# THE RELATIVISTIC TWO-FERMION EQUATIONS (II)

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A new set of relativistic equations for a spin 1/2 fermion-antifermion bound system in the case of instantaneous interaction have been previously considered by the authors (*Acta Phys. Pol.* **B11**, 413 (1980), I). In the present work the 16 amplitudes are re-expressed in terms of three scalars and four vectors which satisfy coupled differential equations. Lorentz, parity and charge conjugation invariance are used to reduce these equations to sets of coupled differential equations according to their parity and total angular momentum. A detailed solution of these equations is given and the positronium case is also studied.

## 1. *Introduction*

One of the authors [1, 2] has derived field equations of the form

$$P_\mu \psi \equiv (S_{\mu\nu} p_\nu + m\gamma_\mu)\psi = p_\mu \psi \tag{1.1}$$

which describe particles of definite mass $m$ and definite spin $s$. These equations have been derived for the finite dimensional representations of the inhomogeneous de Sitter group SO(4, 1) with the invariant $P_A P_A = 0$. $P_A$ is the momentum-energy-mass five vector, with components $P_k$ = Cartesian components of momentum, $P_4 = iP_0$, $P_0$ = energy and $P_5 = m$. The $S_{AB} = (S_{\mu\nu}, S_{\mu 5} = \gamma_\mu)$ are the generators of the homogeneous de Sitter group. Several general field equations have been proposed by different authors[1]. However, equations (1.1) have the advantage that they follow from a symmetry principle. In the previous paper (hereafter it will be referred to as I) the authors studied a generalization of equations (1.1) in the case of a two-fermion system. The selfconsistency of the equations obtained have been studied and the case of the $e^+e^-$ bound system has also been considered. In the next section the radial equations are presented and the states are classi-

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[1] See references [6–22] in reference [2].

fied according to the total angular momentum $J$, $J_3$ and the parity. For a classification of the states into singlets and triplets, we refer the reader to e.g. references [3, 4]. In Section 3 the positronium case is considered and the solution of the radial equations obtained is given. A discussion of the results obtained is given in Section 4.

### 2.1. The wave equations in vectorial form

The notations used here will be the same as (I). Now let us consider the two-body equation with Breit interaction

$$E\psi = H\psi,$$

$$H = \boldsymbol{\alpha}_1 \cdot \boldsymbol{P}_1 + \boldsymbol{\alpha}_2 \cdot \boldsymbol{P}_2 + m_1\gamma_0^{(1)} + m_2\gamma_0^{(2)} + V_{\mathrm{B}},$$

$$V_{\mathrm{B}} = \frac{e_1 e_2}{r}\left[1 - \tfrac{1}{2}\left(\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2 + \frac{(\boldsymbol{\alpha}_1 \cdot \boldsymbol{r})(\boldsymbol{\alpha}_2 \cdot \boldsymbol{r})}{r^2}\right)\right], \qquad \boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2.$$

In the mass-centre frame of reference,

$$\boldsymbol{P}_1 = \boldsymbol{P}, \qquad \boldsymbol{P}_2 = -\boldsymbol{P}, \qquad \boldsymbol{P} = -i\frac{\partial}{\partial \boldsymbol{r}},$$

$$H = (\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2) \cdot \boldsymbol{P} + m_1\gamma_0^{(1)} + m_2\gamma_0^{(2)} + V_{\mathrm{B}},$$

and $E = M$ is the mass of the composite particle. We can classify the states according to the parity $P$ and the total angular momentum $J^2 = j(j+1)$ and $J_z = m$, where the total angular momentum

$$\boldsymbol{J} = \boldsymbol{r} \wedge \boldsymbol{P} + \tfrac{1}{2}(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)$$

and the parity operator

$$P = \gamma_0^{(1)}\gamma_0^{(2)}\pi,$$

where

$$\pi\psi(\boldsymbol{r}) = \psi(-\boldsymbol{r})$$

commute with $H$.

Following the previous paper (I) the wave equation in the mass-centre system

$$M\psi = [(\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2) \cdot \boldsymbol{P} + m_1\gamma_0^{(1)} + m_2\gamma_0^{(2)} + V_{\mathrm{B}}]\psi$$

is written in terms of the tensorial components $a$ and $\psi_{AB}$ as follows:

$$\left(M + \frac{e_1 e_2}{r}\right)\chi_0 = -2\mu a,$$

$$\left(M + \frac{e_1 e_2}{r}\right)\varphi_0 = 2\kappa\eta,$$

$$2i \operatorname{div} \boldsymbol{H} = \left(M - \frac{3e_1e_2}{r}\right)\eta - 2\kappa\phi_0,$$

$$2\nabla\eta = 2i\mu\boldsymbol{\phi} - iM\boldsymbol{H} + \frac{ie_1e_2}{r^2}\, \boldsymbol{r}H_r,$$

$$2i \operatorname{div} \boldsymbol{E} = \left(M - \frac{3e_1e_2}{r}\right)a + 2\mu\chi_0,$$

$$2i\nabla a = 2\kappa\boldsymbol{\chi} + M\boldsymbol{E} - \frac{e_1e_2}{r^2}\, \boldsymbol{r}E_r,$$

$$2i \operatorname{curl} \boldsymbol{\phi} = 2\kappa\boldsymbol{E} + \left(M - \frac{2e_1e_2}{r}\right)\boldsymbol{\chi} + \frac{e_1e_2}{r^2}\, \boldsymbol{r}\chi_r,$$

$$-2i \operatorname{curl} \boldsymbol{\chi} = -2\mu\boldsymbol{H} + \left(M - \frac{2e_1e_2}{r}\right)\boldsymbol{\phi} + \frac{e_1e_2}{r^2}\, \boldsymbol{r}\phi_r,$$

where

$$2\kappa = m_1 + m_2, \quad 2\mu = m_1 - m_2, \quad E_r = \boldsymbol{E}\cdot\boldsymbol{r}/r.$$

The states of definite parity $\varepsilon$, $\gamma_0^{(1)}\gamma_0^{(2)}\psi(-r) = \varepsilon\psi(r)$, are given by the following relations

$$a(-\boldsymbol{r}) = \varepsilon a(\boldsymbol{r}), \qquad \chi_0(-\boldsymbol{r}) = \varepsilon\chi_0(\boldsymbol{r}),$$

$$\varphi_0(-\boldsymbol{r}) = -\varepsilon\varphi_0(\boldsymbol{r}), \qquad \eta(-\boldsymbol{r}) = -\varepsilon\eta(\boldsymbol{r}),$$

$$\boldsymbol{H}(-\boldsymbol{r}) = \varepsilon\boldsymbol{H}(\boldsymbol{r}), \qquad \boldsymbol{\phi}(-\boldsymbol{r}) = \varepsilon\boldsymbol{\phi}(\boldsymbol{r}),$$

$$\boldsymbol{E}(-\boldsymbol{r}) = -\varepsilon\boldsymbol{E}(\boldsymbol{r}), \qquad \boldsymbol{\chi}(-\boldsymbol{r}) = -\varepsilon\boldsymbol{\chi}(\boldsymbol{r}).$$

### 2.2. Charge conjugation for equal masses

For equal masses if we write $\psi = \tilde{\psi}C$, the wave equation reads

$$i\partial_t\tilde{\psi} = \boldsymbol{P}\cdot(\boldsymbol{\alpha}\tilde{\psi} + \tilde{\psi}\boldsymbol{\alpha}) + m(\gamma_0\tilde{\psi} - \tilde{\psi}\gamma_0) + \frac{e_1e_2}{r}(\tilde{\psi}^{\mathrm{T}} + \boldsymbol{\alpha}\cdot\tilde{\psi}\boldsymbol{\alpha}).$$

If we take the transpose of this equation, and multiply by $C$ and $C^{-1}$ from the left and the right and denote

$$\bar{\psi} = C\tilde{\psi}^{\mathrm{T}}C^{-1}$$

we obtain

$$i\partial_t\bar{\psi} = -\boldsymbol{P}\cdot(\boldsymbol{\alpha}\bar{\psi} + \bar{\psi}\boldsymbol{\alpha}) + m(\gamma_0\bar{\psi} - \bar{\psi}\gamma_0) + \frac{e_1e_2}{r}(\bar{\psi} + \boldsymbol{\alpha}\cdot\bar{\psi}\boldsymbol{\alpha}).$$

Thus $\bar{\psi}(-\boldsymbol{r})$ satisfies the same equation as $\tilde{\psi}$.

Thus

$$C\tilde{\psi}^{\mathrm{T}}(-\boldsymbol{r})C^{-1} = \varepsilon_C \psi(\boldsymbol{r}),$$

i.e.

$$a(-\boldsymbol{r})+\tfrac{1}{2}\, CS_{AB}^{\mathrm{T}}C^{-1}\psi_{AB}(-\boldsymbol{r}) = \varepsilon_C(a+\tfrac{1}{2}\, S_{AB}\psi_{AB}).$$

Hence,

$$a(-\boldsymbol{r}) = \varepsilon_C a(\boldsymbol{r}), \qquad \psi_{ab}(-\boldsymbol{r}) = -\varepsilon_C \psi_{ab}(\boldsymbol{r}),$$

and

$$\psi_{a6}(-\boldsymbol{r}) = \varepsilon_C \psi_{a6}(\boldsymbol{r}).$$

Combining these results with spatial parity we find that, if we define

$$\varepsilon = \varepsilon_C \varepsilon_P$$

then for $\varepsilon = 1$ $(\varepsilon_C = \varepsilon_P)$

$$\boldsymbol{H} = \chi_0 = \phi_0 = \eta = 0$$

and for $\varepsilon = -1$ $(\varepsilon_C = -\varepsilon_P)$

$$\boldsymbol{E} = \boldsymbol{\chi} = \boldsymbol{\phi} = a = 0.$$

Thus we are left with the following classifications:

a) $\varepsilon = 1,\ \varepsilon_P = (-1)^j$,

$$\phi_0 = \eta = \chi_0 = \boldsymbol{H} = 0, \qquad a \neq 0,$$

b) $\varepsilon = 1,\ \varepsilon_P = (-1)^{j+1}$,

$$\chi_0 = \phi_0 = \eta = \boldsymbol{H} = 0, \qquad a = 0,$$

c) $\varepsilon = -1,\ \varepsilon_P = (-1)^{j+1}$,

$$\boldsymbol{E} = \boldsymbol{\chi} = \boldsymbol{\phi} = a = 0,$$

and we obtain the splitting into three cases, as discussed in the following sections.

### 2.3. Eigenvectors of definite total angular momentum

The eigenvalue equation $J_z\psi = m\psi$ gives

$$-i\,\frac{\partial a}{\partial \varphi} = ma, \qquad -i\,\frac{\partial E_z}{\partial \varphi} = mE_z,$$

$$-i\,\frac{\partial}{\partial \varphi}(E_x \pm iE_y) = (m\pm 1)(E_x \pm iE_y)$$

with similar equations for the other scalars and vectors. Furthermore, the eigenvalue equation $J^2\psi = j(j+1)\psi$ gives

$$L^2 a = j(j+1)a, \quad L^2 E + 2iL \wedge E + 2E = j(j+1)E.$$

As is well known [5], the eigenfunctions are $a \sim Y_{jm}$ for the scalars, and the three orthogonal vectors $\boldsymbol{r}Y_{jm}$, $\boldsymbol{P}Y_{jm}$ and $\boldsymbol{L}Y_{jm}$ for the vectors of parities $(-1)^{j+1}$, $(-1)^{j+1}$ and $(-1)^j$ respectively.

Now we write

$$a = f(r)Y_{jm}, \quad \eta = g(r)Y_{jm},$$

$$\chi_0 = h(r)Y_{jm}, \quad \phi_0 = k(r)Y_{jm},$$

$$\boldsymbol{\phi} = u_1(r)\boldsymbol{L}Y_{jm} + rv_1(r)\boldsymbol{P}Y_{jm} + \frac{w_1(r)}{r}\boldsymbol{r}Y_{jm},$$

$$\boldsymbol{\chi} = u_2(r)\boldsymbol{L}Y_{jm} + rv_2(r)\boldsymbol{P}Y_{jm} + \frac{w_2(r)}{r}\boldsymbol{r}Y_{jm},$$

$$\boldsymbol{E} = u_3(r)\boldsymbol{L}Y_{jm} + rv_3(r)\boldsymbol{P}Y_{jm} + \frac{w_3(r)}{r}\boldsymbol{r}Y_{jm},$$

$$\boldsymbol{H} = u_4(r)\boldsymbol{L}Y_{jm} + rv_4(r)\boldsymbol{P}Y_{jm} + \frac{w_4(r)}{r}\boldsymbol{r}Y_{jm},$$

and we classify the solutions into two families of definite parities

$$\varepsilon = (-1)^{j+1} \quad \text{and} \quad \varepsilon = (-1)^j.$$

For $\varepsilon = (-1)^{j+1}$, we get

$$a = \chi_0 = 0, \quad \phi_0 = k(r)Y_{jm}(\theta, \varphi), \quad \eta = g(r)Y_{jm}.$$

$$\boldsymbol{E} = u_3\boldsymbol{L}Y_{jm}, \quad \boldsymbol{\chi} = u_2(r)\boldsymbol{L}Y_{jm},$$

$$\boldsymbol{\phi} = \left(rv_1\boldsymbol{P} + \frac{w_1}{r}\boldsymbol{r}\right)Y_{jm}, \quad \boldsymbol{H} = \left(rv_4\boldsymbol{P} + \frac{w_4}{r}\boldsymbol{r}\right)Y_{jm}$$

and for $\varepsilon = (-1)^j$, we get

$$\phi_0 = \eta = 0, \quad a = f(r)Y_{jm}, \quad \chi_0 = h(r)Y_{jm},$$

$$\boldsymbol{\phi} = u_1\boldsymbol{L}Y_{jm}, \quad \boldsymbol{H} = u_4\boldsymbol{L}Y_{jm},$$

$$\boldsymbol{\chi} = \left(rv_2\boldsymbol{P} + \frac{w_2}{r}\boldsymbol{r}\right)Y_{jm}, \quad \boldsymbol{E} = \left(rv_3\boldsymbol{P} + \frac{w_3}{r}\boldsymbol{r}\right)Y_{jm}.$$

Thence, the radial equations for $\varepsilon = (-1)^{j+1}$ are:
Case $(i)$

$$\left(M-\frac{e^2}{r}\right)k(r) = 2\kappa g(r), \qquad 2g = 2\mu r v_1 - M r v_4,$$

$$-2ig' = 2\mu w_1 - \left(M+\frac{e^2}{r}\right)w_4, \qquad 2\kappa u_2 + M u_3 = 0,$$

$$2\left[\frac{-j(j+1)}{r}v_4 + \frac{i}{r^2}\frac{d}{dr}(r^2 w_4)\right] = \left(M+\frac{3e^2}{r}\right)g - 2\kappa k,$$

$$\frac{2j(j+1)}{r}u_2 = -2\mu w_4 + \left(M+\frac{e^2}{r}\right)w_1,$$

$$2i\frac{d}{dr}(ru_2) = -2\mu r v_4 + \left(M+\frac{2e^2}{r}\right)r v_1,$$

and

$$2\left(\frac{w_1}{r} + \frac{i}{r}\frac{d}{dr}(rv_1)\right) = 2\kappa u_3 + \left(M+\frac{2e^2}{r}\right)u_2.$$

The above equations reduce to two coupled equations in $u_2$ and $g$; whereas the other six functions are expressed in terms of these two functions as follows:

$$u_3 = -2\kappa u_2/M, \qquad k = 2\kappa g\Big/\left(M-\frac{e^2}{r}\right),$$

$$rv_1 = 2\left[\mu g - iM\frac{d}{dr}(ru_2)\right]\Big/\left[4\mu^2 - M\left(M+\frac{2e^2}{r}\right)\right],$$

$$rv_4 = 2\left[\left(M+\frac{2e^2}{r}\right)g - 2i\mu\frac{d}{dr}(ru_2)\right]\Big/\left[4\mu^2 - M\left(M+\frac{2e^2}{r}\right)\right],$$

$$w_1 = -2\left[2i\mu g' + \left(M+\frac{e^2}{r}\right)\frac{j(j+1)}{r}u_2\right]\Big/\left[4\mu^2 - \left(M+\frac{e^2}{r}\right)^2\right],$$

$$w_4 = -2\left[i\left(M+\frac{e^2}{r}\right)g' + 2\mu j(j+1)\frac{u_2}{r}\right]\Big/\left[4\mu^2 - \left(M+\frac{e^2}{r}\right)^2\right],$$

and the two coupled equations are

$$\tfrac{1}{4}\left(M+\frac{2e^2}{r}-4\frac{\kappa^2}{M}\right)ru_2=\frac{2i\mu g'+\left(M+\dfrac{e^2}{r}\right)\dfrac{j(j+1)}{r}u_2}{\left(M+\dfrac{e^2}{r}\right)^2-4\mu^2}$$

$$+\frac{d}{dr}\left\{\frac{2i\mu g+M\dfrac{d}{dr}(ru_2)}{4\mu^2-M\left(M+\dfrac{2e^2}{r}\right)}\right\}$$

and

$$\tfrac{1}{4}\left\{M+\frac{3e^2}{r}-\frac{4\kappa^2}{M-\dfrac{e^2}{r}}\right\}r^2g$$

$$=j(j+1)\left[\left(M+\frac{2e^2}{r}\right)g-2i\mu\frac{d}{dr}(ru_2)\right]\Big/\left[M\left(M+\frac{2e^2}{r}\right)-4\mu^2\right]$$

$$+\frac{d}{dr}\left\{\frac{r^2\left[\left(M+\dfrac{e^2}{r}\right)g'-2\mu j(j+1)\dfrac{u_2}{r}\right]}{4\mu^2-\left(M+\dfrac{e^2}{r}\right)^2}\right\}.$$

These coupled equations degenerate into two separate solutions for equal masses $\mu = 0$.
Case (*ii*)

The radial equations for $\varepsilon = (-1)^j$ are

$$\left(M-\frac{e^2}{r}\right)h=-2\mu f,\qquad Mu_4=2\mu u_1,$$

$$-2f=2\kappa rv_2+Mrv_3,\qquad 2if'=2\kappa w_2+\left(M+\frac{e^2}{r}\right)w_3,$$

$$2\left[\frac{-j(j+1)}{r}v_3+\frac{i}{r^2}\frac{d}{dr}(r^2w_3)\right]=\left[M+\frac{3e^2}{r}\right]f+2\mu h,$$

$$-2i\frac{d}{dr}(ru_1)=2\kappa rv_3+\left(M+\frac{2e^2}{r}\right)rv_2,$$

$$-2\frac{j(j+1)}{r}u_1 = 2\kappa w_3 + \left(M+\frac{e^2}{r}\right)w_2,$$

$$-2\left(\frac{w_2}{r}+\frac{i}{r}\frac{d}{dr}(rv_2)\right) = -2\mu u_4 + \left(M+\frac{2e^2}{r}\right)u_1.$$

Here again these eight equations reduce actually to only two coupled differential equations in terms of $u$ and $f$; and all other six radial functions are expressible in terms of these two functions. One verifies easily that

$$h = -2\mu f\Big/\left(M-\frac{e^2}{r}\right), \quad u_4 = 2\mu u_1/M,$$

$$rv_2 = -2\left[2\kappa f - iM\frac{d}{dr}(ru_1)\right]\Big/\left[4\kappa^2 - M\left(M+\frac{2e^2}{r}\right)\right],$$

$$rv_3 = -2\left[2i\kappa\frac{d}{dr}(ru_1) - \left(M+\frac{2e^2}{r}\right)f\right]\Big/\left[4\kappa^2 - M\left(M+\frac{2e^2}{r}\right)\right],$$

$$w_2 = 2\left[2i\kappa f' + \left(M+\frac{e^2}{r}\right)j(j+1)\frac{u_1}{r}\right]\Big/\left[4\kappa^2 - \left(M+\frac{e^2}{r}\right)^2\right],$$

and

$$w_3 = 2\left[2i\kappa f' + \left(M+\frac{e^2}{r}\right)j(j+1)\frac{u_1}{r}\right]\Big/\left[4\kappa^2 - \left(M+\frac{e^2}{r}\right)^2\right],$$

and the two coupled differential equations are

$$\tfrac{1}{4}\left(M+\frac{2e^2}{r}-\frac{4\mu^2}{M}\right)ru_1 = \frac{d}{dr}\left\{\frac{2i\kappa f + M\frac{d}{dr}(ru_1)}{4\kappa^2 - M\left(M+\frac{2e^2}{r}\right)}\right\} - \frac{2i\kappa f' + \left(M+\frac{e^2}{r}\right)j(j+1)\frac{u_1}{r}}{4\kappa^2 - \left(M+\frac{e^2}{r}\right)^2},$$

and

$$\tfrac{1}{4}\left(M+\frac{3e^2}{r}-\frac{4\mu^2}{M-\frac{e^2}{r}}\right)r^2 f = \frac{j(j+1)\left[2i\kappa\frac{d}{dr}(ru_1) - \left(M+\frac{2e^2}{r}\right)f\right]}{4\kappa^2 - M\left(M+\frac{2e^2}{r}\right)} + \frac{d}{dr}\left\{\frac{r^2\left[\left(M-\frac{e^2}{r}\right)f' - 2i\kappa j(j+1)\frac{u_1}{r}\right]}{4\kappa^2 - \left(M+\frac{e^2}{r}\right)^2}\right\}.$$

### 3.1. Positronium case

For positronium $m_1 = m_2 = \kappa$, $\mu = 0$. In this case the solutions are easily separable:
(*i*) For $\varepsilon = (-1)^{j+1}$ we get two separate solutions
(*i.a*) $a = \chi_0 = \boldsymbol{\phi} = \boldsymbol{E} = \boldsymbol{\chi} = 0$,

$$\tilde{\psi} = \left[\beta_5 g - \beta_0 k - i\boldsymbol{\sigma}\cdot\left(rv_4\boldsymbol{P} + \frac{w_4}{r}\boldsymbol{r}\right)\right] Y_{jm},$$

where

$$k = 2\kappa g \Big/ \left(M - \frac{e^2}{r}\right), \quad v_4 = -2g/Mr, \quad w_4 = 2ig'/(M+e^2/r),$$

and

$$\frac{1}{r^2}\frac{d}{dr}\left\{\frac{r^2 g'}{M+\dfrac{e^2}{r}}\right\} + \left[\tfrac{1}{4}\left(M + \frac{3e^2}{r} - \frac{4\kappa^2}{M-\dfrac{e^2}{r}}\right) - \frac{j(j+1)}{Mr^2}\right] g = 0.$$

(*i.b*) $\eta = \phi_0 = \psi_0 = \boldsymbol{H} = 0$,

$$\boldsymbol{\chi} = u_2 \boldsymbol{L} Y_{jm}, \quad \boldsymbol{E} = u_3 \boldsymbol{L} Y_{jm}, \quad \boldsymbol{\phi} = \left(rv_1\boldsymbol{P} + \frac{w_1}{r}\boldsymbol{r}\right) Y_{jm},$$

where

$$u_3 = -2\kappa u_2/M,$$

$$v_1 = 2i\frac{d}{dr}(ru_2)\Big/\left(M+\frac{2e^2}{r}\right), \quad w_1 = 2j(j+1)u_2\Big/ r\left(M+\frac{e^2}{r}\right),$$

and

$$\frac{1}{r}\frac{d}{dr}\left\{\frac{\dfrac{d}{dr}(ru_2)}{M+\dfrac{2e^2}{r}}\right\} + \left[\tfrac{1}{4}\left(M+\frac{2e^2}{r} - \frac{4\kappa^2}{M}\right) - \frac{j(j+1)}{r^2\left(M+\dfrac{e^2}{r}\right)}\right] u_2 = 0.$$

(*ii*) For $\varepsilon = (-1)^j$, we still have two coupled equations, but $\chi_0 = \boldsymbol{H} = 0$,

$$a = fY_{jm}, \quad \boldsymbol{\phi} = u_1 \boldsymbol{L} Y_{jm},$$

$$\boldsymbol{\chi} = \left(rv_2\boldsymbol{P} + \frac{w_2}{r}\boldsymbol{r}\right) Y_{jm}, \quad \boldsymbol{E} = \left(rv_3\boldsymbol{P} + \frac{w_3}{r}\boldsymbol{r}\right) Y_{jm},$$

where

$$\left[4\kappa^2 - M\left(M+\frac{2e^2}{r}\right)\right] rv_2 = 2iM\frac{d}{dr}(ru_1) - 4\kappa f,$$

$$\left[4\kappa^2 - M\left(M + \frac{2e^2}{r}\right)\right] r v_3 = 2\left(M + \frac{2e^2}{r}\right) f - 4i\kappa \frac{d}{dr}(r u_1),$$

$$\left[4\kappa^2 - \left(M + \frac{2e^2}{r}\right)^2\right] w_2 = 4i\kappa f' + 2\left(M + \frac{e^2}{r}\right) j(j+1) \frac{u_1}{r},$$

$$\left[4\kappa^2 - \left(M + \frac{e^2}{r}\right)^2\right] w_3 = -4\kappa j(j+1) \frac{u_1}{r} - 2i\left(M + \frac{e^2}{r}\right) f',$$

and the two coupled differential equations are

$$\tfrac{1}{4}\left(M + \frac{2e^2}{r}\right) r u_1 = \frac{d}{dr}\left\{\frac{2i\kappa f + M \dfrac{d}{dr}(r u_1)}{4\kappa^2 - M\left(M + \dfrac{2e^2}{r}\right)}\right\} - \frac{2i\kappa f' + \left(M + \dfrac{e^2}{r}\right) j(j+1) \dfrac{u_1}{r}}{4\kappa^2 - \left(M + \dfrac{e^2}{r}\right)^2},$$

and

$$\tfrac{1}{4}\left(M + \frac{3e^2}{r}\right) r^2 f = \frac{j(j+1)\left[2i\kappa \dfrac{d}{dr}(r u_1) - \left(M + \dfrac{2e^2}{r}\right) f\right]}{4\kappa^2 - M\left(M + \dfrac{2e^2}{r}\right)} + \frac{d}{dr}\left\{\frac{r^2\left[\left(M + \dfrac{e^2}{r}\right) f' - 2i\kappa j(j+1) \dfrac{u_1}{r}\right]}{4\kappa^2 - \left(M + \dfrac{2e^2}{r}\right)^2}\right\}.$$

### 3.2. Solution of the radial equation for the case (*i.a*)

As $r \to \infty$, the differential equation for $g$ gives $g_\infty \sim e^{-Kr}$ where $K = \left(\kappa^2 - \dfrac{M^2}{4}\right)^{1/2}$, whereas for $j \neq 0$ and $r \to 0$ we do not obtain a regular solution. Thus we are left with the $j = 0$ case.

Let $g_0 = r^\sigma$ (as $r \to 0$) hence

$$\sigma = -1 \pm \sqrt{1 - \frac{3\alpha^2}{4}}.$$

The root with the negative sign diverges nearly quadratically with $\frac{1}{r}$ and we exclude it and are left with the very weakly divergent solution

$$\sigma = \left(1 - \frac{3\alpha^2}{4}\right)^{1/2} - 1 \approx - \frac{3\alpha^2}{8} \sim 10^{-5}.$$

However, in this case $v_4 = \frac{-2g}{Mr}$ will be singular, leading to a non-normalizable solution. Thence we are left with the spherically symmetric solution

$$\eta = g(r), \quad \phi_0 = 2\kappa g \Big/ \left(M - \frac{\alpha}{r}\right),$$

$$\tilde{\psi} = i\gamma_5 \left(g - \frac{2\kappa g}{M - \frac{\alpha}{r}} \gamma_0\right) - \frac{2g'}{Mr+\alpha} \boldsymbol{\sigma} \cdot \boldsymbol{r}$$

with parity $\varepsilon = -1$.

Writing $g = Ge^{-Kr}$, we can obtain the energy levels, for it is sufficient to consider the asymptotic approximation $G_\infty$ of $G$. We keep only orders of $r^2$ and $r$, and neglect orders $O(1/r^2)$. Then, the asymptotic differential equation is

$$M^2 r^2 G''_\infty + (2M^2 r - 2M^2 K r^2) G'_\infty + [(2M^2 K - 2\alpha M K^2 + \tfrac{1}{2} \alpha M^2) r$$
$$- \tfrac{1}{4} (\alpha^2 M^2 - 2\alpha^2 K^2 - \alpha M K)] G_\infty = 0.$$

This equation holds for $r \gg \frac{\alpha}{M} \approx \frac{\alpha}{2\kappa}$. In fact, for $r < \frac{\alpha}{2\kappa}$, virtual annihilation and creation of the positronium takes place, and the behaviour of the wave function in this region differs very much from the solution of our differential equation for $G$. Writting $G_\infty = \sum_{n=0}^{N} A_n r^{n+\sigma}$. The coefficient of $A_0$ gives the equation

$$M^2 \sigma(\sigma+1) + \tfrac{1}{4} (\alpha^2 M^2 - 2\alpha^2 K^2 - \alpha M K) = 0.$$

On the other hand, since $A_{n+1}/A_n \to \frac{2K}{n}$ as $n \to \infty$, the convergence, as $r \to \infty$, requires that $G_\infty$ should be a polynomial, of degree $N+\sigma$, say. Hence, we get the equation for energy levels

$$\left(\frac{M_N}{2\kappa}\right)^2 = \tfrac{1}{2} + \tfrac{1}{2} \sqrt{1 - \frac{\alpha^2}{(N+\sigma-1)^2 + \alpha^2}}$$

in the first order of $\alpha^2$, we get the usual Rydberg's energy levels, with $N-1$ instead of $N$:

$$M_N \approx 2\kappa - \frac{\kappa\alpha^2}{4(N-1)^2}$$

and using the recurrence relation we obtain

$$g_\infty = r^\sigma L_N^{(2\sigma+1)}(2Kr)e^{-Kr},$$

where $L_n^{(\alpha)}$ is the Laguerre polynomial [6]. The nonrelativistic expression, obtained from Schrodinger's equation is given by

$$g_\infty = L_N^{(1)}(2K_0 r)e^{-K_0 r},$$

where

$$K_0 = \frac{\alpha\kappa}{2N}.$$

We note that $K \approx K_0$ (except that $N-1$ replaces $N$ in the expression for $K$), and that we obtain Schrodinger's solution if we put $\sigma = 0$. In fact, in the lowest order of $\alpha$, we have $M = 2\kappa$, $K = \dfrac{\kappa\alpha}{2(N-1)}$ and $\sigma \simeq -\alpha^2\left(1-\dfrac{1}{4(N-1)}\right)$, which we can neglect.

On the other hand, for $r \ll \dfrac{\alpha}{M}$, we approximate the differential equation for $g$ neglecting the order $O(r^2)$. Then, we write

$$g_0 = \sum A_n r^{n+\sigma},$$

which converges for $r \leqslant \dfrac{\alpha}{M} \approx \dfrac{\alpha}{2\kappa}$, neglecting $\alpha^2$ and thus also $\sigma$ as compared with $n$, we obtain

$$g_0 \approx \sum_{n=1}^{\infty} \frac{\left(-\dfrac{Mr}{\alpha}\right)^{n+\sigma}}{n(n+1)(n+2)}$$

which is the approximate function $g$ near the origin.

Now, we study the behaviour near $r = \dfrac{\alpha}{M}$, since $\phi_0 = \dfrac{2\kappa g}{M - \dfrac{\alpha}{r}}$, then we require that $g \to 0$ as $r \to \dfrac{\alpha}{M}$. Consider the solution around the point $r = \dfrac{\alpha}{M}$. Let $r = x + \dfrac{\alpha}{M}$.

Then in the first degree in $x$ (neglecting $o(x^2)$), we get

$$\frac{d^2 g}{dx^2} = \frac{2\alpha\kappa^2}{M}\frac{g}{x}.$$

The solution for $x \to 0$ is

$$g = e^{\lambda x \ln x} = x^{\lambda x}, \qquad g'' \cong \frac{\lambda g}{x},$$

hence, $\lambda = \frac{2\kappa^2\alpha}{M}$. As $x \to 0$ we find that $g \to 1$. Thus an analytic solution around $x = 0$ leads to $g$ finite and thus $\phi_0 \to \infty$ as $x \to 0$. This is excluded physically and one cannot have an analytic continuation at $x = 0$. We should then have two solutions which are valid for $r < \frac{\alpha}{M}$ and the other is valid for $r > \frac{\alpha}{M}$ such that $g = 0$ at $r = \frac{\alpha}{M}$ on both sides[2].

Since $r = \frac{\alpha}{M}$ is a singular point we have to distinguish between two solutions; the interior solution for $r < \frac{\alpha}{M}$ and the exterior solution for $r > \frac{\alpha}{M}$. We seek the solutions which have the same behaviour at $r \sim \frac{\alpha}{M}$.

Consider now the exterior solution which vanishes at infinity, we write

$$G(r) = \sum_n G_n(Kr)^{n+S}$$

using the recurrence relation we obtain for $n \to +\infty$

$$G_n \sim \frac{2^n}{n!}.$$

Thus

$$\sum_{n=0}^{\infty} G_n(Kr)^{n+\sigma} = (Kr)^\sigma e^{2Kr},$$

which means that $g \to +\infty$. Hence, we have to cut the series at some $n = +N, N > 0$. Also if one cuts the series at $n = 0$, we get two recurrence series — one from below and one from above (in ascending and in decreasing powers). One verifies easily that this recurrenoe relation leads to two additional conditions on the coefficients, which are not self-consistent. Thus we have to allow for $n < 0$ and take the infinite series for negative $n$. That is

$$G(r) = \sum_{n=-\infty}^{n=+N} G_n(Kr)^{n+S}.$$

By considering the asymptotic coefficients $G_n$ as $n \to -\infty$ and using the recurrence relation, we obtain

$$\sum_{n=-\infty} G_n(Kr)^{n+\sigma} \approx \frac{\pm\left(\frac{Mr}{\alpha}\right)^\sigma}{1 \mp \frac{\alpha}{Mr}} \sim \frac{(Mr)^{S+1}}{Mr \mp \alpha},$$

[2] N.B. In fact, we cannot find a solution of the form $g = X^S$ for this equation.

which converges for $r > \frac{\alpha}{M}$. This gives the exterior solution, which has either a singular $G$ at $r = \frac{\alpha}{M}$ or a finite solution.

The energy levels are obtained from the recurrence relation taking $n = N$,

$$\left(\frac{M_N}{2\kappa}\right)^2 = \tfrac{1}{2}+\tfrac{1}{2}\sqrt{1-\frac{\alpha^2}{(N+S-1)^2+\alpha^2}},$$

since

$$S = \sigma-1 = \sqrt{1-\frac{3\alpha^2}{4}}-2,$$

which gives in the nonrelativistic limit

$$M_N = 2\kappa-\frac{\kappa\alpha^2}{2(N-2)^2}.$$

Now we consider the interior solution for $r < \frac{\alpha}{M}$. Here the origin is included, but $r = \infty$ is excluded. Thus we consider the solution as a power series

$$g = \sum_{n=0}^{\infty} g_n(Kr)^{n+\sigma},$$

where $\sigma$ is to be determined. The indicial equation with the condition $g_0 \neq 0$ gives

$$\sigma = -1\pm\sqrt{1-\frac{3\alpha^2}{4}}.$$

The weakly singular solution is

$$\sigma = \sqrt{1-\frac{2\alpha^2}{4}}-1 \approx -\frac{3\alpha^2}{8}$$

($\sigma \approx 0$ in nonrelativistic theory).

By considering the asymptotic recurrence relation as $n \to \infty$ we seek a solution which gives the same asymptotic behaviour at $r = \frac{\alpha}{M}$ as the exterior solution. The recurrence relation takes the form

$$\nu^2 g_{n-2} = \alpha^2 g_n, \qquad \nu = M/K.$$

Thus

$$\sum_{n=0}^{\infty} g_n(Kr)^{n+\sigma} = \frac{\left(\frac{Mr}{\alpha}\right)^\sigma}{1\mp\frac{Mr}{\alpha}} \sim \frac{(Mr)^\sigma}{Mr\mp\alpha}$$

for $r < \frac{\alpha}{M}$ which has the same asymptotic behaviour at $r = \frac{\alpha}{M}$, then,

$$\sigma = S+1 \quad \text{or} \quad S = \sigma - 1.$$

This connects the interior and exterior solutions.

## 4. *Conclusion*

In the previous section we discussed in detail the solution of the differential equation for $g$ (case *i.a*). Similarly, we obtained closed form expressions for the solutions of the differential equation for $u_2$ (case *i.b*) and the coupled differential equations for $u_1$ and $f$ (case *ii*). For $u_2$ we obtained a solution of this form

$$u_2 = \frac{U}{r} e^{-Kr},$$

where $U$ is a polynomial in $r$

$$U_\infty = r^S L_N^{2S-1}(2Kr)e^{-Kr}.$$

Here again there is a singularity at $r = \frac{2\alpha}{M}$ and solutions have been obtained to match each other at that point with

$$S = \sigma - 1$$

for the interior and exterior solutions.

For the coupled differential equations for $u_1$ and $f$ we assumed the form

$$u_1 = \frac{U}{Kr} e^{-Kr} \quad \text{and} \quad f = Fe^{-Kr}.$$

In this case we have two singularities: at

$$r_1 = \frac{\alpha M}{2K^2},$$

and

$$r_2 = \frac{\alpha}{2\kappa - M} \quad \text{(note: } r_1 \text{ is less than } r_2\text{)}.$$

A regular solution in the whole line $r$ is obtained with $S = \sigma - 1$ for the interior solution $\left(r \leqslant \frac{\alpha}{M}\right)$ and the exterior solution $\left(r \geqslant \frac{\alpha}{M}\right)$. In this case we notice that

$$r_1 = \frac{\alpha M}{2K^2}, \quad r_2 = \frac{\alpha}{2\kappa - M}, \quad r_0 = \frac{\alpha}{M}$$

are such that

$$\frac{r_1}{r_0} = \frac{M^2}{2K^2} \gg 1, \quad \frac{r_2}{r_1} = \frac{M}{2\kappa - M} \gg 1,$$

and $r_0 < r_1 < r_2$. That is, the singularities of the differential equation (not of the wave function) are in the exterior solution.

Finally, we note that the equations proposed look promising for studying other bound state systems, e.g. the $q\bar{q}$ system and the confinement of the quarks [8]. Also, a great advantage of the equations obtained over the Bethe–Salpeter formalism lies in the fact that we were able to avoid all difficulties connected with the relative time (relative energy) problem as well as an easier and analytically solvable set of differential equations are obtained in our case.

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