EXTERNAL CHARGES IN SU(2) GAUGE THEORY AND TOPOLOGY*

BY H. ARODŹ

Department of Field Theory, Institute of Physics, Jagellonian University, Cracow**

(Received January 10, 1980)

We observe that the external charges in SU(2) gauge theory can be classified accordingyl to their Hopf index. The relationship of this topological classification and the winding number classification of gauge transformations is indicated. The Coulomb solution of the Yang–Mills equations with external charges is obtained with the aid of the gauge transformations characterised by a nonzero winding number.

1. Introduction

Recently, one observes an increasing interest in solutions of classical Yang–Mills equations with external charges [1–8]. While relevance of these solutions for quantum theory is still not clear [4], they nevertheless provide a very interesting insight into interactions of color charges. There have been discovered several types of solutions of Yang–Mills equations for a given external source (for a review see [4]). For continuous external charge distributions these are: the static Coulomb solution and its time dependent generalizations [2], the so called non-abelian Coulomb solution and its time dependent generalizations [4]. There exist also bifurcating solutions — they appear only if the external source is sufficiently strong [4]. Only the static Coulomb solution and its time dependent generalizations are known in an exact form. The others are constructed only in the form of a perturbative series (in the external source). Apart from the great progress in the field, there is a number of problems unsettled as yet. In this note we address ourselves to one such undiscussed problem, namely the question of whether any topological numbers are involved or not.

We observe that there exists a class of continuous external charge distributions that admit a topological number, namely the Hopf index [9]. This class of external sources is characterised in detail in Section 3. The most important feature of it is that the color direction of the external source approaches a constant vector when $r \to \infty$ (this is equivalent

* This work was supported in part by the Polish Ministry of Higher Education, Science and Technology, Project M.R.I.7.

** Address: Instytut Fizyki UJ, Reymonta 4, 30-059 Kraków, Poland.

to a compactification of R^3 to S^3 — the standard procedure required when introducing topological numbers). We also observe that the usual method [2, 8] of obtaining the static Coulomb solution, by means of a gauge transformation, works also in the case of the external charge distributions with the nonzero Hopf index. However, the method now requires the use of the so called large gauge transformations characterised by a nonzero winding number. We argue that the external charge distributions with nonzero Hopf index lead in a natural way to the non-abelian Coulomb solution of Jackiw, Jacobs and Rebbi [4].

The plan of this note is the following. In Section 2 we briefly describe the method of obtaining the static Coulomb solution and we fix the notations. In Section 3 we describe the external charge distributions with nonzero Hopf index and we show that the corresponding Yang–Mills potentials can be obtained with the aid of the gauge transformations characterised by a nonzero winding number. Section 4 is devoted to some final remarks. In the Appendix we describe briefly a definition of the Hopf index.

2. *The static Coulomb solution*

Classical Yang–Mills equations (the gauge group is SU(2)) have the following form

$$\partial_\nu \hat{F}^\nu{}_\mu - ig[\hat{A}_\nu, \hat{F}^\nu{}_\mu] = -g\hat{j}_\mu, \tag{1}$$
$$\hat{F}_{\nu\mu} = \partial_\nu \hat{A}_\mu - \partial_\mu \hat{A}_\nu - ig[\hat{A}_\mu, \hat{A}_\nu],$$

where the current $\hat{j}_\mu$ is assumed to be of the form

$$\hat{j}_\mu(x) = \delta_{\mu 0}\hat{\varrho}(\vec{x}). \tag{2}$$

From (1) it follows that

$$\partial_\mu \hat{j}^\mu - ig[\hat{A}_\mu, \hat{j}^\mu] = 0. \tag{3}$$

The current (2) describes static external color charges, with charge density $\vec{\varrho}(\vec{x})$. Here we use the following notations

$$\hat{A}_\mu = \tfrac{1}{2}\sigma^a A^a_\mu, \quad \hat{\varrho}(\vec{x}) = \tfrac{1}{2}\sigma^a \varrho^a(\vec{x}) = \tfrac{1}{2}\vec{\sigma}\vec{\varrho}, \tag{4}$$

where σ^a are Pauli matrices, $a = 1, 2, 3$.

Equations (1), (2) are covariant under the gauge transformations

$$\hat{A}'_\mu = \omega^{-1}\hat{A}_\mu\omega + \frac{i}{g}\,\omega^{-1}\partial_\mu\omega,$$
$$\hat{F}'_{\mu\nu} = \omega^{-1}\hat{F}_{\mu\nu}\omega, \quad \hat{j}'_\mu = \omega^{-1}\hat{j}_\mu\omega, \tag{5}$$

where $\omega = \omega(t, \vec{x})$ is an SU(2) matrix valued function of $(t, \vec{x})$. This covariance can be used to rotate the only nonvanishing component of the current to the so called (4) abelian frame, where

$$\hat{j}'_0(\vec{x}) = \frac{\sigma^3}{2}\,\varrho'^3(\vec{x}). \tag{6}$$

In this frame Eqs. (1), (3) can be easily satisfied with the Ansatz

$$\hat{A}'_\mu = \delta_{\mu 0} A'^3_0(\vec{x}) \frac{\sigma^3}{2} .$$

All nonlinear terms vanish and one is left with the usual Poisson equation

$$\Delta A'^3_0(\vec{x}) = g\varrho'^3(\vec{x}) \tag{7}$$

which, in the case of $\varrho'^3(\vec{x})$ vanishing at infinity, has the Coulomb solution (we assume the usual boundary conditions at infinity)

$$\begin{aligned}\hat{A}'_0(\vec{x}) &= -\frac{g}{4\pi}\int \frac{1}{|\vec{x}-\vec{x}'|}\,\hat{\varrho}'(\vec{x}')d^3\vec{x}' \\ &= -\frac{g}{4\pi}\int \frac{1}{|\vec{x}-\vec{x}'|}\,\omega^{-1}(\vec{x}')\hat{j}_0(\vec{x}')\omega(\vec{x}')d^3\vec{x}'.\end{aligned} \tag{8}$$

The constraint (3) is satisfied automatically in the abelian frame. Coming back to the initial frame one gets

$$\hat{A}_\mu(\vec{x}) = \omega(\vec{x})\hat{A}'_\mu(\vec{x})\omega^{-1}(\vec{x}) - \frac{i}{g}\,\partial_\mu\omega(\vec{x})\omega^{-1}(\vec{x}), \tag{9}$$

where

$$\hat{A}'_\mu(\vec{x}) = A'^3_0 \frac{\sigma^3}{2}\,\delta_{\mu 0}$$

is given by (8).

The abelian frame solution (8) vanishes when $\hat{\varrho}' \to 0$. In Ref. [4] it is shown that there exist solutions which do not vanish when $\hat{\varrho}' \to 0$ in the abelian frame — they are called the non-abelian Coulomb solutions.

The only constraint $\omega(\vec{x})$ is subject to is

$$\omega^{-1}(\vec{x})\hat{j}_0(\vec{x})\omega(\vec{x}) = \varrho'^3(\vec{x})\frac{\sigma^3}{2} . \tag{10}$$

Neither $\omega(\vec{x})$ nor $\varrho'^3(\vec{x})$ is determined by (10) uniquely. Thus, formula (9) describes a family of solutions, enumerated by allowed values of $\omega(\vec{x})$ and $\varrho'^3(\vec{x})$. In particular, solutions with different $\varrho'^3(\vec{x})$ have in general different energies

$$\mathcal{E} = \tfrac{1}{2}\int d^3\vec{x}[E^{ia}E^{ia} + B^{ia}B^{ia}], \qquad E^{ia} \equiv F^a_{i0}, \qquad B^{ia} \equiv \tfrac{1}{2}\,\varepsilon^{ikl}F^a_{kl}$$

(see, e.g., [5]).

The above method of solving of Yang–Mills equations with external charges was proposed in Ref. [2] for continuous charges and in Ref. [8] for a set of pointlike charges. For a more detailed description of the solution in the case of a continuous charge distribution we refer the reader to papers [2–4].

The most essential step in the above procedure is the choice of ω satisfying (10). We argue in the next section that there exist $\hat{\varrho}(\vec{x})$ such that the corresponding $\omega(\vec{x})$ is topologically nontrivial, i.e., it is characterised by a nonzero winding number. Such $\hat{\varrho}(\vec{x})$ themselves are characterised by the nonzero Hopf index.

3. The external charge distributions and the Hopf index

Let us consider the external charge distributions of the following kind:
a) $\hat{\varrho}(\vec{x}) \neq 0$ for any finite $\vec{x}$,
b) $\hat{\varrho}(\vec{x}) \to 0$ when $|\vec{x}| \to \infty$ (localizability),

c)
$$\lim_{|\vec{x}|\to\infty} \frac{\vec{\varrho}(\vec{x})}{\sqrt{\vec{\varrho}^{\,2}(\vec{x})}} = \vec{c}, \tag{11}$$

where $\vec{c}$ is a constant vector, independent of the direction along which $|\vec{x}| \to \infty$.

When $\hat{\varrho}(\vec{x})$ belongs to the class (11) of the external charges, one can introduce the normalized vector field

$$\vec{e}(\vec{x}) \equiv \frac{\vec{\varrho}(\vec{x})}{\sqrt{\vec{\varrho}^{\,2}(\vec{x})}}, \tag{12}$$

which is constant at infinity. Therefore $\vec{e}(\vec{x})$ can be regarded as a normalized to unity vector field on the 3-dimensional sphere S^3. Then $\vec{x}$ is identified with the stereographic projection coordinates on S^3. The set of possible values of $\vec{e}(\vec{x})$ can be identified with S^2. Thus, $\vec{e}(\vec{x})$ defines a map from S^3 into S^2. Such maps are known to be classified accordingly to their Hopf index [9].

Some of the charges which do not belong to the class (11) can also be classified according to the Hopf index — the only requirement is that the set of $\vec{x}$ such that $\vec{\varrho}(\vec{x}) \neq 0$ should allow a compactification to S^3. For example, if $\vec{\varrho}(\vec{x}) = 0$ outside a ball B, then it is sufficient that $\vec{e}(\vec{x}) =$ constant for $\vec{x}$ from the surface of B.

The Hopf index was already applied, for instance, in discussion of closed vortices in classical field theory [10], magnetohydrodynamics [11a] and nematic liquids [11b]. The vector field $\vec{e}(\vec{x})$ with the Hopf index n can be described as follows [11a]. Consider the constant unit vector field $\vec{e}_z(\vec{x})$ parallel to the z-axis. Then rotate $\vec{e}_z(\vec{x})$ around the radial direction defined by $\vec{n} = \vec{x}/|\vec{x}|$. The angle of rotation $\varphi(r)$, ($r = |\vec{x}|$), should satisfy the conditions

$$\varphi(0) = 0, \qquad \varphi(\infty) = 2\pi n. \tag{13}$$

For example, one can take $\varphi(r) = 4n \operatorname{arctg} r$. The resulting vector field $\vec{e}(\vec{x})$ has the Hopf index n. Such an $\vec{e}(\vec{x})$ forms a torus-like structure in S^3 (hence in R^3).

One can also introduce the topological charge density and the corresponding current [10]. Namely, one can introduce an antisymmetric tensor

$$f_{\mu\nu} = \vec{e}(\partial_\mu \vec{e} \times \partial_\nu \vec{e}), \tag{14}$$

and the corresponding potentials

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu. \tag{15}$$

The current is given by

$$i^\mu = \tilde{f}^{\mu\nu} a_\nu, \tag{16}$$

where $\tilde{f}_{\nu\mu} = \frac{1}{2}\varepsilon_{\mu\nu\varrho\sigma} f^{\varrho\sigma}$ is the dual of $f^{\mu\nu}$. In particular,

$$4\pi n = \int d^3\vec{x}\, i^0 = \int d^3\vec{x}\vec{a}\,\mathrm{rot}\,\vec{a}. \tag{17}$$

Of course, in our static case $\vec{e}$ does not depend on time. Therefore one can take $a_0 = 0$ and then $i^k = 0$, i.e., in the static case only $\imath^0 \neq 0$.

The formulae (14)–(16) make it possible to extend the topological classification also to time-dependent external sources. We will not discuss this problem here.

The crucial point is to what extent the classification of the external charges according to the Hopf index is gauge invariant. Because the gauge transformations (5) of the source $\hat{\varrho}(\vec{x})$ are just $\vec{x}$-dependent rotations of the vector field $\varrho^a(\vec{x})$, it is obvious that one can choose $\omega(\vec{x})$ such that it will unfold the configuration obtained with the prescription (13), thus reducing its Hopf index to zero. For instance, we may take

$$\omega(\vec{x}) = \exp\left[-i\,\frac{2\pi n\vec{x}}{\sqrt{\vec{x}^2+a^2}}\,\vec{\sigma}\right]. \tag{18}$$

The axis of rotation is $\vec{x}/|\vec{x}|$, the angle of rotation

$$\varphi(r) = -2\pi n\frac{r}{\sqrt{r^2+a^2}}$$

satisfies (13) with $n \to -n$.

However the gauge transformation (18) has nontrivial topological properties. It is an element of the n-th Pontriagin class of the gauge transformations [12] related to a classification of mappings from S^3 to S^3. Thus, we obtain the result that the external charge distributions with the Hopf index n are obtained from the charge distribution (6) in the abelian frame by a gauge transformation with the winding number n. As a corollary we obtain the result that, similarly as the winding number, the Hopf index is invariant under small gauge transformations, that is gauge transformations with zero winding number.

Let us recall that the large gauge transformations (i.e., those with $n \neq 0$) have to be treated on a different footing than the small gauge transformations. The reason is that the large gauge transformations cannot be directly implemented in the quantum theory through the Gauss law constraint, which is the generator of the small gauge transformations. In fact, they require to introduce the θ-vacuum, and only after that they are implementable as multiplication by the factor exp $(in\theta)$, [12]. The presence of the θ-vacuum is the physical effect caused by the large gauge transformations.

The Coulomb solution of the Yang–Mills equations with the external charge characterised by $n \neq 0$ is given by (8), (9), where $\omega(\vec{x})$ is given by (18) or any gauge transformation satisfying (10) and homotopically equivalent to (18).

Let us observe that from (9) it follows that $\hat{A}_\mu$ does not vanish when the external charge $\hat{\varrho}(\vec{x})$ tends to zero. It becomes the pure gauge $\omega^{-1}\partial_\mu\omega$. On the other hand one can think of constructing a solution which vanishes when $\hat{\varrho} \to 0$. This could be done by a perturbative expansion in the source and presently we are trying to find it in an exact form. Then, from (8), (9) it follows that the corresponding solution $\hat{A}'_\mu$ in the abelian frame would not vanish when $\hat{\varrho}' \to 0$. But this is the defining feature of the non-abelian Coulomb solution of Jackiw, Jacobs and Rebbi [4]. Thus, the charge distributions with the nonzero Hopf index seem to lead in a very natural way to the non-abelian Coulomb solution for gauge potentials.

4. *Final remarks*

We have found out that there exists a gauge invariant (we mean the small gauge transformations) topological number characterising the external charges in the Yang–-Mills theory. Strictly speaking, the topological classification works for the normalized external charge $\vec{e}(\vec{x}) = \vec{\varrho}(\vec{x})/|\vec{\varrho}|$. Thus, all $\vec{\varrho}(\vec{x})$ which differ by a normalisation factor belong to the same class. This classification of the external charges is intimately related to the large gauge transformations with nonzero winding number.

Because the large gauge transformations imply the θ-vacuum, it is natural to think of a clasical gauge theory which somehow takes into account the presence of this vacuum. Unfortunately, such a classical theory is not constructed as yet. The only thing we dare to say at the moment is that one can expect that the external charge in such a theory is a superposition of the original charge with $n = 0$ and its "copies" with all values of the Hopf index. Thus, the topologically nontrivial charges would be indispensable in that theory. It is not impossible that such a theory would exhibit the commonly believed confining property of QCD.

Another very interesting problem is to generalize the presented topological classification of the external charges in SU(2) gauge theory to SU(n), $n > 2$, gauge group.

APPENDIX

Definition of the Hopf index

Let $\vec{e}(\vec{x})$ be a unit vector field, $|\vec{e}(\vec{x})| = 1$, on 3-dimensional sphere S^3 ($\vec{x}$ are the stereographic coordinates on S^3). The values of $\vec{e}(\vec{x})$ we regard as points of S^2. Let $\vec{e}_0(\vec{x})$ denote some fixed vector field. The vector equation

$$\vec{e}(\vec{x}) = \vec{e}_0(\vec{x}) \tag{A1}$$

is the equation of a closed curve C on S^3. If, for a given $\vec{e}(\vec{x})$, the equation (A1) does not define a closed curve on S^3, then there exists another $\vec{e}(\vec{x})$, homotopically equivalent to $\vec{e}(\vec{x})$, for which (A1) does define a closed curve. Now, if Ω is a closed, connected surface on S^3 having C as the border, then $\vec{e}(\vec{x})$ maps Ω on the sphere S^2. The Hopf index of the vector field $\vec{e}(\vec{x})$ is defined as the number of times $\vec{e}(\vec{x})$ maps Ω onto S^2. It can be shown that the value of the Hopf index does not depend on the particular choice of $\vec{e}_0(\vec{x})$. We take $\vec{e}_0(\vec{x}) = \vec{e}_z$, where $\vec{e}_z$ denotes a constant unit vector parallel to the z-axis. The Hopf index has the same value for all homotopically equivalent $\vec{e}(\vec{x})$.

REFERENCES

[1] J. Mandula, *Phys. Rev.* **D14**, 3497 (1976); M. Magg, *Phys. Lett.* **74B**, 246 (1978).

[2] P. Sikivie, N. Weiss, *Phys. Rev. Lett.* **40**, 1411 (1978); P. Sikivie, N. Weiss, *Phys. Rev.* **D18**, 3809 (1978).

[3] P. Pirilä, P. Prešnajder, *Nucl. Phys.* **B142**, 229 (1978).

[4] R. Jackiw, L. Jacobs, C. Rebbi, *Phys. Rev.* **D20**, 474 (1979); R. Jackiw, P. Rossi, *Stability and Bifurcation in Yang Mills Theory*, MIT preprint, October 1979 (submitted to Phys. Rev.).

[5] E. Corrigan, *Phys. Lett.* **82B**, 407 (1979).

[6] M. Kalb, *Phys. Rev.* **D18**, 2909 (1978).

[7] R. A. Freedman, L. Wilets, S. D. Ellis, E. M. Henley, University of Washington (Seattle) preprint, July 1979.

[8] H. Arodź, *Phys. Lett.* **78B**, 129 (1978).

[9] H. Flanders, *Differential Forms with Applications to the Physical Sciences*, Academic Press, New York, London 1963; P. J. Hilton, *Introduction to Homotopy Theory*, Cambridge 1953.

[10] H. J. de Vega, *Phys. Rev.* **D18**, 2945 (1978); J. Hertel, DESY preprint 76/59 (1976).

[11] a) D. Finkelstein, D. Weil, Yeshiva University preprint (1977); b) R. Shankar, Harvard University preprint, HUTP-77/A043 (1977).

[12] C. G. Callan, R. Dashen, D. J. Gross, *Phys. Rev.* **D17**, 2717 (1978); R. Jackiw, C. Rebbi, *Phys. Rev. Lett.* **37**, 172 (1976).