}_b+(1-n)\gamma_a^2(1\\ -\boldsymbol{v}_a\cdot\boldsymbol{v}_b c^{-2})\boldsymbol{x}_a\times\boldsymbol{v}_b]\omega_{ab}^{n-1}W_s(r_{ab}) + D^{-1}(D_b^{2s}-D_a^{2s})\Big[\boldsymbol{x}_a\times\boldsymbol{x}_b\Big(1\\ -\frac{\boldsymbol{v}_a\cdot\boldsymbol{v}_b}{c^2}\Big)\frac{1}{r_{ab}}\frac{dW_s(r_{ab})}{dr_{ab}} + \frac{n}{c^2}\boldsymbol{v}_a\times\boldsymbol{v}_b W_s(r_{ab})\Big]\omega_{ab}^{n-1}\Big\},\end{aligned} \tag{34}$$

$$\begin{aligned}\boldsymbol{K} = -t\boldsymbol{P} + \sum_a m_a\gamma_a\boldsymbol{x}_a + \sum_{a<b}\sum g_a g_b \sum_{s=0}^{\infty} c^{-2s}[(2s)!]^{-1}\Big\{c^{-2}D_a^{2s}\boldsymbol{x}_a\gamma_a^2[1\\ -nv_a^2c^{-2}+(n-1)\boldsymbol{v}_a\cdot\boldsymbol{v}_b c^{-2}]\omega_{ab}^{n-1}W_s(r_{ab}) + c^{-2}D_a^{2s}\boldsymbol{x}_b\gamma_b^2[1-nv_b^2c^{-2}\\ +(n-1)\boldsymbol{v}_a\cdot\boldsymbol{v}_b c^{-2}]\omega_{aa}^{n-1}W_s(r_{ab}) + D^{-1}\Big[c^{-2}(D_a^{2s}-D_b^{2s})\boldsymbol{v}_{ab}n\omega_{ab}^{n-1}W_s(r_{ab})\end{aligned}$$

$$-c^{-2}D_b^{2s+1}\boldsymbol{x}_a(1-\boldsymbol{v}_a\cdot\boldsymbol{v}_bc^{-2})\omega_{ab}^{n-1}W_s(r_{ab})-c^{-2}D_a^{2s+1}\boldsymbol{x}_b(1-\boldsymbol{v}_a\cdot\boldsymbol{v}_bc^{-2})\omega_{ab}^{n-1}W_s(r_{ab})$$

$$+D^{-1}(D_b^{2s}-D_a^{2s})r_{ab}(1-\boldsymbol{v}_a\cdot\boldsymbol{v}_bc^{-2})\omega_{ab}^{n-1}\frac{1}{r_{ab}}\frac{dW_s(r_{ab})}{dr_{ab}}\Bigg]\Bigg\}. \tag{35}$$

In formulae (32)–(34) the standard free-particle expressions entering Eqs. (26)–(28) are marked by index f. We note that series (32)–(35) include only even powers of c^{-1}. For Wheeler-Feynman's electrodynamics we have

$$g_a = e_a, \quad n = 1, \quad W_s(r) = r^{2s-1} \tag{36}$$

and the above formulae give the results of paper [5].

Concluding this section we discuss one more aspect of the problem of motion integrals in the relativistic theory of direct interactions. Expressions (26)–(29) or generally (15), are integrals for equations of motion of infinitely high order. It was pointed out in [1] from all sets of solutions of the latter, in some cases the subset may be picked out (just this subset is often of physical interest), satisfying also the equations of motion of the second order. Clearly, the integrals of the latter are obtained by excluding in initial equations (15) all derivatives higher than the first order ones using the same equations of motion. The functions of particle position and velocities obtained in that way are the series in terms of coupling constant powers having a very complicated structure. Some of their general properties including differential and integrofunctional equations which should be satisfied by them were discussed by a number of authors in a four-dimensional formalism of the predictive relativistic mechanics (PRM) [11]. The investigation of this problem in the framework of our fomalism demands a separate paper.

6. *Expansion in* c^{-1}

The exact formulae obtained for the integrals of motion of the relativistic system which is described by Lagrangian (9) may serve as a basis for finding various approximate expressions. As an example, we consider here expansion of the energy, linear and angular momentum, and integral of the center-of-mass motion (32)–(35) for the particle system with tensor interaction up to the order c^{-4}. For this purpose it is necessary to take the first three terms of the sum over s in formulae (32)–(35), cancel simultaneously all the operators D^{-1}, perform a simple series development in v_a^2/c^2 for the expressions involved and neglect all terms of the order higher than c^{-4}. By means of recurrence relations (I.23) between functions W_s all necessary expressions may be written in terms of the two functions W_0 and W_1. Finally, we get:

$$E = \sum_a \left\{m_ac^2+\frac{m_av_a^2}{2}+\frac{3m_av_a^4}{8c^2}+\frac{5m_av_a^6}{16c^4}\right\}+\sum_{a<b}\sum g_ag_b\left\{W_0(r_{ab})\right.$$

$$+\frac{1}{2c^2}\left\{[\boldsymbol{v}_a\cdot\boldsymbol{v}_b+(1-n)v_{ab}^2]W_0(r_{ab})-(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a)(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b)\frac{1}{r_{ab}}\frac{dW_0(r_{ab})}{dr_{ab}}\right\}$$

$$+\frac{1}{8c^4}\Big\{\{3(1-n^2)(v_a^4+v_b^4)-2(3n^2-2n-1)v_a^2v_b^2+(12n^2-10n-1)(\boldsymbol{v}_a\cdot\boldsymbol{v}_b)(v_a^2+v_b^2)$$

$$-4(3n^2-2n+1)(\boldsymbol{v}_a\cdot\boldsymbol{v}_b)^2+(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a)[2(1-2n)\boldsymbol{v}_a\cdot\dot{\boldsymbol{v}}_b+(8n-3)\boldsymbol{v}_b\cdot\dot{\boldsymbol{v}}_a$$

$$+(4n-1)\boldsymbol{v}_b\cdot\dot{\boldsymbol{v}}_b-\boldsymbol{r}_{ab}\cdot\ddot{\boldsymbol{v}}_b]+(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b)[(1-4n)\boldsymbol{v}_a\cdot\dot{\boldsymbol{v}}_a+(3-8n)\boldsymbol{v}_a\cdot\dot{\boldsymbol{v}}_b+2(2n-1)\boldsymbol{v}_b\cdot\dot{\boldsymbol{v}}_a$$

$$-\boldsymbol{r}_{ab}\cdot\ddot{\boldsymbol{v}}_a]+(4n-3)[(\boldsymbol{r}_{ab}\cdot\dot{\boldsymbol{v}}_a)(\boldsymbol{v}_{ab}\cdot\boldsymbol{v}_b)+(\boldsymbol{r}_{ab}\cdot\dot{\boldsymbol{v}}_b)(\boldsymbol{v}_{ab}\cdot\boldsymbol{v}_a)]$$

$$+(\boldsymbol{r}_{ab}\cdot\dot{\boldsymbol{v}}_a)(\boldsymbol{r}_{ab}\cdot\dot{\boldsymbol{v}}_b)\}W_0(r_{ab})+\{(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a)^2[(4n-3)(\boldsymbol{v}_{ab}\cdot\boldsymbol{v}_b)+\boldsymbol{r}_{ab}\cdot\dot{\boldsymbol{v}}_b]$$

$$+(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b)^2[(3-4n)(\boldsymbol{v}_{ab}\cdot\boldsymbol{v}_a)-\boldsymbol{r}_{ab}\cdot\dot{\boldsymbol{v}}_a]+(\boldsymbol{v}_{ab}\cdot\boldsymbol{v}_a)(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b)[4(n-1)\boldsymbol{v}_a\cdot\boldsymbol{v}_b$$

$$-(2n+1)(v_a^2+v_b^2)-3\boldsymbol{r}_{ab}\cdot\dot{\boldsymbol{v}}_{ab}]\}\frac{1}{r_{ab}}\frac{dW_0(r_{ab})}{dr_{ab}}-(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a)(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b)[(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a)^2$$

$$+(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b)^2+(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a)(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b)]\left(\frac{1}{r_{ab}}\frac{d}{dr_{ab}}\right)^2W_0(r_{ab})+(1-4n)(\dot{\boldsymbol{v}}_a\cdot\dot{\boldsymbol{v}}_b$$

$$-\boldsymbol{v}_a\cdot\ddot{\boldsymbol{v}}_b-\boldsymbol{v}_b\cdot\ddot{\boldsymbol{v}}_a)W_1(r_{ab})\Big\}\Big\}+O(c^{-6}), \tag{37}$$

$$\boldsymbol{P}=\sum_a m_a\boldsymbol{v}_a\left(1+\frac{v_a^2}{2c^2}+\frac{3v_a^4}{8c^4}\right)+\frac{1}{2c^2}\sum_{a<b}\sum g_ag_b\Big[(\boldsymbol{v}_a+\boldsymbol{v}_b)W_0(r_{ab})$$

$$-\boldsymbol{r}_{ab}(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a+\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b)\frac{1}{r_{ab}}\frac{dW_0(r_{ab})}{dr_{ab}}\Big]+\frac{1}{8c^4}\sum_{a<b}\sum g_ag_b\Big\{\{\dot{\boldsymbol{v}}_a[(8n-3)\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a$$

$$-\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b]+\dot{\boldsymbol{v}}_b[\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a+(3-8n)\boldsymbol{v}_{ab}\cdot\boldsymbol{v}_b]+\boldsymbol{v}_a[(3-2n)v_a^2+(1-2n)v_b^2+2(2n-1)\boldsymbol{v}_a\cdot\boldsymbol{v}_b$$

$$+(4n-3)\boldsymbol{r}_{ab}\cdot(\dot{\boldsymbol{v}}_a+\dot{\boldsymbol{v}}_b)]+\boldsymbol{v}_b[(1-2n)v_a^2+(3-2n)v_b^2+2(2n-1)\boldsymbol{v}_a\cdot\boldsymbol{v}_b+(3-4n)\boldsymbol{r}_{ab}\cdot(\dot{\boldsymbol{v}}_a+\dot{\boldsymbol{v}}_b)]$$

$$+\boldsymbol{r}_{ab}[(1-4n)\boldsymbol{v}_{ab}\cdot(\dot{\boldsymbol{v}}_a+\dot{\boldsymbol{v}}_b)-\boldsymbol{r}_{ab}\cdot(\ddot{\boldsymbol{v}}_a+\ddot{\boldsymbol{v}}_b)]\}W_0(r_{ab})+\{\boldsymbol{v}_a[-2(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a)(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b)$$

$$+(4n-3)(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_{ab})\boldsymbol{r}_{ab}\cdot(\boldsymbol{v}_a+\boldsymbol{v}_b)]+\boldsymbol{v}_b[-2(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a)(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b)+(3-4n)(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_{ab})\boldsymbol{r}_{ab}\cdot(\boldsymbol{v}_a+\boldsymbol{v}_b)]$$

$$+\boldsymbol{r}_{ab}(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a)[(1-2n)(v_b^2-2\boldsymbol{v}_a\cdot\boldsymbol{v}_b)-(1+2n)v_a^2-\boldsymbol{r}_{ab}\cdot(3\dot{\boldsymbol{v}}_a-\dot{\boldsymbol{v}}_b)]+\boldsymbol{r}_{ab}(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b)[(1-2n)(v_a^2$$

$$-2\boldsymbol{v}_a\cdot\boldsymbol{v}_b)-(1+2n)v_b^2-\boldsymbol{r}_{ab}\cdot(\dot{\boldsymbol{v}}_a-3\dot{\boldsymbol{v}}_b)]\}\frac{1}{r_{ab}}\frac{dW_0(r_{ab})}{dr_{ab}}-\boldsymbol{r}_{ab}[(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a)^2$$

$$+(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b)^2]\boldsymbol{r}_{ab}\cdot(\boldsymbol{v}_a+\boldsymbol{v}_b)\left(\frac{1}{r_{ab}}\frac{d}{dr_{ab}}\right)^2W_0(r_{ab})+(4n-1)(\ddot{\boldsymbol{v}}_a+\ddot{\boldsymbol{v}}_b)W_1(r_{ab})\Big\}+O(c^{-6}), \tag{38}$$

$$\boldsymbol{J}=\sum_a m_a\boldsymbol{x}_a\times\boldsymbol{v}_a\left(1+\frac{v_a^2}{2c^2}+\frac{3v_a^4}{8c^4}\right)+\sum_{a<b}\sum g_ag_b\Big\{\frac{1}{2c^2}\Big\{[\boldsymbol{x}_a\times\boldsymbol{v}_b+\boldsymbol{x}_b\times\boldsymbol{v}_a$$

$$+2(1-n)\boldsymbol{r}_{ab}\times\boldsymbol{v}_{ab}]W_0(r_{ab})+\boldsymbol{x}_a\times\boldsymbol{x}_b(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a+\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b)\frac{1}{r_{ab}}\frac{dW_0(r_{ab})}{dr_{ab}}\Big\}$$

$$+\frac{1}{8c^4}\Big\{\{\boldsymbol{x}_a\times\dot{\boldsymbol{v}}_b[(3-8n)\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a+\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b]+\boldsymbol{x}_b\times\dot{\boldsymbol{v}}_a[(8n-3)\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b-\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a]$$

$$+4\boldsymbol{x}_a\times\boldsymbol{v}_a[n(n-1)(2\boldsymbol{v}_a\cdot\boldsymbol{v}_b-v_b^2)+(1-n^2)v_a^2+n\boldsymbol{r}_{ab}\cdot\dot{\boldsymbol{v}}_b]+4\boldsymbol{x}_b\times\boldsymbol{v}_b[n(n-1)(2\boldsymbol{v}_a\cdot\boldsymbol{v}_b-v_a^2)$$

$$+(1-n^2)v_b^2-n\boldsymbol{r}_{ab}\cdot\dot{\boldsymbol{v}}_a]+\boldsymbol{x}_a\times\boldsymbol{v}_b[(4n^2-6n-1)(v_a^2-2\boldsymbol{v}_a\cdot\boldsymbol{v}_b)+(4n^2-2n-1)v_b^2$$

$$+(1-4n)\boldsymbol{r}_{ab}\cdot\dot{\boldsymbol{v}}_b]+\boldsymbol{x}_b\times\boldsymbol{v}_a[(4n^2-6n-1)(v_b^2-2\boldsymbol{v}_a\cdot\boldsymbol{v}_b)+(4n^2-2n-1)v_a^2+(4n-1)\boldsymbol{r}_{ab}\cdot\dot{\boldsymbol{v}}_a]$$

$$+\boldsymbol{x}_a\times\boldsymbol{x}_b[(4n-3)\boldsymbol{v}_{ab}\cdot(\dot{\boldsymbol{v}}_a+\dot{\boldsymbol{v}}_b)-\boldsymbol{r}_{ab}\cdot(\ddot{\boldsymbol{v}}_a+\ddot{\boldsymbol{v}}_b)]+2(2n-1)\boldsymbol{v}_a\times\boldsymbol{v}_b(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a+\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b)\}W_0(r_{ab})$$

$$+\{4(1-n)\boldsymbol{x}_a\times\boldsymbol{v}_a(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b)^2+4(1-n)\boldsymbol{x}_b\times\boldsymbol{v}_b(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a)^2+\boldsymbol{x}_a\times\boldsymbol{v}_b[(4n-3)(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b)^2$$

$$-(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a)^2-2(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a)(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b)]+\boldsymbol{x}_b\times\boldsymbol{v}_a[(4n-3)(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a)^2-(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b)^2-2(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a)(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b)]$$

$$+\boldsymbol{x}_a\times\boldsymbol{x}_b(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a)[2n-1)(v_b^2-2\boldsymbol{v}_a\cdot\boldsymbol{v}_b)+(2n+1)v_a^2+\boldsymbol{r}_{ab}\cdot(\dot{\boldsymbol{v}}_a-3\dot{\boldsymbol{v}}_b)]$$

$$+\boldsymbol{x}_a\times\boldsymbol{x}_b(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b)[(2n-1)(v_a^2-2\boldsymbol{v}_a\cdot\boldsymbol{v}_b)+(2n+1)v_b^2+\boldsymbol{v}_{ab}\cdot(3\dot{\boldsymbol{v}}_a-\dot{\boldsymbol{v}}_b)]\}\frac{1}{r_{ab}}\frac{dW_0(r_{ab})}{dr_{ab}}$$

$$+\boldsymbol{x}_a\times\boldsymbol{x}_b[(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a)^2+(\boldsymbol{v}_{ab}\cdot\boldsymbol{v}_b)^2]\boldsymbol{r}_{ab}\cdot(\boldsymbol{v}_a+\boldsymbol{v}_b)\left(\frac{1}{r_{ab}}\frac{d}{dr_{ab}}\right)^2W_0(r_{ab})$$

$$+(1-4n)(\boldsymbol{x}_a\times\ddot{\boldsymbol{v}}_b+\boldsymbol{x}_b\times\ddot{\boldsymbol{v}}_a)W_1(r_{ab})\Big\}\Big\}+O(c^{-6}), \tag{39}$$

$$\boldsymbol{K}=-t\boldsymbol{P}+\sum_a m_a\boldsymbol{x}_a\left(1+\frac{v_a^2}{2c^2}+\frac{3v_a^4}{8c^4}\right)+\sum\sum_{a<b} g_a g_b\Big\{\frac{\boldsymbol{x}_a+\boldsymbol{x}_b}{2c^2}W_0(r_{ab})$$

$$+\frac{1}{8c^4}\Big\{\{\boldsymbol{x}_a[(5-6n)v_a^2+(2n-1)(v_b^2+2\boldsymbol{v}_a\cdot\boldsymbol{v}_b)-\boldsymbol{r}_{ab}\cdot(\dot{\boldsymbol{v}}_a+\dot{\boldsymbol{v}}_b)]+\boldsymbol{x}_b[(5-6n)v_b^2$$

$$+(2n-1)(v_a^2+2\boldsymbol{v}_a\cdot\boldsymbol{v}_b)+\boldsymbol{r}_{ab}\cdot(\dot{\boldsymbol{v}}_a+\dot{\boldsymbol{v}}_b)]+2(2n-3)\boldsymbol{v}_{ab}(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a+\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b)\}W_0(r_{ab})$$

$$-[2(\boldsymbol{x}_a+\boldsymbol{x}_b)(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a)(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_b)+\boldsymbol{r}_{ab}(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_{ab})(\boldsymbol{r}_{ab}\cdot\boldsymbol{v}_a+\boldsymbol{v}_{ab}\cdot\boldsymbol{v}_b)]\frac{1}{r_{ab}}\frac{dW_0(r_{ab})}{dr_{ab}}$$

$$+(4n-1)(\dot{\boldsymbol{v}}_a+\dot{\boldsymbol{v}}_b)W_1(r_{ab})\Big\}\Big\}+O(c^{-6}). \tag{40}$$

Clearly, these expressions may be obtained also directly from the single-time Lagrangian (1.38) for a tensor interaction taken up to order c^{-4}. Just in this way, by using various forms of the particle system Lagrangian up to order c^{-2}, the integrals of motion were found in this approximation in [12, 13]. The terms of order c^{-2} in our formulae (37)–(40) are in full agreement with these results.

The terms of order c^{-4} include already the second and third derivatives of particle coordinates. To exclude them for the purpose of obtaining the integrals of equations of motion of second order, it is sufficient to make use of the nonrelativistic equations of motion Then, the expressions for the integrals of motion will include besides double sums, the triple ones and the energy will include also quadruple sums i.e. the terms of the second and third order with regard to the coupling constant. The comparison of the expressions of order c^{-4} with the results known previously is very poor because the only calculations known to us including c^{-4} terms concern electrodynamics. Substituting (36) in formulae (37)–(40) we obtain the expressions which agree with the results of [14] and [15] up to order c^{-4}. In the former they were obtained by solution of the PRM equations expressing the transformation properties and the predictivity conditions of the equations of motion and in the latter the integrals of motion being obtained directly from the four-dimensional Fokker's action (without the terms of the third order with regard to the coupling constant) were calculated.

7. Conclusion

The investigation of the single-time Lagrangians which correspond to the action integrals of the Fokker-type performed in the present paper shows a possibility to formulate a symmetry theory for them in terms of the representation of a group by Lie–Bäcklund transformations. The application of this theory to the Poincaré group allows us to write the conditions of relativistic covariance of the equations of motion, defined by the single--time interaction Lagrangian. The fact that the Lagrangian L obtained by the transformation of the Poincaré invariant action integral of the Fokker-type to single-time form does satisfy these conditions means that four-dimensional many-time formulation of relativistic Fokker's dynamics and three-dimensional single-time one are equivalent in the sense of agreement with the Poincaré–Einstein relativity principle.

These results may be regarded from another point of view. Namely, the Lagrangian found on the basis of the action integral of the Fokker-type and determined by Eqs. (9) and (24), represents a closed form of a wide class of the solution of the set of equations (4) and (5), expressing (if it is written for the Poincaré group) the requirements of relativistic invariance for the Lagrangian formulation of classical mechanics of the interacting particle system. Finding such solutions by means of direct integration of the mentioned set of equations seems to be a very difficult problem.

As it follows from the method proposed for constructing a single-time Lagrangian the particle variables x_a characterizing their space localization preserve their covariant character. This feature is an essential advantage of the Lagrangian formalism, since in the Hamiltonian formalism the known no-interaction theorems forbid using the covariant coordinates as canonical variables.

The methods proposed in Sections 2–3 are suitable for investigating the covariance not only under Poincaré group, but also under an arbitrary one for which there exists a representation by Lie–Bäcklund transformations (for example, conformal group) as well as for finding corresponding conservation laws.

We note finally that studying the special case of tensor interaction makes it possible to find the integrals of motion of the particle system in which the interaction is described usually by the field-theoretical methods.

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