PLEBAŃSKI CLASSIFICATION OF THE TENSOR OF MATTER*

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This work presents a unified approach to the Plebański classification of the tensor of matter by showing how all the polynomial, tensorial, and spinorial objects used by Plebański arise from the study of a single linear operator, a derivation, on the real Clifford–Dirac algebra. In particular, we show that the classification of the tensor of matter is equivalent to the Petrov–Penrose classification of the conformal Weyl tensor, by way of the principal correlation which exists between them, which gives a positive answer to Plebański's question of whether such a correlation exists. A new interpretation of spinors as elements of a complex projective plane of bivectors emerges. Our approach makes extensive use of the method of simplicial and multivector differentiation, and this method is explained in a series of appendices.

Introduction

In his by now classical paper [1], Plebański thoroughly studies the structure of the tensor of matter considered as a symmetric, traceless matrix, and as a spinorial object, recognizing the relationship of this problem to the problem of the classification of the Weyl tensor using the spinorial methods of Penrose [2]. In [3], we have shown that a far more powerful formalism[1] can be successfully employed in the Petrov–Penrose classification of the Weyl tensor, considered as a bivector operator, and more generally in studying properties of the Riemann curvature tensor. The purpose of this paper is to show that the virtues of our formalism are even more apparent when applied to this related but more complicated problem; it allows us to find the principal correlation whose existence was suspected

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[1] We are referring to the *Spacetime Algebra (STA)* of Hestenes [4, 5], and more generally to the multivector calculus extensively developed in the book *From Clifford Algebra to Geometric Calculus: A Unified Language for Mathematics and Physics*, [6].

by Plebański. It will be seen that our methods are distinctly different than those employed by all others, for example [7, 8], and whereas the results of others can be elegantly restated in our formalism, the converse is not true.

In Section 1, we geometrically extend the inner and outer vector products in the real Dirac algebra so that they apply equally well to "complex" vectors in the real Dirac algebra. By a "complex" vector we mean that vectors and pseudovectors (or trivectors) in the real Dirac algebra are treated as a single mixed quantity. This geometric extension is necessary because we wish to retain the geometric interpretation of complex eigenvalues of symmetric operators that was given in [3] when studying the bivector operator equivalent of the conformal curvature tensor. The extended inner product, when considered on the space of complex vectors, turns this space into a complex 4-dimensional Euclidean space. We take steps to reveal the relationship between this complex 4-dimensional space and Penrose's twistor theory [9].

In Section 2, we study a symmetric trace-free vector operator by extending it to a derivation (see Ref. [10]) on the full Dirac algebra. This derivation, when considered as a bivector operator, is anti-dual symmetric, which corresponds to an anti-linear transformation from the space of "dotted" spinors to "undotted" spinors in the spinor formalism. The derivation, when considered as an operator on complex vectors, is dual symmetric and its characteristic equation is the same as for the trace-free vector operator, but now "complex" eigenvalues take on the geometric meaning of "scalar + pseudoscalar" in the Dirac algebra. The importance of this geometric interpretation is that it allows us to find the principal correlation of this operator to the Petrov–Penrose classification of a dual symmetric bivector operator constructed on a subalgebra orthogonal to a real space-like eigenvector.

In Section 3, we study basic properties of Hermitian forms, such as the law of inertia, both on the complex 4-dimensional Euclidean space of complex vectors, and on the complex 3-dimensional Euclidean space of bivectors. By the Riesz theorem, which is directly established by multivector differentiation, to each Hermitian form there corresponds a unique anti-dual symmetric operator. We give examples of Hermitian forms which are preserved by the respective isometry groups: U(4), U(3, 1), U(2, 2), U(3), U(2, 1), and U(2, 0). The unitary group U(2, 2) is associated with Penrose's twistor theory [9]. See also references [11, 12, and 13].

A rather extensive series of appendices is included. The author feels that this is justified by the fact that most readers are unfamiliar with the mathematical details of the Spacetime Algebra (STA) formalism used here. In fact, some of the material was worked out in a thesis [14], and has not been published elsewhere. Appendix A gives the Pauli algebra homomorphism which is utilized in Section 2. Appendix B explains the method of simplicial differentiation which was first developed in [14], and later more extensively in [6]. It is shown how the method of bivector differentiation employed in [3] is a special case of the more general multidifferentiation defined herein. Appendix C relates the differential scalars of the derivation, defined in Section 2, back to the differential scalars of the trace-free symmetric vector operator in terms of which it is defined. Appendix D relates formulas in the STA formalism to more familiar equivalent formulas in other formalisms, and in particular, to formulas used by Plebański. In addition, a new interpretation of spinors as elements of a complex 2-dimension projective plane of bivectors is given.

In conclusion, whereas Plebański studies properties as reflected at the spinorial or tensorial level, or in the "world" of polynomials, there is a natural, simple and direct relationship between these various worlds, as is exhibited in the study of so simple an object as a derivation on the Dirac algebra. It would seem reasonable to expect, therefore, that whenever the former methods are employed in Physics, great advantage can be gained by exploiting the Spacetime algebra techniques used in this paper.

(In this paper we assume the familiarity of the reader with the results and symbolism used in [3].)

1. Geometric extension of operations in Spacetime Algebra

In this section we briefly review and geometrically extend operations on vectors in the Dirac algebra to "complex" vectors in the Dirac algebra.

By a *complex scalar* in the Dirac algebra $\mathscr{D}$, we mean

$$\tau = \alpha + I\beta, \tag{1.1}$$

where α and β are real numbers and I is the unit pseudoscalar element of $\mathscr{D}$ with the familiar property $I^2 = -1$. By $\bar{\tau}$, we mean the complex conjugate of τ. If e_0 is a unit timelike vector ($e_0^2 = 1$) in $\mathscr{D}$, then we can define the conjugate of τ in terms of e_0 by

$$\bar{\tau} = e_0 \tau e_0 = \alpha - I\beta. \tag{1.2}$$

By a *complex vector* in $\mathscr{D}$, we mean an element $Z \in \mathscr{D}^- \equiv \mathscr{D}_{1+3}$, i.e.,

$$Z = u + Iv = \sum_{u=0}^{3} \tau^u e_u = \text{vector} + \text{trivector}, \tag{1.3}$$

where $u, v \in \mathscr{D}_1$, and $\{e_u\}$ is an orthonormal basis of $\mathscr{D}_1$ (see [3; (I.1)]). The complex conjugate of Z is best defined in terms of the operation of reversion (+), which reverses the order of the geometric products of vectors in terms of which Z is expressed. A discussion of the operation of reversion in STA can be found in [4]. For our purposes, we need only note that

$$\bar{Z} \equiv Z^+ = u - Iv = \bar{\tau}^u e_u \text{ (summation convention)}. \tag{1.4}$$

Note, that when the operation of reversion is applied to a bivector $B \in \mathscr{D}_2$, we get

$$\bar{B} \equiv B^+ = (a \wedge b)^+ = b \wedge a = -a \wedge b = -B. \tag{1.5}$$

Complex conjugation in $\mathscr{D}^-$ satisfies the following important properties:

$$\tau Z = Z\bar{\tau}, \tag{1.6}$$

$$\overline{\tau Z} \equiv (\tau Z)^+ = \bar{Z}\tau = \bar{\tau}\bar{Z} \text{ (note that } \tau^+ = \tau), \tag{1.7}$$

and

$$\overline{Z_1 Z_2} \equiv (Z_1 Z_2)^+ = \bar{Z}_2 \bar{Z}_1, \tag{1.8}$$

which are easy consequences of the definitions (1.2) and (1.4), and the property that $Iu = -uI$, i.e., vectors and pseudoscalars in the Dirac algebra anticommute.

We now define

$$Z_1 \cdot Z_2 \equiv \langle Z_1 \bar{Z}_2 \rangle_{0+4} = \tfrac{1}{2}(Z_1\bar{Z}_2 + Z_2\bar{Z}_1) = \tau_1^u \tau_2^v g_{uv}, \tag{1.9}$$

where $g(e_u, e_v) \equiv e_u \cdot e_v = g_{uv}$. Next, we define

$$Z_1 \wedge Z_2 \equiv \langle Z_1 \bar{Z}_2 \rangle_2 = \tfrac{1}{2}(Z_1\bar{Z}_2 - Z_2\bar{Z}_1) = \tau_1^u \tau_2^v e_u \wedge e_v. \tag{1.10}$$

It is easy to check that for $Z_k = u_k + Iv_k$ $(k = 1, 2)$, that

$$Z_1 \cdot Z_2 = u_1 \cdot u_2 - v_1 \cdot v_2 + I(u_1 \cdot v_2 + u_2 \cdot v_1) \tag{1.11}$$

and

$$Z_1 \wedge Z_2 = u_1 \wedge u_2 - v_1 \wedge v_2 + I(u_1 \wedge v_2 - u_2 \wedge v_1). \tag{1.12}$$

Combining (1.9) and (1.10), we find that

$$Z_1 \bar{Z}_2 = Z_1 \cdot Z_2 + Z_1 \wedge Z_2 = u_1 u_2 - v_1 v_2 + I(u_1 v_2 + v_1 u_2), \tag{1.13}$$

which reduces to the familiar geometric product of vectors, [3;(1.5)]:

$$u_1 u_2 = u_1 \cdot u_2 + u_1 \wedge u_2,$$

when Z_1 and Z_2 are real Dirac vectors, i.e., when $Z_k = \bar{Z}_k$ for $k = 1, 2$.

The complex inner product (1.9) satisfies the following important properties:

$$C(Z_1, Z_2) \equiv Z_1 \cdot Z_2 = Z_2 \cdot Z_1 \equiv C(Z_2, Z_1), \tag{1.14}$$

$$(\tau Z_1) \cdot Z_2 = \tau Z_1 \cdot Z_2 = Z_1 \cdot (\tau Z_2), \tag{1.15}$$

$$\overline{Z_1 \cdot Z_2} = \bar{Z}_1 \cdot \bar{Z}_2, \tag{1.16}$$

and turns $\mathscr{D}^-$ into a complex 4-dimensional Euclidean space with the symmetric complex metric $C(Z_1, Z_2)$. It follows from (1.11) that (local) Lorentz transformations preserve the metric $C(Z_1, Z_2)$. Twistors can be identified with the elements of this complex 4-dimensional space endowed with a Hermitian form with signature $(++--)$. Hermitian forms on our complex 4-dimensional space will be discussed in Section 3.

The outer product (1.10) of complex vectors satisfies the following important properties:

$$(\tau Z_1) \wedge Z_2 = \tau Z_1 \wedge Z_2 = Z_1 \wedge (\tau Z_2), \tag{1.17}$$

$$Z_1 \wedge Z_2 = -Z_2 \wedge Z_1 = \overline{Z_1 \wedge Z_2}, \tag{1.18}$$

which can be easily established from the definition (1.10), and (1.5).

For $Z = u + Iv$, we calculate, using (1.11) and (1.12),

$$Z\bar{Z} = Z \cdot Z = u^2 - v^2 + 2Iu \cdot v, \quad \text{and} \quad Z \wedge Z = 0 \tag{1.19}$$

and

$$Z^2 = u^2+v^2-2Iu \wedge v \Leftrightarrow Z \cdot \bar{Z} = u^2+v^2, \quad Z \wedge \bar{Z} = -2Iu \wedge v. \tag{1.20}$$

An important consequence of (1.19) is that

$$Z \cdot Z = 0 \Leftrightarrow u^2 = v^2 \quad \text{and} \quad u \cdot v = 0. \tag{1.21}$$

There are basically two kinds of *complex null vectors*, by which we mean a complex vector Z satisfying (1.21). By inspection, we find that

$$Z_1 = \varrho e^{I\theta} n, \quad \text{and} \quad Z_2 = \varrho(a_1+Ia_2) \quad \text{for} \quad \theta, \varrho \in \mathbf{R}, \tag{1.22}$$

where n is a null vector ($n^2 = 0$), and a_1 and a_2 are orthogonal unit space-like vectors ($a_k^2 = -1$ and $a_1 a_2 = -a_2 a_1$).

Similarly, with the help of (1.20), we find

$$Z^2 = Z \cdot \bar{Z}+Z \wedge \bar{Z} = 0 \Leftrightarrow Z = \tau n = \varrho e^{I\theta} n, \tag{1.23}$$

where n is a null vector. Thus, (1.23) corresponds to a complex null vector of the first kind in (1.22), and is a more restrictive condition than (1.21). Note also that

$$Z \cdot Z = 0 = Z \cdot \bar{Z} \Leftrightarrow Z = \tau n = \varrho e^{I\theta} n, \tag{1.24}$$

and hence the left hand sides of (1.23) and (1.24) are equivalent. To prove (1.24), note that the left hand side of (1.24) is equivalent to

$$Z \cdot (Z \pm \bar{Z}) = 0 \Leftrightarrow u^2 = 0 = v^2 \quad \text{and} \quad u \cdot v = 0,$$

which says that u and v are orthogonal null vectors, but this is only possible in Minkowski space-time when they are colinear. For an equivalent argument, see [1; p. 980]. The relationship (1.23) or (1.24) can be used to define the notion of a complex null line as used in [9; p. 258].

Finally, we note that any formula involving the operations of dot ($\cdot$) and wedge ($\wedge$) can be appropriately extended to apply to complex vectors. For example,

$$\begin{aligned} Z_1 \cdot (Z_2 \wedge Z_3) &= Z_1 Z_2 Z_3 - Z_1 Z_3 Z_2 \\ &= \tfrac{1}{4}(Z_1 \bar{Z}_2 Z_3 - Z_1 \bar{Z}_3 Z_2 - Z_2 \bar{Z}_3 Z_1 + Z_3 \bar{Z}_2 Z_1). \end{aligned} \tag{1.25}$$

Similarly,

$$Z_1 \wedge (Z_2 \wedge Z_3) = \tfrac{1}{4}(Z_1 \bar{Z}_2 Z_3 - Z_1 \bar{Z}_3 Z_2 + Z_2 \bar{Z}_3 Z_1 - Z_3 \bar{Z}_2 Z_1). \tag{1.26}$$

Definitions (1.25) and (1.26) imply

$$Z_1 \bar{Z}_2 \bar{Z}_3 = Z_1 \cdot (Z_2 \wedge Z_3) + Z_1 \wedge Z_2 \wedge Z_3. \tag{1.27}$$

Great care must be taken when using the extended dot and wedge, since they only reduce to the ordinary dot and wedge when applied to real vectors. In terms of components, letting $Z_k = \tau_k^u e_u$, (1.26) can be expressed by

$$Z_1 \wedge Z_2 \wedge Z_3 = \tau_1^u \tau_2^v \tau_3^w e_u \wedge e_v \wedge e_w = \begin{vmatrix} \tau_1^0 & \tau_1^1 & \tau_1^2 \\ \tau_2^0 & \tau_2^1 & \tau_2^2 \\ \tau_3^0 & \tau_3^1 & \tau_3^2 \end{vmatrix} e_0 \wedge e_1 \wedge e_2$$

$$+|\;\;| e_0 \wedge e_1 \wedge e_3 + |\;\;| e_0 \wedge e_2 \wedge e_3 + |\;\;| e_1 \wedge e_2 \wedge e_3. \tag{1.28}$$

It follows that Z_1, Z_2, Z_3 are linearly independent iff (1.28) is non-vanishing.

2. *Derivation of a trace-free symmetric operator*

Plebański begins his work [1] by expressing the Riemann tensor in the form:

$$R^{rs}_{uv} = W^{rs}_{uv} - \tfrac{1}{2}\,\delta^{rst}_{uvw}(R^w_t - \tfrac{1}{4}\,\delta^w_t) + \tfrac{1}{12}\,\delta^{rs}_{uv} R.$$

In [3; (5.18)], we have shown the equivalence of this expression to the following expression involving bivector operators:

$$R(B) = W(B) + U(B) + S(B), \text{ (slightly different notation)} \tag{2.1}$$

where

$$W(B) = R(B) - \tfrac{1}{2}\,B \cdot \partial_v[R(v) - \tfrac{1}{6}\,Rv] \tag{2.2}$$

is a dual symmetric bivector operator, equivalent to the Weyl tensor, and

$$U(B) = \tfrac{1}{2}\,B \cdot \partial_v[R(v) - \tfrac{1}{4}\,Rv] \tag{2.3}$$

is an anti-dual symmetric bivector operator, and

$$S(B) = \tfrac{1}{12}\,RB \tag{2.4}$$

is the identity bivector operator times 1/12[th] the scalar curvature R, and is thus, also, dual symmetric. Therefore, the study of the algebraic structure of the curvature tensor splits into the study of these three bivector operators. The dual-symmetric Weyl operator was studied in [3], together with the bivector operator $S(B)$. From the expression (2.3), we see that the study of $U(B)$ reduces to the study of the vector operator $t: \mathscr{D}_1 \to \mathscr{D}_1$,

$$t(v) \equiv R(v) - \tfrac{1}{4}\,Rv \Leftrightarrow U^r_s = R(e_s) \cdot e^r - \tfrac{1}{4}\,\delta^r_s R, \tag{2.5}$$

where $R(v) \equiv \partial_a \cdot R(a \wedge v)$ is the Ricci operator, and R is the scalar curvature, see [3; (5.14), (5.17)]. From [3; (3.28)], it follows that $t(v)$ is trace-free and symmetric, i.e.,

$$\partial_v t(v) = 0 \Leftrightarrow \partial_v \cdot t(v) = 0 = \partial_v \wedge t(v). \tag{2.6}$$

Before we start the classification of $t(v)$, recall [3; (3.34)], that a general bivector operator can be expressed in the form

$$F(B) = W(B) + S(B) + U(B) + J(B) + D(B).$$

The operators $W(B)$, $S(B)$, and $J(B)$ ([3; (4.9)]), were classified in [3], and from [3; (3.31)] it follows that the operator $D(B)$ can also be classified by studying a trace-free symmetric vector operator $t'(v)$. It therefore follows that the general classification of a bivector operator $F(B)$ is complete, once the classification of trace-free symmetric vector operators is known.

Our study of the symmetric, trace-free vector operator $t(v)$ will be based neither on spinorial, nor matrix, nor Sach's null-leg techniques, as used by Plebański, but rather on the extension of $t(v)$ to a derivation (w.r.t the outer product) on the full Dirac algebra $\mathscr{D}$. This extension is defined by $T: \mathscr{D} \to \mathscr{D}$,

$$T(A) = A^+ \cdot \partial_v t(v) \quad \text{for all} \quad A \in \mathscr{D}. \tag{2.7}$$

Note that the definition uses the operation $(+)$ of reversal, also called the main antiautomorphism of $\mathscr{D}$ and hence, strictly speaking, the extension $T(A)$ is the outer-derivation of $t(v)$ followed by a reversion. This is evident in the relationship

$$T(A^+) = A \cdot \partial_v t(v) = [T(A)]^+, \tag{2.8}$$

which is easily established from the definition (2.7).

For scalars and pseudoscalars, we find

$$T(1) = 0 \quad \text{and} \quad T(I) = I \cdot \partial_v t(v) = I\partial_v t(v) = 0, \tag{2.9}$$

as follows from (2.6) and definition (2.7). For bivectors $a \wedge b$, we find by using (2.3), (2.6), and (2.7),

$$U(a \wedge b) = -T(a \wedge b) = (a \wedge b) \cdot \partial_v t(v) = t(a) \wedge b + a \wedge t(b). \tag{2.10}$$

Finally, for complex vectors $Z = a + Ib$, we calculate

$$T(Z) = T(a) + T(Ib) = t(a) + It(b). \tag{2.11}$$

The second equality is a consequence of the following steps:

$$T(Ib) = -(Ib) \cdot \partial_v t(v) = -Ib \wedge \partial_v t(v) = -Ib\partial_v t(v) + Ib \cdot \partial_v t(v) = It(b).$$

The extension of $t(v)$ to the derivation $T(A)$ should be contrasted with the extension of $t(v)$ to an outermorphism $\underline{t}(A)$, which is given in Appendix B.

From properties (2.11) and (2.9), and the linearity of $T(A)$, it follows that

$$T(\tau Z) = \tau T(Z) \text{ and } T(Z_1) \cdot Z_2 = Z_1 \cdot T(Z_2), \tag{2.12}$$

i.e., $T(Z)$ is dual symmetric when considered as an operator on complex vectors $Z \in \mathscr{D}^-$. On the other hand, when considered as a bivector operator on $\mathscr{D}_2$, we find that T is anti-dual symmetric, i.e.,

$$T(IB) = -IT(B) \quad \text{and} \quad T(A) \bigcirc B = \overline{A \bigcirc T(B)}, \tag{2.13}$$

as has already been proven in [3], or as can be directly established in steps similar to those following (2.11). In Section 3, we shall study the Hermitian form $Q(A, B) = A \bigcirc T(B)$, which $T(B)$ defines. An immediate consequence of (2.13) is that

$$T^2(IB) = IT^2(B) \quad \text{and} \quad T^2(A) \bigcirc B = A \bigcirc T^2(B) \tag{2.14}$$

which says that T^2 is dual symmetric, and hence the Petrov–Penrose classification as worked out in [3] applies to T^2. Plebański has used this fact to obtain a classification of $t(v)$, [1; p. 1001]. Although our formalism can be used to simplify Plebański's considerations, we will only refer the reader to Appendix D, where we discuss the relationship of our formalism to that used by Plebański and others.

An important result obtained by Plebański and others (see [7] for relevant historical comments and recent references to the literature), and that will be used in the following consideration is that $t(v)$, and hence $T(Z)$, will always have at least one real space-like eigenvector; there will always exist a vector $a_3 \in \mathscr{D}_1$, and a real eigenvalue τ_3 such that

$$T(a_3) = t(a_3) = \tau_3 a_3 \quad \text{and} \quad a_3^2 = -1. \tag{2.15}$$

Although there may be more than one eigenvector satisfying (2.15), different selections lead to essentially the same classification scheme.

Let us now show how the Petrov–Penrose classification worked out in [3] can be applied to $t(v)$ to obtain what Plebański has referred to as the principal correlation of $t(v)$, [1; p. 1018]. First, we obtain a factorization of the pseudoscalar I by writing

$$i = Ia_3 = a_2a_1a_0 = -a_3I \text{ or } I = a_3i, \tag{2.16}$$

where a_2 and a_1 are orthonormal space-like vectors orthogonal to the unit time-like vector a_0. Thus, i is a unit trivector orthogonal to the space-like vector a_3, and it is easy to check that $i^2 = -1$. The vectors a_0, a_1, a_2 generate a Clifford subalgebra of the Dirac algebra $\mathscr{D}$ which we denote by $\mathscr{S}$. This subalgebra is homomorphic to the Pauli subalgebra of the time-like vector a_0, as is explained in Appendix A. We will now define an operator on $\mathscr{S}$ which has the same structure as a dual symmetric bivector operator on the Pauli algebra by way of the homomorphism.

First, we define

$$t'(v) \equiv (t+\tfrac{1}{3}\tau_3)(v) = t(v)+\tfrac{1}{3}\tau_3 v. \tag{2.17}$$

Next, we define the vector derivative on $\mathscr{S}$ by setting

$$\partial_v' = \partial_v + a_3a_3\cdot\partial_v. \tag{2.18}$$

The vector derivative ∂_v' can also be defined as the projection of the vector derivative ∂_v of the Dirac algebra, i.e.,

$$\partial_v' = P_i(\partial_v) = -ii\cdot\partial_v.$$

Applying the operator equation (2.18) to (2.17), and using (2.6) and (2.15), and (B. 5) from Appendix B, we find that

$$\partial_v' t'(v) = [\partial_v + a_3a_3\cdot\partial_v]\,[t(v)+\tfrac{1}{3}\tau_3 v] = 0, \tag{2.19}$$

so $t'(v)$ is a trace-free symmetric operator on $\mathscr{S}$. Just as $t(v)$ was extended by (2.7) to a derivation on $\mathscr{D}$, we can use (2.7) to extend $t'(v)$ to a derivation

$$T'(A) = A^+\cdot\partial_v t'(v), \tag{2.20}$$

on all of $\mathscr{D}$. However, we are most interested in the structure of T' when restricted to the subalgebra $\mathscr{S}$. In the same way that we proved the properties (2.9) and (2.11) of $T(A)$, we can show that

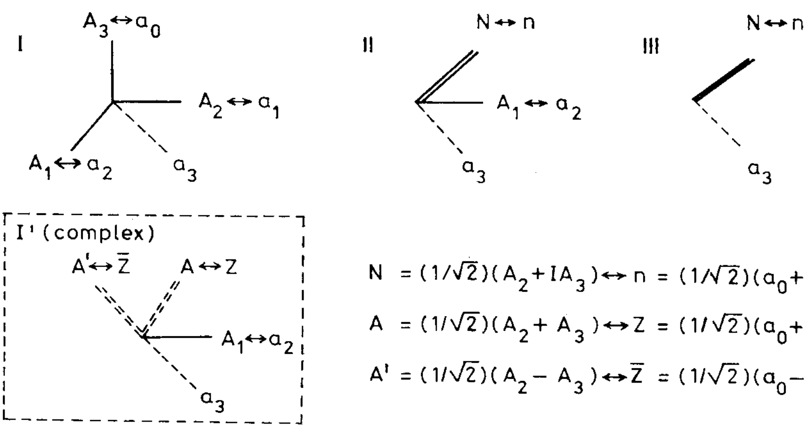
$$T'(1) = 0 = T'(i), \tag{2.21}$$

$$T'(iA) = iT'(A) \quad \text{for all} \quad A \in \mathscr{S}. \tag{2.22}$$

Because $t(v)$ is symmetric, T' is also symmetric. Property (2.22) shows that T' is self-dual on *both* vectors and bivectors, and this should be contrasted with the corresponding properties (2.12) and (2.13) of T. Because of the algebra homomorphism, given in Appendix A, T' has the structure of a dual symmetric bivector operator, and thus the Petrov–Penrose classification as worked out in [3] can be applied to T'.

Since $t'(v)$, as defined in (2.17), and $t(v)$ have the same eigenvectors (but with possibly different eigenvalues), the classification of $t'(v)$ and $t(v)$ is essentially the same. A discussion of the exact relationship between the eigenvectors and values of t and t', and of the eigen-multivectors and values of the corresponding extended operators T and T', is deferred to Appendices B and C where the characteristic equations of these operators is discussed. Here, it is only necessary to point out that complex eigenmultivectors (= vector + bivector) of T' correspond to complex eigenvectors of T.

We close this section with a diagram of the Petrov–Penrose classification of the dual symmetric bivector operator which is the ppl. correlation of the complex vector operator $T(Z)$ by way of the construction involving T' and the algebra homomorphism given in Appendix A.



$$N = (1/\sqrt{2})(A_2 + IA_3) \leftrightarrow n = (1/\sqrt{2})(a_0 + a_1)$$

$$A = (1/\sqrt{2})(A_2 + A_3) \leftrightarrow Z = (1/\sqrt{2})(a_0 + Ia_1)$$

$$A' = (1/\sqrt{2})(A_2 - A_3) \leftrightarrow \bar{Z} = (1/\sqrt{2})(a_0 - Ia_1)$$

Diagram 1

Notes about diagram 1

a) The A_k's represent orthonormal time-like eigenbivectors with real eigenvalues of the dual symmetric bivector operator which is ppl. correlated to $T(Z)$. By way of the algebra homomorphism they correspond to real eigenvectors of $T(Z)$, e.g., $A_3 \rightarrow a_0$, etc.

b) Null eigenbivectors and the corresponding real null eigenvectors of $T(Z)$ are represented by double solid lines at an angle of 45°.

c) The real eigenvector a_3 of $T(Z)$ is dashed-in because it is outside the Petrov–Penrose diagram of the ppl. correlation of $T(Z)$; thus a_3 does not correspond to an eigenbivector.

d) The subcase type I' applies only when there are complex eigenvalues, and is in the dashed box. In this case, the orthonormal time-like eigenbivectors A and A' correspond to 2 complex unit time-like eigenvectors Z and $\bar{Z}$ of $T(Z)$.

3. *Hermitian forms*

In the last section, we were interested in classifying the complex vector operator $T(Z)$ by studying its ppl. correlation to a dual symmetric bivector operator. We have seen how this reduces the problem of the classification of a dual symmetric operator on complex 4-dimensional space to the classification of a dual symmetric operator on complex 3-dimensional space, once a real eigenvector of $t(v)$ has been found. In this section, we will see what properties of $t(v)$ can be learned by studying properties of the Hermitian form associated with $t(v)$ by way of its derivation $T(B)$. Plebański [1] has used the method of Hermitian forms as the basis for an alternative classification scheme of $t(v)$, but our objective will be only to study the role played by Hermitian forms, both on the complex 4-dim Euclidean space of complex vectors, and on the 3-dim complex Euclidean space of bivectors. We will first give several examples of Hermitian forms to point out the various well-known unitary and pseudo-unitary structures involved.

Hermitian forms are associated with anti-dual symmetric operators. If L is an anti-dual symmetric operator $L: \mathscr{D}^- \to \mathscr{D}^-$, then by definition L satisfies the properties

$$L(\tau Z) = \bar{\tau} L(Z) \quad \text{and} \quad L(Z_1) \cdot Z_2 = \overline{Z_1 \cdot L(Z_2)}, \tag{3.1}$$

and the Hermitian form associated with L is

$$H(Z_1, Z_2) = Z_1 \cdot L(Z_2) = \overline{L(Z_1) \cdot Z_2} = \overline{H(Z_2, Z_1)}. \tag{3.2}$$

The well-known property that

$$H(\tau Z_1, Z_2) = \tau H(Z_1, Z_2) = H(Z_1, \bar{\tau} Z_2) \tag{3.3}$$

then follows trivially from (3.1), and (1.15).

The complex vector derivative ∂_Z is very useful in getting back the anti-dual symmetric operator L of the Hermitian form H as given in (3.2). The following relationship is easily established from (B. 24) in Appendix B:

$$\overline{2L(Z_1)} = \partial_Z H(Z, Z)|_{Z=Z_1} = \partial_Z H(Z, Z_1), \quad \text{and} \quad \partial_Z H(Z_1, Z) = 0. \tag{3.4}$$

So, once properties are proven for Hermitian forms, by using the relationship (3.4), they can be easily translated into properties of anti-dual symmetric operators.

Properties (3.1) through (3.4) have been stated for Hermitian forms and their corresponding anti-dual symmetric operators on the complex 4-dim Euclidean space of complex vectors $\mathscr{D}^-$. However, these properties remain equally valid when stated on the complex 3-dim Euclidean space of bivectors. Thus, for the anti-dual symmetric bivector operator $T(B)$, given in (2.13), we have the Hermitian form

$$Q(A, B) = A \bigcirc T(B) = \overline{T(A) \bigcirc B} = \overline{Q(B, A)}. \tag{3.5}$$

Applying the bivector derivative ∂_B to (3.5), we find that

$$T(A) = \tfrac{1}{2}\, \partial_B Q(B)|_{B=A} = \partial_B Q(B, A), \quad \text{and} \quad \partial_B Q(A, B) = 0, \tag{3.6}$$

which is the analog of property (3.4) for complex bivector space.

Consider now the three anti-dual symmetric operators $L_k\colon \mathscr{D}^- \to \mathscr{D}^-$, defined by

$$L_1(Z) = e_0 Z e_0,\ L_2(Z) = \bar{Z},\ L_3(Z) = e_3 Z e_3. \tag{3.7}$$

To each of these operators there corresponds a Hermitian form:

$$H_k(Z_1, Z_2) = Z_1 \cdot L_k(Z_2) = \overline{L_k(Z_1) \cdot Z_2} = \overline{H_k(Z_2, Z_1)}. \tag{3.8}$$

The Hermitian form H_1 has signature $(+\,+\,+\,+)$ and makes $\mathscr{D}^-$ into the unitary space U(4). The Hermitian form H_2 makes $\mathscr{D}^-$ into the pseudo-unitary space $U(1, 3)$. The Hermitian form H_3 has signature $(+\,+\,-\,-)$ and makes $\mathscr{D}^-$ into the pseudo-unitary space U(2,2), which is known as twistor space. See references [13; p. 318], [9; p. 256].

Similarly, by defining the anti-dual symmetric bivector operators

$$T_1(B) = -e_0 B e_0,\ T_2(B) = -e_1 B e_1, \text{ and } T_3(B) = -e_1 E_1 \times (E_1 \times B) e_1 \tag{3.9}$$

and the corresponding Hermitian forms

$$Q_k(A, B) = A \bigcirc T_k(B) = \overline{T_k(A) \bigcirc B} = \overline{Q_k(B, A)}, \tag{3.10}$$

we find the unitary spaces U(3), U(2,1), and U(2,0). The connection between U(2,0) $\simeq$ U(2) and Spinors is well-known [11; p. 43]; in appendix D we shall give a new geometric identification of spinor space as a projected complex 2-dim bivector space.

Let us now study the properties of the Hermitian form determined by the anti-dual symmetric bivector operator (2.13). Recall that $T(B)$ is defined entirely in terms of the traceless symmetric vector operator $t(v)$ as given in (2.10). In particular, we will use well--known methods to reduce $Q(B)$ to a sum of squares, [12; pp. 299, 334]. Although we will prove these properties for the Hermitian form defined on bivector space, we want to emphasize that all properties can be immediately carried over to the setting of the complex 4-dim vector space $\mathscr{D}^-$.

First note, that because $\partial_a \wedge T(a \wedge b) = 0$, it follows that $Q(B) = 0$ for all simple bivectors B is equivalent to the statement that $Q(B)$ is identically 0, [3; (5.13)]. Also, note that $T(B)$ does not necessarily map simple bivectors into simple bivectors, as the following simple example shows:

$$t(e_0) = -e_1, \quad t(e_1) = e_0, \quad t(e_2) = e_2, \quad t(e_3) = -e_3,$$

from which it follows that

$$T(e_1 \wedge (e_2+e_3)) = (e_2-e_3) \wedge e_1+(e_2+e_3) \wedge e_0.$$

Suppose now that $Q \neq 0$, then from the above remarks it follows that there exists a bivector A_1 with the property that $\alpha_1 \equiv Q(A_1) \neq 0$. Now define Q' and T' by

$$Q(B) = (1/\alpha_1)Q(B, A_1)\overline{Q(B, A_1)}+Q'(B) \tag{3.11}$$

and

$$T'(B) = \tfrac{1}{2}\,\partial_B Q'(B) = T(B)-(1/\alpha_1)\overline{Q(B, A_1)}T(A_1), \tag{3.12}$$

and note that $Q'(A_1) = 0 = T'(A_1)$. Next we calculate

$$\partial_a T'(a \wedge b) = \partial_a \cdot T(a \wedge b)+(1/\alpha_1)T(A_1)bT(A_1),$$

which shows that

$$\partial_a \wedge T'(a \wedge b) = 0,$$

so that (3.11) and (3.12) can be used to define a Q'' and T'' in terms of Q' and T'. Repeating these steps, we eventually find that

$$Q(B) = \sum_{k=1}^{3} (1/\alpha_k)B\bigcirc C_k\overline{B\bigcirc C_k}, \tag{3.13}$$

where $C_1 = T(A_1)$, $C_2 = T'(A_2)$, and $C_3 = T''(A_3)$, and $\alpha_2 = Q'(A_2)$, $\alpha_3 = Q''(A_3)$. From (3.13), by using (3.6), we calculate

$$T(B) = \tfrac{1}{2}\,\partial_B Q(B) = \sum_{k=1}^{3} (1/\alpha_k)\overline{B\bigcirc C_k}C_k, \tag{3.14}$$

and

$$t(b) = \partial_a \cdot T(a \wedge b) = \sum_{k=1}^{3} (1/\alpha_k)C_k b C_k. \tag{3.15}$$

The (sgn (α_1), sgn (α_2), sgn (α_3)) is called the signature of Q, and was used by Plebański in an alternative classification of $t(v)$, see [1; p. 1003]. The relationship (3.15) shows that any trace-free symmetric vector operator (any real symmetric second-order trace-free tensor) can be written in terms of at most three bivectors, which generalizes the case of the Maxwell energy momentum tensor. This result can be found in [1; p. 1004], and more recently in [7; p. 61].

Finally, we mention the group $\mathscr{U}$ of "generalized duality rotations" ([1; p. 1004]) defined by

$$S \in \mathscr{U} \quad \text{iff} \quad Q(S(A), S(B)) = Q(A, B), \tag{3.16}$$

for all bivectors $A, B \in \mathscr{D}_2$. We see that $\mathscr{U}$ is the unitary group of the Hermitian form $Q(A, B)$. Equivalently,

$$S \in \mathscr{U} \quad \text{iff} \quad T = S^{+}TS, \tag{3.17}$$

where S^+ is defined by the equation

$$S(A) \bigcirc B = A \bigcirc S^+(B) \text{ for all bivectors } A, B.$$

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APPENDIX A

Pauli algebra homomorphism

The homomorphism between the Pauli algebra generated by the unit timelike bivectors $\{A_k = a_k \wedge a_0 |$ for $k = 1, 2, 3\}$, and the subalgebra $\mathcal{S} \subset \mathcal{D}$ generated by the vectors $\{a_u | \; u = 0, 1, 2\}$ that was used in Section 2, is given by

$$a_0 \leftrightarrow A_3, \quad ia_1 \leftrightarrow A_2, \quad ia_2 \leftrightarrow -A_1. \tag{A.1}$$

The homomorphism (A.1) can be verified by comparing the multiplication charts of corresponding elements given below:

	a_0	ia_1	ia_2
a_0	1	$-a_2$	a_1
ia_1	a_2	1	ia_0
ia_2	$-a_1$	ia_0	1

	A_3	A_2	$-A_1$
A_3	1	$-IA_1$	$-IA_2$
A_2	IA_1	1	IA_3
$-A_1$	IA_2	$-IA_3$	1

$$i = a_2 a_1 a_0 = (-ia_2)(ia_1)a_0 \leftrightarrow A_1 A_2 A_3 = I$$

The above homomorphism shows that a dual symmetric operator on $\mathcal{S}$ corresponds to a dual symmetric bivector operator, and hence may be classified by the Petrov–Penrose classification worked out in [3].

APPENDIX B

Simplicial derivatives and characteristic polynomials

We begin by introducing the notions of a simplicial k-variable $v_{(k)}$, and the operation $\partial_{(k)}$ of differentiation w.r.t a simplicial k-variable, for $k = 0, 1, 2, 3, 4$.

$$v_{(0)} \equiv t, \quad \text{and} \quad \partial_{(0)} \equiv \left.\frac{d}{dt}\right|_{t=0}, \tag{B.1a}$$

where t is a real scalar variable, and

$$v_{(k)} \equiv v_1 \wedge \ldots \wedge v_k, \quad \text{and} \quad \partial_{(k)} \equiv \frac{1}{k!} \partial_k \wedge \ldots \wedge \partial_1, \tag{B.1b}$$

where v_i, and $\partial_i = \partial_{v_i}$ are respectively vector variables and vector derivatives w.r.t those variables, as defined in [3; §2].

The most important elementary formula (that is formulas involving derivatives of the identity mapping) for simplicial derivatives is the following combinatorial-like identity:

$$\partial_{(j)}v_{(j)}a_{(k)} = \partial_{(j)}v_{(j)} \wedge a_{(k)} + \partial_{(j)}\langle v_{(j)}a_{(k)}\rangle_{j+k-2} + \dots + \partial_{(j)}v_{(j)} \cdot a_{(k)}, \tag{B.2a}$$

which is termwise equivalent to

$$\binom{4}{j} a_{(k)} = \binom{k}{0}\binom{4-k}{j} a_{(k)} + \binom{k}{1}\binom{4-k}{j-1} a_{(k)} + \dots + \binom{k}{j}\binom{4-k}{0} a_{(k)}, \tag{B.2b}$$

where $\binom{i}{j} \equiv 0$ for $i < j$. Thus, for example, we have

$$a_{(k)} \wedge \partial_{(j)}v_{(j)} = \binom{4-k}{j} a_{(k)} = \partial_{(j)}v_{(j)} \wedge a_{(k)} \tag{B.3}$$

and

$$a_{(k)} \cdot \partial_{(j)}v_{(j)} = \binom{k}{j} a_{(k)} = \partial_{(j)}v_{(j)} \cdot a_{(k)}. \tag{B.4}$$

Equating the left hand sides of (B.2a) and (B.2b), we find

$$\partial_{(j)}v_{(j)} = \binom{4}{j}. \tag{B.5}$$

Formulas (B.3) and (B.4) generalize (2.8) and (2.6) in [3], and formula (B.5) generalizes formulas (2.7) and (2.14) in [3]. Actually, only the right hand sides of (B.3) and (B.4) are immediate consequences of (B.2); the left hand sides follow from the analogous left handed identity to (B.2). The method of proof of (B.2) is to find an orthogonal basis $\{b_i\}$ of $\mathscr{D}_1$, with the property that $a_{(k)} = b_{(k)} = b_1 b_2 \dots b_k$, and then to express all simplicial derivatives and variables in terms of this basis.

Simplicial derivatives are to be contrasted with multivector derivatives. Simplicial derivatives are composed of the outer product of vector derivatives w.r.t several vector variables. The definition of a multivector derivative w.r.t a vector or bivector variable was given in [3] by formulas (2.1)–(2.4), and the general definition for the multivector derivative w.r.t an r-vector variable V_r is given by analogous formulas. Multivector and simplicial derivatives are basic tools for the coordinate free study of n-dimensional linear algebra and differential geometry. The development of these tools was initiated in [14], and later much more extensively in [6]. For our purposes here, we wish only to point out that for linear functions of an r-vector V_r, r-vector derivatives and r-simplicial derivatives coincide, that is

$$\partial_{V_r}F(V_r) = \partial_{(r)}F(v_{(r)}). \tag{B.6}$$

Identity (B.6) is established by noting that for all $a_{(r)}$,

$$a_{(r)} \cdot \partial_{V_r}F(V_r) = F(a_{(r)}) = F(a_{(r)} \cdot \partial_{(r)}v_{(r)}) = a_{(r)} \cdot \partial_{(r)}F(v_{(r)}).$$

The first equality on the left is a direct consequence of the definition of the r-vector derivative, and the second and third equalities make use of (B.4) and the linearity of F.

In terms of the simplicial derivative, the theory of linear operators can be efficiently reformulated. That part of the theory that is needed here will now be briefly given. Let $f(v)$ be a vector operator (by this we mean that f is a linear mapping of Dirac vectors into Dirac vectors). We can extend f to a homomorphism on $\mathscr{D}$([10; p. 221]), which we call an *outermorphism*, by defining for $r \neq 0$

$$\underline{f}(a_{(r)}) \equiv a_{(r)} \cdot \partial_{(r)} f_{(r)} = f(a_1) \wedge \ldots \wedge f(a_r), \tag{B.7}$$

where

$$a_{(r)} = a_1 \wedge \ldots \wedge a_r \quad \text{and} \quad f_{(r)} = f(v_1) \wedge \ldots \wedge f(v_r).$$

In the case $r = 0$, it is assumed that $f(t) = t$, for the scalar variable t.

We can now derive the characteristic polynomial of a vector operator $f(v)$. First define

$$f' \equiv f'(v) = f(v) - \tau v \equiv (f - \tau)(v). \tag{B.8}$$

Then,

$$\begin{aligned} \varphi(\tau) &\equiv \det(f') = \partial_{(4)}(f-\tau)_{(4)} = \tfrac{1}{24}\,\partial_4 \wedge \ldots \wedge \partial_1(f_1 - \tau v_1) \wedge \ldots \wedge (f_4 - \tau v_4) \\ &= \ldots \\ &= \tau^4 - \partial \cdot f \tau^3 + \partial_{(2)} \cdot f_{(2)} \tau^2 - \partial_{(3)} \cdot f_{(3)} \tau + \partial_{(4)} \cdot f_{(4)}. \end{aligned} \tag{B.9}$$

Equation (B.9) shows that the characteristic polynomial of a linear operator is just a differential identity involving the differential scalars of f. If $\tau_1, \tau_2, \tau_3, \tau_4$ are the characteristic roots of $\varphi(\tau) = 0$, then we can write

$$\begin{aligned} \varphi(\tau) = \prod_{k=1}^{4} (\tau - \tau_k) &= \tau^4 - (\tau_1 + \ldots + \tau_4)\tau^3 + (\tau_1\tau_2 + \tau_1\tau_3 + \ldots + \tau_3\tau_4)\tau^2 \\ &\quad - (\tau_1\tau_2\tau_3 + \ldots + \tau_2\tau_3\tau_4)\tau + \tau_1\tau_2\tau_3\tau_4. \end{aligned} \tag{B.10}$$

Comparing (B.9) and (B.10) shows that

$$\partial_{(k)} \cdot f_{(k)} = \partial_{V_k} \cdot \underline{f}(V_k) = \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant 4} \tau_{i_1} \ldots \tau_{i_k}, \tag{B.11}$$

a formula which is well known.

There is an important identity relating simplicial k-derivatives of a vector operator to traces of compositions of the operator:

$$\partial_{(k)} \cdot f_{(k)} = 1/k \sum_{s=1}^{k} (-1)^{s+1} \partial_{(k-s)} \cdot f_{(k-s)} \partial \cdot f^s. \tag{B.12}$$

This formula can either be directly derived as a differential identity involving the differential scalars of f, or surmised from the equivalent formula in the more usual approaches. Identities (B.11) and (B.12) together imply that in terms of the characteristic roots

$$\partial \cdot f^k = \partial_v \cdot f^k(v) = \tau_1^k + \ldots + \tau_4^k, \tag{B.13}$$

as is also well known [13; p. 87].

A vector operator $f(v)$ is said to be *trace-free* if

$$\partial \cdot f = \tau_1+\tau_2+\tau_3+\tau_4 = 0. \tag{B.14}$$

Given a vector operator $g(v)$, we can always define a trace-free operator $f(v)$ which will have the same eigenvectors as $g(v)$. Define

$$f(v) = [g-\tfrac{1}{4}\,\partial \cdot g]\,(v) = g(v)-\tfrac{1}{4}\,\partial \cdot gv. \tag{B.15}$$

If for an eigenvector a, $g(a) = \tau_a a$, then we find

$$f(a) = (\tau_a-(\partial \cdot g)/4)a. \tag{B.16}$$

An operator $f(v)$ is said to be *symmetric* if

$$\partial \wedge f = 0 \Leftrightarrow (a \wedge b)\cdot(\partial \wedge f) \equiv a \cdot f(b)-f(a)\cdot b = 0, \tag{B.17}$$

for all $a, b \in \mathscr{D}_1$. Combining (B.14) and (B.17), we see that an operator $t(v)$ is trace-free and symmetric iff

$$\partial t = \partial \cdot t+\partial \wedge t = 0. \tag{B.18}$$

Using (B.14) and (B.12) to simplify (B.9), we find that the characteristic polynomial of a trace-free vector operator simplifies to

$$\varphi(\tau) = \tau^4-\tfrac{1}{2}\,\partial \cdot t^2\tau^2-\tfrac{1}{3}\,\partial \cdot t^3\tau+\tfrac{1}{8}\,[(\partial \cdot t^2)^2-2\partial \cdot t^4]. \tag{B.19}$$

The characteristic roots of (B.9) or (B.19) may be real or complex. In [3], we saw how complex characteristic roots are best interpreted as "scalar + pseudoscalar" quantities, but (2.14) shows that the present considerations are not independent of the results in [3], and therefore, we retain this geometric interpretation of complex characteristic roots here. Thus, we must reinterpret and generalize all of our considerations of vector operators so that they apply to complex vector operators. By a *complex vector operator* we mean a real linear mapping $F: \mathscr{D}^- \to \mathscr{D}^-$. By writing

$$F(Z) = F_+(Z)+F_-(Z), \tag{B.20}$$

where

$$F_+(Z) \equiv \tfrac{1}{2}(F(Z)-IF(IZ)) \quad \text{satisfies} \quad F_+(IZ) = IF_+(Z),$$

and

$$F_-(Z) \equiv \tfrac{1}{2}(F(Z)+IF(IZ)) \quad \text{satisfies} \quad F_-(IZ) = -IF_-(Z),$$

we decompose $F(Z)$ into *dual* and *anti-dual* parts, just as we did for bivector operators in [3; (3.3)]. We are interested here in the theory of complex dual vector operators, although examples of anti-dual operators were given in Section 3 in connection with Hermitian forms. In Section 2, we saw how a dual symmetric complex vector operator (2.12) arises in the study of the derivation (2.7).

We now define differentiation w.r.t the complex vector $Z = u+Iv$ in terms of vector differentiation:

$$\partial_Z = \partial_u+I\partial_v. \tag{B.21}$$

We have the following formulas for differentiation,

$$\partial_Z Z = 8 \Leftrightarrow \partial \cdot \bar{Z} = 8, \quad \text{and} \quad \partial \wedge \bar{Z} = 0, \tag{B.22}$$

and

$$\partial_Z \bar{Z} = 0 \Leftrightarrow \partial \cdot Z = 0, \quad \text{and} \quad \partial \wedge Z = 0, \tag{B.23}$$

which can be easily verified by using (B.5) and (1.13). Note also the formulas

$$Z_1 \cdot \bar{\partial}_Z Z = 2Z_1 = \partial_Z Z \cdot \bar{Z}_1, \tag{B.24}$$

and

$$\bar{\partial}_Z Z \cdot \bar{Z}_1 = 0 = Z_1 \cdot \partial_Z Z. \tag{B.25}$$

For F_+ and F_- defined in (B.20), we find by using (B.24) and (B.25),

$$Z_1 \cdot \bar{\partial}_Z F_+(Z) = 2F_+(Z_1) = 2Z_1 \cdot \partial_v F_+(v), \quad Z_1 \cdot \bar{\partial}_Z F_-(Z) = 0, \tag{B.26}$$

and

$$\partial_Z F_+(Z) = 2\partial_v F_+(v), \quad \partial_Z F_-(Z) = 0. \tag{B.27}$$

The above considerations lead us to define

$$\partial_{(K)} = (\tfrac{1}{2})^k (1/k!) \partial_{Z_k} \wedge \ldots \wedge \partial_{Z_1}, \quad \text{and} \quad Z_{(K)} = Z_1 \wedge \ldots \wedge Z_k, \tag{B.28}$$

in analogy to (B.1). We then find that

$$\partial_{(K)} F_{+(K)} = \partial_{(k)} F_{+(k)}, \tag{B.29}$$

which generalizes (B.27). Because of (B.29), we can conclude that the characteristic equation of the complex dual vector operator F_+ is equivalent to (B.9); it is defined entirely in terms of the vector derivative.

Let us now consider in more detail properties of the complex vector operator $T(Z)$ given in (2.11). The operator $T(Z)$ has characteristic polynomial (B.19), whose characteristic roots we will denote by $\tau_0, \tau_1, \tau_2, \tau_3$. Suppose that τ_3 is the real eigenvalue of the spacelike eigenvector a_3, as given in (2.15). Now recall the vector operator t' defined in (2.17). The general characteristic equation of an operator on the 3-dimensional subspace $\mathscr{S}_1$ is

$$\varphi'(\tau') = \tau'^3 - \partial' \cdot t' \tau'^2 + \partial'_{(2)} \cdot t'_{(2)} \tau - \partial'_{(3)} \cdot t'_{(3)}, \tag{B.30}$$

analogous to (B.9). But, since by (2.19), $t'(v)$ is trace-free, and using (B.12), (B.30) reduces to

$$\varphi'(\tau') = \tau'^3 - \tfrac{1}{2} \partial' \cdot t'^2 \tau' - \tfrac{1}{3} \partial' \cdot t'^3. \tag{B.31}$$

Using (2.17) and (2.18), we can express (B.31) in terms of the differential scalars of t:

$$\varphi'(\tau') = \tau'^3 - (\tfrac{1}{2} \partial \cdot t^2 - \tfrac{2}{3} \tau_3^2)\tau' - \tfrac{1}{3}(\partial \cdot t^3 + \tau_3 \partial \cdot t^2 - \tfrac{20}{9} \tau_3^3).$$

Let us denote the 3 roots of this equation by $\tau'_0, \tau'_1, \tau'_2$, of which at least one, say τ'_2, will be real since the equation has real coefficients. It follows from (2.17), that any real eigenvector a of t' with eigenvalue τ'_a will satisfy

$$t'(a) = \tau'_a a = (\tau_a + \tau_3/3)a,$$

or

$$\tau'_a = \tau_a + \tau_3/3, \tag{B.32}$$

where τ_a is the corresponding eigenvalue of (B.19).

The relationship between complex characteristic roots τ_c and τ'_c of (B.19) and (B.31) is more complicated. Suppose $Z = a_0 + ia_1$ satisfies

$$T(Z) = \tau_c Z \Leftrightarrow t(a_0) = \tau_r a_0 - \tau_{im} a_1, \quad t(a_1) = \tau_r a_0 + \tau_{im} a_1,$$

where $\tau_c = \tau_r + I\tau_{im}$. Now define $z = a_0 + ia_1$, and $\tau'_c = (\tau_r + \tau_3/3) + \tau_{im} i$. Then one can easily check that

$$T'(z) = \tau'_c z, \tag{B.33}$$

where T' is the derivation of t' as given in (2.20). One can also check that

$$T(\bar{Z}) = \bar{\tau}_c \bar{Z}, \quad \text{and} \quad T'(\bar{z}) = \bar{\tau}'_c \bar{z},$$

so that $\bar{Z}$ and $\bar{z}$, the complex conjugates of Z and z, are also complex eigenvectors of T and T' respectively. Since $\tau_c \neq \bar{\tau}_c$, and $\tau'_c \neq \bar{\tau}'_c$, it follows that

$$Z \cdot \bar{Z} = 0 = z \cdot \bar{z} \Leftrightarrow a_1^2 = -a_0^2. \tag{B.34}$$

The equivalent condition on the right-hand side follows from (1.20). Furthermore, we are free to set

$$Z \cdot Z = 2 = z \cdot z \Leftrightarrow a_1^2 = -a_0^2 = -1, \quad \text{and} \quad a_1 \cdot a_0 = 0, \tag{B.35}$$

as follows from (1.19), and thus we see that a_0 is a unit time-like vector orthonormal to the unit space-like vector a_1. See [I; p. 981].

Finally, we wish to show how the formulas involving the differential scalars of a dual bivector operator, as worked out in [3], are related to the above formulas. Define the (simplicial) bivector variables $B_{(K)}$ by

$$B_{(I)} = B_1, \quad B_{(II)} = B_1 \times B_2, \quad B_{(III)} = B_1 \times B_2 \bigcirc B_3, \tag{B.36}$$

and the bivector derivatives $\partial_{(K)}$ by

$$\partial_{(I)} = \tfrac{1}{2}\,\partial_{B_1}, \quad \partial_{(II)} = (\tfrac{1}{2})^2 \frac{1}{2!} \partial_{B_2} \times \partial_{B_1}, \quad \text{and}$$

$$\partial_{(III)} = (\tfrac{1}{2})^3 \frac{1}{3!} \partial_3 \bigcirc \partial_2 \times \partial_1. \tag{B.37}$$

Then for a dual bivector operator $L(B)$, with

$$L_{(I)} = L(B_1), \quad L_{(II)} = L(B_1) \times L(B_2), \quad L_{(III)} = L(B_1) \times L(B_2) \bigcirc L(B_3),$$

its characteristic polynomial, given by (3.24) in [3], takes the form

$$\psi(\lambda) = \lambda^3 - \partial_{(I)} \bigcirc L_{(I)} \lambda^2 + \partial_{(II)} \bigcirc L_{(II)} \lambda - \partial_{(III)} \bigcirc L_{(III)}, \tag{B.38}$$

which is exactly the form of the characteristic equation (B.30) for the vector operator t' on the 3-dim space $\mathscr{S}_1$. Analogous to (B.11), we find

$$\partial_{(K)} \bigcirc L_{(K)} = \sum_{1 \leq i_1 < \dots < i_k \leq 3} \lambda_{i_1} \dots \lambda_{i_k} \quad \text{for} \quad k = 1, 2; 3, \tag{B.39}$$

where λ_1, λ_2, λ_3 are the characteristic roots of (B.38). Analogous to (B.12), we have

$$\partial_{(K)} \bigcirc L_{(K)} = 1/k \sum_{s=1}^{k} (-1)^{s+1} \partial_{(k-s)} \bigcirc L_{(k-s)} \partial_{(1)} \bigcirc L_{(1)}^{s}. \tag{B.40}$$

Using (B.40), (B.38) can be expressed in the form

$$\begin{aligned} \psi(\lambda) = {} & \lambda^3 - \partial_{(1)} \bigcirc L_{(1)} \lambda^2 + \tfrac{1}{2} [(\partial_{(1)} \bigcirc L_{(1)})^2 - \partial_{(1)} \bigcirc L_{(1)}^2] \lambda \\ & - \tfrac{1}{6} [2 \partial \bigcirc L^3 - 3 \partial \bigcirc L \partial \bigcirc L^2 + (\partial \bigcirc L)^3] \end{aligned} \tag{B.41}$$

which is equivalent to (3.24) in [3].

If some of the above formulas seem unfamiliar, it may be of help for the reader to refer to Appendix D where the symbolism is compared to the more standard formalism used by Plebański.

APPENDIX C

Calculations involving differential scalars

In this appendix we want to relate all differential scalars of the derivation T, as defined in Section 2, back to the differential scalars of the symmetric trace-free operator $t(v)$, in terms of which it was defined by (2.8). We will also discuss the relationships between the eigenvectors and eigenvalues of t, and those of its derivation T.

Recall formula (2.10), that

$$T(a \wedge b) = t(b) \wedge a + b \wedge t(a). \tag{C.1}$$

Applying the Leibniz formula, [10; p. 147], to (C.1), we find that

$$T^k(a \wedge b) = (-1)^k \sum_{r=0}^{k} \binom{k}{r} t^r(a) \wedge t^{k-r}(b). \tag{C.2}$$

Noting that

$$\partial_a t^r(a) \wedge t^s(b) = \partial \cdot t^r t^s(b) - t^{r+s}(b),$$

we can now calculate

$$\partial_a T^k(a \wedge b) = (-1)^k \sum_{r=0}^{k} \binom{k}{r} [\partial \cdot t^r t^{k-r}(b) - t^k(b)], \tag{C.3}$$

and

$$\partial_b \partial_a T^k(a \wedge b) = (-1)^k \sum_{r=0}^{k} \binom{k}{r} [\partial \cdot t^r \partial \cdot t^{k-r} - \partial \cdot t^k],$$

from which it follows that

$$\partial_B T^k(B) = \partial_{(2)} T(v_{(2)}) = (-1)^k \tfrac{1}{2} \sum_{r=0}^{k} \binom{k}{r} [\partial \cdot t^r \partial \cdot t^{k-r} - \partial \cdot t^k], \tag{C.4}$$

by using (B.6). From (C.3) and (C.4), and using the Cayley–Hamilton theorem to express traces $\partial \cdot t^k$ in terms of traces $\partial \cdot t^r$ with $r \leqslant 4$, we find after straight forward calculations:

$$\partial_a T^2(a \wedge b) = \partial \cdot t^2 b, \quad \partial_B T^2(B) = 2\partial \cdot t^2, \tag{C.5}$$

$$\partial_a T^4(a \wedge b) = \partial \cdot t^4 b + 4\partial \cdot t^3 t(b) + 6\partial \cdot t^2 t^2(b) - 12 t^4(b), \tag{C.6a}$$

$$\partial_B T^4(B) = -4\partial \cdot t^4 + 3(\partial \cdot t^2)^2, \tag{C.6b}$$

and

$$\partial_B T^6(B) = -6\partial \cdot t^4 \partial \cdot t^2 + \tfrac{2}{3}(\partial \cdot t^3)^2 - \tfrac{7}{2}(\partial \cdot t^2)^3. \tag{C.7}$$

Using formulas (C.5)–(C.7) in [3; (3.24)], or in (B.41) by taking care to properly normalize the bivector derivatives with factors of ($\frac{1}{2}$), we can express the characteristic polynomial of the bivector operator $L(B) = T^2(B)$ entirely in terms of traces of the operator $t(v)$. Let us denote by λ_1^2, λ_2^2, λ_3^2 the characteristic roots of this characteristic polynomial.

Let us now see how the characteristic values λ_k^2 of the bivector operator $T^2(B)$ are related to the characteristic values τ_u of the vector operator $t(v)$. Recall from (2.15) that τ_3 is the real characteristic value of the eigenvector a_3, and let Z_u denote the real or complex eigenvector of the real or complex eigenvalue τ_u, for $u \neq 3$. Then we calculate

$$T(Z_u \wedge a_3) = t(a_3) \wedge Z_u + a_3 \wedge T(Z_u) = -(\tau_u + \tau_3) Z_u \wedge a_3, \tag{C.8}$$

for $u = 0, 1, 2$, from which it follows from (2.13) that

$$T(IZ_u \wedge a_3) = -IT(Z_u \wedge a_3) = (\tau_u + \tau_3) Z_u \wedge a_3 I. \tag{C.9}$$

The identities (C.8) and (C.9) give up to 6 distinct eigenbivectors of T with eigenvalues $\pm\lambda_{u+1}$, where

$$\lambda_{u+1} \equiv -(\tau_u + \tau_3) \quad \text{for} \quad u = 0, 1, 2. \tag{C.10}$$

From (C.8) we calculate

$$T^2(Z_u \wedge a_3) = (\tau_u + \tau_3)^2 Z_u \wedge a_3,$$

from which it follows that

$$\lambda_{u+1}^2 = (\tau_u + \tau_3)^2 \quad \text{for} \quad u = 0, 1, 2, \tag{C.11}$$

are the eigenvalues of T^2.

APPENDIX D

Relationship to other formalisms

The following is a table of how some of the differential scalars, discussed in Appendices B and C, are related to parameters used by Plebański:

TABLE I

Plebański [1]	Conversion	STA
$U = \|\|U^{\alpha}_{\beta}\|\|$ (p. 971)	$U^{\alpha}_{\beta} = t(e^{\alpha}) \cdot e_{\beta}$	(B. 18) $t(v)$
$\overset{p}{U} = \mathrm{Tr}\,(U^{p})$ (p. 971)	$\overset{p}{U} = \partial \cdot t^{p}$	(B. 13) $\partial_{v} \cdot t^{p}(v)$
$\underset{[k]}{U}$ (p. 971)	$\underset{[k]}{U} = \partial_{(k)} \cdot t_{(k)}$	(B. 11) $\partial_{(k)} \cdot t_{(k)}$
$\underset{[k]}{Q}$ (p. 973)	$\underset{[k]}{Q} = \partial_{(K)} \bigcirc \mathrm{T}^{2}_{(K)}$	(B. 39) $\partial_{(K)} \bigcirc \mathrm{T}^{2}_{(K)}$

Next, we give a table showing how various spinorial objects find direct expression in the STA formalism. Note that dotted indices always correspond to complex conjugation, or to a Hermitian form. Also, note the relationship between upper and lower spinor and bivector differentiation.

TABLE II

Spinor [1]	Intermediary complex form	STA
$\varphi^{\alpha\beta}$ (p. 1002)	$B \bigcirc A$	A
$\varphi_{\alpha\beta}$	$A \bigcirc \partial_{B}$	A
$\varphi^{\dot{\alpha}\dot{\beta}}$	$\overline{B \bigcirc A}$	A
$\varphi_{\dot{\alpha}\dot{\beta}}$	$\overline{A \bigcirc \partial_{B}}$	A
$U^{\dot{\alpha}\dot{\beta}\delta\gamma}$ (p. 964) (p. 1002)	$T(A) \bigcirc B$ (3.5), (3.6)	$T(A)$
$U_{\dot{\alpha}\dot{\beta}}{}^{\delta\gamma}$	$T(\partial_{A}) \bigcirc B$	$T(A)$

From the above table we can construct other spinorial objects and their counterparts in the STA formalism. For example, Plebański [1; p. 991] uses

$$Q^{\theta\omega\delta\gamma} \equiv U_{\dot{\alpha}\dot{\beta}}{}^{\delta\gamma} U^{\dot{\alpha}\dot{\beta}\theta\omega}. \tag{D.1}$$

Using the corresponding complex forms from the *Intermediary* column of the above table, we find with the help of [3; (2.13)],

$$B \bigcirc T(\partial_{A}) T(A) \bigcirc C = \overline{T(B) \bigcirc \partial_{A} A \bigcirc T(C)} = 2\overline{T(B) \bigcirc T(C)} = 2B \bigcirc T^{2}(C).$$

Thus, we have the correspondence

$$Q^{\theta\omega\delta\gamma} \leftrightarrow B \bigcirc T^2(C) \leftrightarrow T^2(C). \tag{D.2}$$

We will now identify spinors and spinor space in a new way. Let E_1, E_2, E_3 be a time-like orthonormal basis of the bivector space $\mathscr{D}_2$, as given in [3]. Next, define the projection P of the complex 3-dim space $\mathscr{D}_2$ onto the complex 2-dim subspace of bivectors orthogonal (w.r.t. the complex metric A $\bigcirc$ B) to the bivector E_3. This projection is best defined by

$$B' = P(B) = (B \times E_3) \times E_3 = B - B \bigcirc E_3 E_3 \tag{D.3}$$

and we will denote the subspace of such elements by $\mathscr{P}' = \{B'\}$. The subspace $\mathscr{P}'$ is called the *spinor space of the bivector* E_3, and the elements of $\mathscr{P}'$ are called *spinors*.

With respect to the basis $\{E_k\}$, any null bivector can be uniquely expressed in the form

$$N = \lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3, \tag{D.4}$$

where $N^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 0$. Now define the *null bivector cone* $\mathscr{N}$ by

$$\mathscr{N} \equiv \{N : N^2 = 0\}.$$

The Cartan-Whittaker null bivector of the spinor $B' = \alpha_1 E_1 + \alpha_2 E_2$ is defined by

$$N \equiv 2\alpha_1\alpha_2 E_1 - (\alpha_1^2 - \alpha_2^2)E_2 - (\alpha_1^2 + \alpha_2^2) I E_3. \tag{D.5}$$

If we choose the null bivector $N_0 = (1 + E_1)E_2$ as a point of reference on the null cone $\mathscr{N}$, then (D.5) takes the direct, equivalent expression

$$N = B' N_0 B'. \tag{D.6}$$

From (D.5) and (D.6) we conclude that any point N on the null cone can be reached by the reflection of the reference point N_0 through an appropriate spinor B' in the spinor plane $\mathscr{P}'$.

The geometric interpretation of spinors in physics has been the concern of many authors, for example [2, 5, 15, 16], but in the view of the present author, (D.6) offers the most direct route to the geometric understanding of the spinor concept, as will be explored in a forthcoming paper entitled "Geometry of Null Bivectors". We complete this sketch of ideas by giving the spinor space a sympletic structure, which is accomplished by defining the sympletic matric

$$A' \square B' \equiv (A' \times B') \bigcirc E_3 = A' \bigcirc (B' \times E_3), \quad \text{for} \quad A', B' \in \mathscr{P}'. \tag{D.7}$$

Let us now determine the most general dual bivector operator L which preserves the form $A' \square B'$, i.e., which satisfies

$$L(A') \square L(B') = A' \square B'. \tag{D.8}$$

By inspection, we see that L is of the form

$$L(B') = L_2 L_1(B') = L_1 L_2(B'), \text{ (polar decomposition)} \tag{D.9}$$

where

$$L_1(B') = \exp(\tfrac{1}{2}\tau E_3)B' = B' \exp(-\tfrac{1}{2}\tau E_3),$$

and

$$L_2(B') = \lambda B' \bigcirc A_1 A_1 + 1/\lambda B' \bigcirc A_2 A_2.$$

We can choose A_1, A_2 to be an orthonormal time-like basis of spinor space with orientation satisfying $A_1 A_2 E_3 = I$. The operator L is completely determined by τ (2 parameters), λ (2 parameters), and A_1 and A_2 (2 parameters are needed to fix A_1 and A_2), a total of 6 parameters. The equation (D.9) determines a Lorentz transformation in spinor space.

We give the following table summarizing the relationship between (usual) spinors, and spinors as we have defined them above.

TABLE III

Spinor	Complex form	STA
φ^{α}	$B' \bigcirc A'$	A'
φ_{α}	$A' \square \partial_{B'}$	$E_3 \times A'$
$\varphi^{\dot{\alpha}}$	$\overline{B' \bigcirc A'}$	A'
$\varphi_{\dot{\alpha}}$	$\overline{A' \square \partial_{B'}}$	$E_3 \times A'$

Let φ^{α}, and ψ^{α} be spinors corresponding to the respective bivectors A' and C'. Then the scalar product of the spinors φ^{α} and ψ^{α} corresponds to

$$\varphi_{\alpha}\psi^{\alpha} = \tfrac{1}{2} A' \square \partial_{B'} B' \bigcirc C' = \tfrac{1}{2} A' \square (\partial_{B'} B' \bigcirc C') = A' \square C'. \tag{D.10}$$

An alternative definition of spinors as elements of a minimal ideal in the Pauli algebra, as discussed in [4; p. 37], deserves mentioning.

Finally, consider the decomposition of a complex vector Z given by

$$Z = Z \cdot (e_3 \wedge e_0) e_3 \wedge e_0 + Z \cdot (e_1 \wedge e_3) e_2 \wedge e_1. \tag{D.11}$$

By noting that

$$e_3 \wedge e_0 = \tfrac{1}{2}(e_3 + e_0) \wedge (e_3 - e_0), \qquad e_1 \wedge e_2 = \tfrac{1}{2}(e_1 + I e_2) \wedge (e_1 - I e_2),$$

we see that (D.11) is equivalent to expanding Z in terms of the familiar Sach's leg

$$\{\eta_u\} = \{e_0 + e_3, e_0 - e_3, e_1 + I e_2, e_1 - I e_2\},$$

and its reciprocal

$$\{\eta^u\} = \{\tfrac{1}{2}(e_0 - e_3), \tfrac{1}{2}(e_0 + e_3), -\tfrac{1}{2}(e_1 - I e_2), -\tfrac{1}{2}(e_1 + I e_2)\};$$

i.e.,

$$Z = Z \cdot \eta^u \eta_u. \tag{D.12}$$

The decomposition (D.11) or (D.12) is equivalent to decomposing Z into its component spinors determined by the dual spinor planes $e_3 \wedge e_0$ and $e_2 \wedge e_1$.

A complete discussion of spinors and twistors as elements of complex projective planes will be given in another paper, but the basic ingredients have been set down in the above considerations. The table below gives the correspondence between spinors with a dotted and an undotted index, and complex vectors:

TABLE IV

Spinor	Complex form	STA
$\varphi^{\alpha\dot{\beta}}$	$Z \cdot Z_1$	Z_1

In reference [4; p. 37], four-component spinors are looked upon as elements of a minimal ideal in the Dirac algebra, and in [5], spinors are given the geometric interpretation of being an even multivector. Reference [17] introduces twistor space as a complex 4-dimensional space with a Hermitian form, but does not identify this space with the space of complex vectors $\mathscr{D}^-$.

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